The Journal of
Mathematics and Computer Science

Available online at
http://www.TJMCS.com

The Journal of Mathematics and Computer Science Vol .2 No.3 (2011) 448-452

Some Normal Edge-transitive Cayley Graphs on Dihedral Groups

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Received: July 2010, Revised: October 2010
Online Publication: January 2011

Abstract

Let G be a group and S a subset of G such that $1_G \notin S$ and $S = S^{-1}$. Let $\Gamma = Cay(G,S)$ be a Cayley graph on G relative to. Then Γ is said to be normal edge-transitive, if $N_{Aut(\Gamma)}(G)$ acts transitively on edges. In this paper we determine all normal edge-transitive Cayley graphs on a dihedral Group D_{2n} of valency n . In addition we classify normal edge-transitive Cayley graphs $\Gamma = Cay(D_{2p}, S)$ of valency four, for a prime p and give some normal edge-transitive Cayley graphs $\Gamma = Cay(D_{2n}, S)$ of valency four that n is not a prime .

Keyword: Cayley graph, normal edge-transitive, Dihedral groups ,

1 Introduction

For a given graph Γ, we denote by $V(\Gamma), E(\Gamma), Aut(\Gamma)$ the vertex set, edge set and automorphism group, respectively. Let G be a group and let S be a subset of G such that $1_G \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = Cay(G,S)$ on G relative to S is defined by $V(\Gamma) = G, E(\Gamma) = \{\{g, sg\} | g \in G, s \in S\}$. The graph $\Gamma = Cay(G,S)$ is vertex-transitive, since $Aut(\Gamma)$ contains the right regular representation G.Thus $G \leq Aut(Cay(G,S))$ and this action of

G is regular on vertices, that is, G is transitive on vertices and only the identity element of G fixes a vertex. A Cayley graph $\Gamma = Cay(G, S)$ is said to be edge-transitive if $Aut(\Gamma)$ is transitivwe on edges. In this paper graphs are finite , simple connected and undirected. It is difficult to find the full automorphism group of a graph in general, and so this makes it difficult to decide whether it is edge-trasitive. On the other hand we often have sufficien information about the group G to determine $N = N_{Aut(\Gamma)}(G)$, because $N$is the semidirect product $N = G.Aut(G, S)$, where $Aut(G, S) = \{\sigma \in Aut(G): S^{\sigma} = S\}$. Hence we focus attention on those graphs for which $N_{Aut\,(\Gamma)}(G)$ is transitive on edges. Such a graph is said to be normal edge-transitive. Thus it is often possible to determine whether $Cay(G, S)$ is normal edge-transitive. In [5] Praeger gave an approach to analyzing normal edge-transitive Cayley graphs. Houlis in [4] determined the isomorphism types of all connected normal edge-transitive undirected Cayley graphs for Z_{pq} where p, q are primes, and for $G = Z_p \times Z_q$, p a prime. In this paper we determined all normal edge-transitive Cayley graphs on a dihedral Group D_{2n} of valency n. In addition we classify normal edge-transitive Cayley graphs $\Gamma = Cay(D_{2p}, S)$ of valency four, for a prime p and give some normal edge-transitive Cayley graphs $\Gamma = Cay(D_{2n}, S)$ of valency four such that n is not a prime .

The group –and graph- theoretic notation and terminology are standard; see [2], [3], and [6] for example.

The rest of this paper is organized as follows: In the section 2 we give some preliminaries. In the section 3 we give all normal edge-transitive Cayley graphs on a dihedral group of valency n and also we classify normal edge-transitive Cayley graphs $\Gamma = Cay(D_{2p}, S)$ of valency four, for a prime p.

2 Preliminaries

The following Propositions are basic for Cayley graphs.

Propositions 2.1. Let $\Gamma = Cay(G, S)$ be a Cayley graph on G relative to S. Then Γ is connected if and only if $G = \langle S \rangle$.

Let $\Gamma = Cay(G, S)$ be a Cayley graph on G relative to S, and let $A = Aut(\Gamma)$. Obviously, $A \geq G.Aut(G, S)$. It is easy to prove the following.

Propositions 2.2. [2]

(1) $N_A(G) = G.Aut(G, S)$,

(2) $A = G.Aut(G, S)$ is equivalence to $G \lhd A$.

Propositions 2.3. [6] Let $\Gamma = Cay(G, S)$ be a Cayley graph on a finite group G. Then Γ is normal edge-transitive if and only if $Aut(G, S)$ is either transitive on S or has two orbits in S which are inverses of each other.

3 Normal edge-transitive Cayley graphs on dihedral groups

Throughout this section, let $D_{2n} = \langle a, b : a^2 = b^n = 1, a^{-1}ba = b^{-1} \rangle$ denote the dihedral group of order $2n$. For $n \geq 3$, any automorphism of D_{2n} can be expressed by $\sigma(r,s)$, where $b^{\sigma(r,s)} = b^r, (n,r) = 1$ and $a^{\sigma(r,s)} = ab^s$.

Lemma 3.1 Let $\Gamma = Cay(D_{2n}, S)$ be a normal edge-transitive Cayley graph on the dihedral group D_{2n} . Then $Aut(D_{2n}, S)$ is transitive on .

Proof. Let $\Gamma = Cay(D_{2n}, S)$ be a normal edge-transitive Cayley graph for a dihedral group D_{2n}. By Proposition 2.3, $Aut(D_{2n}, S)$ is either transitive on S or has two orbits in S which are inverses of each other. We show that the latter case dose not arise. Let $Aut(D_{2n}, S)$ has two orbits T and T^{-1} such that $= T \cup T^{-1}$. Suppose that S contains a^t for some t, and $a^t \in T$. Since $Aut(D_{2n}, S)$ is transitive on T and for every $\sigma \in Aut(D_{2n})$, $(a^t)^\sigma \in < a >$, we have $T \subset \langle a \rangle$. Thus $S = T \cup T^{-1} \subseteq \langle a \rangle$. This is contradiction with connectivity of Γ. Hence elements of S have order2, and we get $T = T^{-1}$, a contradiction.

Propositions 3.2. Let $\Gamma = Cay(D_{2n}, S)$ be a Cayley graph on the dihedral group D_{2n}. Let k be a positive integer with $k \geq 2$. If $S = \{ a, ab, ab^{1+l}, ab^{1+l+l^2}, \ldots, ab^{1+l+\cdots+l^{k-2}} \}$, for an integer l satisfying $1 + l + \cdots + l^{k-1} \equiv 0 (mod\ n)$, then Γ is normal edge transitive.

Proof. Let $\Gamma = Cay(D_{2n}, S)$ be a Cayley graph in which $S = \{ a, ab, ab^{1+l}, ab^{1+l+l^2}, \ldots, ab^{1+l+\cdots+l^{k-2}} \}$, for an integer l satisfying $1 + l + \cdots + l^{k-1} \equiv 0 (mod\ n)$. Consider mapping $\alpha: D_{2n} \to D_{2n}$ by $b \to b^l, a \to ab$. Clearly, we get that $\sigma \in Aut(D_{2n}, S)$ and $< \alpha >$ acts transitively on S. Then by Proposition 2.3, $\Gamma = Cay(D_{2n}, S)$ is normal edge transitive.

Theorem 3.3. Let $\Gamma = Cay(D_{2n}, S)$ be a Cayley graph on the dihedral group D_{2n} of valency . Then Γ is normal edge-transitive if and only if $S = \{ a, ab, ab^2, \ldots, ab^{n-1} \}$.

Proof. Let $\Gamma = Cay(D_{2n}, S)$ such that $S = \{ a, ab, ab^2, \ldots, ab^{n-1} \}$. Then $Aut(D_{2n}, S) = Aut(D_{2n})$. We show that $Aut(D_{2n})$ is transitive on S. For any $ab^i, 0 \leq i \leq n-1$ there exist $\sigma(r,s) \in Aut(D_{2n})$ such that $a^{\sigma(r,i)} = ab^i$. Hence $a^{Aut(D_{2n})}$ =S and this implies that $Aut(D_{2n})$ is transitive on S . Conversely, let $\Gamma = Cay(D_{2n}, S)$, $| S | = n$ be normal edge transitive. If S contains b^t, for some t, then $S = (b^t)^{Aut(D_{2n})} \subseteq \langle b \rangle$ and hence $D_{2n} = \langle S \rangle \subseteq \langle b \rangle$, a contrary. Thus $\subseteq D_{2n} - \langle b \rangle$. Since $| S | = | D_{2n} - \langle b \rangle | = n$, $S = D_{2n} - \langle b \rangle = \{ a, ab, \ldots., ab^{n-1} \}$.

Theorem 3.4. Let $\Gamma = Cay(D_{2n}, S)$ be a Cayley graph on the dihedral group D_{2n} of valency four. If $S = \{ a, ab, ab^i, ab^{1-i} \}$ such that $(n, 2i-1) = 1, 2i(1-i) \equiv 0 (mod\ n)$ then Γ is normal edge transitive.

Proof. Let $\Gamma = Cay(D_{2n}, S)$ be a Cayley graph on the dihedral group D_{2n} of valency four, for $S = \{ a, ab, ab^2, \ldots, ab^{n-1}\}$ such that $(n, 2i-1) = 1, 2i(1-i) \equiv 0 (mod\, n)$.We show that $Aut(D_{2n}, S)$ is transitive on S. There are automorphisms $\sigma_1 = \sigma(n-1, 1)$, $\sigma_2 = \sigma(n-(2i-1), i)$, , and $\sigma_3 = \sigma(2i-1, 1-i)$ such that $a^{\sigma_1} = ab, a^{\sigma_2} = ab^j$ and $a^{\sigma_3} = ab^{1-i}$. Also we have $\sigma_1, \sigma_2, \sigma_3 \in Aut(G, S)$. Hence $Aut(D_{2n}, S)$ is transitive on S, and by Proposition 2.3 Γ is normal edge transitive.

Theorem 3.5. Let $\Gamma = Cay(D_{2p}, S)$ be a Cayley graph on the dihedral group D_{2p} of valency four, where p is a prime number. Then Γ is normal edge-transitive, if and only if Γ is isomorphic with $\Gamma = Cay(D_{2p}, S)$ where $S = \left\{ a, ab, ab^{1+l}, ab^{1+l+l^2}\right\}$ for an integer l satisfying $1 + l + l^2 + l^3 \equiv 0 (mod\, n)$.

Proof. By Proposition 3.2, if $S = \left\{ a, ab, ab^{1+l}, ab^{1+l+l^2}\right\}$ for an integer l satisfying $1 + l + l^2 + l^3 \equiv 0 (mod\, p)$, then graph $Cay(D_{2p}, S)$ is normal edge- transitive. Conversely, let $\Gamma = Cay(D_{2p}, S)$ be a normal edge-transitive Cayley graph on the dihedral group D_{2p} of valency four. By lemma 3.1, $Aut\big(D_{2p}, S\big)$ is transitive on S. Since for every automorphism α of $Aut\big(D_{2p}\big)$, $\langle b\rangle^\alpha = \langle b\rangle$, S is not contain $b^t, 0 \le t \le p-1$, thus we may *assume that* $S = \{ a, ab^i, ab^j, ab^k\}$. *We have only two transitive permutation group on S that are following :*

(I) $\quad Aut\big(D_{2p}, S\big) = \langle \alpha = (a, ab^i, ab^j, ab^k)\rangle$

(II) $\quad Aut\big(D_{2p}, S\big) = \langle \alpha_1 = (a, ab^i)(ab^j, ab^k), \alpha_2 = (a, ab^j)(ab^i, ab^k) \rangle$

In the case (I), we have $a^\alpha = ab^i, (ab^i)^\alpha = ab^j, (ab^j)^\alpha = ab^k$ and $(ab^k)^\alpha = a$. It follows that $(a.ab^i)^\alpha = ab^i.ab^j$, $(ab^i.ab^j)^\alpha = ab^j.ab^k$, $(ab^j.ab^k)^\alpha = ab^k.a$ and $(ab^k.a)^\alpha = a.ab^i$, that is $(b^i)^\alpha = b^{j-i}$, $(b^{j-i})^\alpha = b^{k-j}$, $(b^{k-j})^\alpha = b^{-k}$ and $(b^{-k})^\alpha = b^i$. Sinse α is a group automorphism of $\langle b\rangle$, This implies that $\langle b^j\rangle \subseteq \langle b^i\rangle, \langle b^k\rangle \subseteq \langle b^i\rangle$, and hence $\langle b^i\rangle \subseteq \langle b\rangle$. Thus $(i, p) = 1$ and without loss of generality,we may assume that $i = 1$ and $S = \{ a, ab, ab^j, ab^k\}$.So we have $(a)^\alpha = ab$, $(ab)^\alpha = ab^j$, $(ab)^\alpha = ab^j$,$(ab^j)^\alpha = ab^k$ and $(ab^k)^\alpha = a$. Hence $(b)^\alpha = (a.ab)^\alpha = (a)^\alpha(ab)^\alpha = ab.ab^{i_2} = b^{j-1}$ and $ab^k = (ab^j)^\alpha = ab\, b^{j(j-1)} = ab^{j(j-1)+1}, a = (ab^k)^\alpha = ab.b^{k(j-1)} = ab^{j(j-1)^2+(j-1)+1}$. Let $= j-1$. Then we have $b^k = b^{l^2+l+1}$ and $1 + l + l^2 + l^3 \equiv 0 (mod\, p)$. Thus $S = \left\{ a, ab, ab^{1+l}, ab^{1+l+l^2}\right\}$ and $1 + l + l^2 + l^3 \equiv 0 (mod\, p)$.

We show that the case (II) dos not arise. Let $\alpha_1 = \sigma(r, i)$, which $(r, p) = 1$. Then we have $(ab^i)^{\alpha_1} = ab^i.b^{ri} = a$. Hence $b^{i+ri} = e$ and we conclude that $|\ i(1+r)$. It follows that $p = 1 + r$ and so $\alpha_1 = \sigma(p-1, i)$. Furthermore, we have $ab^k = (ab^j)^{\alpha_1} = ab^i b^{j(p-1)} = ab^{i-j}$. This implies that $k \equiv i - j (mod\, p)$. Similarly, we get that$\alpha_2 = \sigma(p-1, j)$ and $k \equiv j - i (mod\, p)$. Hence $i - j \equiv j - i (mod\, p)$. This implies that $|\ 2(i-j)$, contrary.

In the end we give the following conjecture .

Conjecture. If Γ be a normal edge transitive Cayley graph on dihedral group of valency four , then Γ is isomorphism with $Cay(D_{2n}, S)$ in which either $S = \left\{ a, ab, ab^{1+l}, ab^{1+l+l^2}\right\}$ for an integer l satisfying$1 + l + l^2 + l^3 \equiv 0 (mod\, p)$ or $S = \{ a, ab, ab^i, ab^{i-1}\}$ such that $(n, 2i-1) = 1, 2i(1-i) \equiv 0 (mod\, n)$.

Proposition (3.3) shows that this conjecture holds for prime number n.

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