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On the existence for impulsive fuzzy nonlinear integro-differential equations with nonlocal condition

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Abstract

This work is involved with first-order impulsive functional nonlinear fuzzy integro-differential equations with nonlocal condition in Banach space by using the concept of fuzzy numbers whose values are upper semicontinuous, normal, convex, and compact. The result is obtained by using the Leray-Schauder alternative fixed point theorem. Finally, an application is given that supports us to validate the result.

Keywords: Nonlocal condition, Leary-Schauder alternative fixed point theorem, fuzzy nonlinear integro-differential equations, mild solution.

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1. Introduction

The study of impulsive integro-differential equations has attracted the attention of several researchers from all over the world in recent years [4, 5, 8, 10], which can be used as modeling for many problems in the field of physics, economics, biology, medicine, mechanics, and many other fields. To learn more about impulsive functional differential equations, we refer to the studies [2, 17]. Fuzzy differential equations provide an appropriate framework for the mathematical representation of real-world issues when uncertainty or ambiguity is pervasive [13, 14]. To our best knowledge, there are not many studies on theory of impulsive fuzzy differential equations. Some papers can be found in [7, 9, 11, 12, 18].

In [3] Benchohra et al. studied existence of fuzzy solution for impulsive differential equations

$$\rho'(\tau) = \mathcal{P}(\tau, \rho_\tau), \qquad \rho(\tau_0) = a \in \mathbb{X}^n, \qquad \Delta\rho(\tau_n) = \mathcal{I}_n\rho(\tau_n), \;\; \tau \neq t_n, \;\; n = 1, 2, \ldots, k,$$

by using a fixed point theorem for absolute retract, and in [16], Ramesh et al. studied existence and uniqueness of a solution of the fuzzy impulsive differential equation by using the method of successive approximation.

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Motivated by above work, in this paper we study the fuzzy nonlocal impulsive integro-differential equations of the type:

$$\begin{aligned}
&\rho'(\tau) = \mathcal{A}\rho(\tau) + \mathcal{P}(\tau, \rho_\tau, \int_0^\tau \mathcal{H}(\tau, \mu, \rho_\mu)d\mu), \quad \tau \in (0, \mathcal{T}],\\
&\Delta\rho(\tau_n) = \mathcal{I}_n \rho(\tau_n), \tau \neq t_n, \quad n = 1, 2, \dots, k,\\
&\rho(\tau) + \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(\tau) = \psi(\tau), \quad \tau \in [-\mathfrak{r}, 0],
\end{aligned} \tag{1.1}$$

where $\mathcal{A} : [0, \mathcal{T}] \to \mathbb{X}^n$ is the fuzzy coefficient, $\mathbb{X}^n$ is the set of all normal, convex, and upper semicontinuous fuzzy numbers with bounded α-levels, $\mathcal{P} : [0, \mathcal{T}] \times \mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$, $\mathcal{H} : [0, \mathcal{T}] \times [0, \mathcal{T}] \times \mathbb{X}^n \to \mathbb{X}^n$ and $\mathfrak{h} : (\mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n)^q \to \mathbb{X}^n$ are regular fuzzy nonlinear functions, $\mathcal{I}_n \in \mathcal{C}(\mathbb{X}^n, \mathbb{X}^n)$, and $\psi : [-\mathfrak{r}, 0] \to \mathbb{X}^n$ are bounded functions. $\Delta\rho(\tau_n) = \rho(\tau_n^+) - \rho(\tau_n^-)$, $\rho(\tau_n^+) = \lim_{h\to 0^+} \rho(t_n + h), \rho(\tau_n^-) = \lim_{h\to 0^+} \rho(t_n - h)$ represents the left and right limits of $\rho(\tau)$ at $\tau = t_n$, respectively, $n = 1, 2, \dots, k$. For any function ρ defined on $[-\mathfrak{r}, \mathcal{T}]$ and any $\tau \in [0, \mathcal{T}]$, we denote ρ_τ the element of $\mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n)$ defined by $\rho_\tau(w) = \rho(\tau + w); w \in [-\mathfrak{r}, 0]$. Here, $\rho_\tau(.)$ represents the history of the state from time $\tau - \mathfrak{r}$, up to the present time τ.

The objective of the article is to investigate the existence of mild solutions of equation (1.1). We use the Leray- Schauder alternative, and the generalization of Grownwall-type inequality to derive the result. We are improving and generalizing the results mentioned in [3, 16].

This work is structured as follows. We deal with preliminaries and hypotheses in Section 2. We present the statement and proof of the main result in Section 3. We given example to validate the result in Section 4. Finally we conclude the study in Section 5.

2. Preliminaries and hypotheses

Let $P_r(\mathfrak{R}^n)$ be the family consisting of all nonempty, convex, and compact subsets of $\mathfrak{R}^n$. Denote by $\mathbb{X}^n = \{\vartheta : \mathfrak{R}^n \to [0, 1]$ such that ϑ satisfies (1)-(4) as bellow$\}$.

(1) ϑ is normal, that is, there exists an $\rho_0 \in \mathfrak{R}^n$ such that $\vartheta(\rho_0) = 1$.
(2) ϑ is fuzzy convex, that is, for $\rho, \nu \in \mathfrak{R}^n$ and $0 < \lambda \leqslant 1$, $\vartheta(\lambda\rho + (1 - \lambda)\nu) \geqslant \min\{\vartheta(\rho), \vartheta(\nu)\}$.
(3) ϑ is upper semicontinuous.
(4) $[\vartheta]^0 = \overline{\{\rho \in \mathfrak{R}^n : \vartheta(\rho) > 0\}}$ is compact.

For $0 < \alpha \leqslant 1, [\vartheta]^\alpha = \{\rho \in \mathfrak{R}^n : \vartheta(\rho) \geqslant \alpha\}$. Then from (1)–(4), it follows that the α-level sets $[\vartheta]^\alpha \in P_r(\mathfrak{R}^n)$. If $\mathfrak{h} : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ is a function, then by using Zadeh's extension principle, we can extend $\mathfrak{h}$ to $\mathbb{X}^n \times \mathbb{X}^n \to \mathbb{X}^n$ by the equation $[\mathfrak{h}(\vartheta, \sigma)(w)] = \sup_{w=\mathfrak{h}(\rho,\nu)} \min\{\vartheta(\rho), \sigma(\nu)\}$. It is well knowledge that $[\mathfrak{h}(\vartheta, \sigma)]^\alpha = \mathfrak{h}([\vartheta]^\alpha, [\sigma]^\alpha)$, $\forall \vartheta, \sigma \in \mathbb{X}^n$,$0 \leqslant \alpha \leqslant 1$ and the function $\mathfrak{h}$ is a continuous. In addition, we have

$$[\vartheta + \sigma]^\alpha = [\vartheta]^\alpha + [\sigma]^\alpha, \quad [a\vartheta]^\alpha = a[\vartheta]^\alpha,$$

where

$$\vartheta, \sigma \in \mathbb{X}^n, \quad 0 \leqslant \alpha \leqslant 1, \quad a \in \mathfrak{R}.$$

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two nonempty bounded subsets of $\mathfrak{R}^n$. The distance between $\mathcal{B}_1$ and $\mathcal{B}_2$ is determined by using the Hausdorff metric

$$\mathcal{H}_d(\mathcal{B}_1, \mathcal{B}_2) = \max\left\{ \sup_{b_1\in\mathcal{B}_1} \inf_{b_2\in\mathcal{B}_2} \|b_1 - b_2\|, \sup_{b_2\in\mathcal{B}_2} \inf_{b_1\in\mathcal{B}_1} \|b_1 - b_2\| \right\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathfrak{R}^n$. Then $(P_r(\mathfrak{R}^n), \mathcal{H}_d)$ is a separable and complete metric space [15]. We define the complete metric d_∞ on $\mathbb{X}^n$ by

$$d_\infty(\vartheta, \sigma) = \sup_{0<\alpha\leqslant 1} \mathcal{H}_d([\vartheta]^\alpha, [\sigma]^\alpha) = \sup_{0<\alpha\leqslant 1} [\vartheta_l^\alpha - \sigma_l^\alpha, \vartheta_r^\alpha - \sigma_r^\alpha],$$

for all $\vartheta, \sigma \in \mathbb{X}^n$. $(\mathbb{X}^n, d_\infty)$ is a complete metric space. Also $\forall \vartheta, \sigma, \mu \in \mathbb{X}^n$ and $\lambda \in \mathfrak{R}$, we have $d_\infty(\vartheta + \mu, \sigma + \mu) = d_\infty(\vartheta, \sigma)$ and $d_\infty(\lambda\vartheta, \lambda\sigma) = |\lambda| d_\infty(\vartheta, \sigma)$. We define $\hat{0} \in \mathbb{X}^n$ as $\hat{0}(\rho) = 1$ if $\rho = 0$ and $\hat{0}(\rho) = 0$ if $\rho \neq 0$. The supremum metric $\mathcal{H}_1$ on $C([0,1], \mathbb{X}^n)$ is defined by

$$\mathcal{H}_1(\vartheta, \sigma) = \sup_{0 \leqslant \tau \leqslant \mathcal{T}} d_\infty(\vartheta(\tau), \sigma(\tau)).$$

Hence $(\mathcal{C}([0,1], \mathbb{X}^n), \mathcal{H}_1)$ is a complete metric space.

Definition 2.1. A family of functions $(\mathcal{D}(\tau))_{\tau \geqslant 0}$ of continuous linear operators on $\mathbb{X}^n$ is called fuzzy $\mathcal{C}_0$-semigroup if

1. for all $\rho \in \mathbb{X}^n$ the mapping $\mathcal{D}(\tau)(\rho) : \mathfrak{R}^+ \to \mathbb{X}^n$ is continuous with respect to $\tau \geqslant 0$;
2. $\mathcal{D}(\tau + \mu) = \mathcal{D}(\tau)\mathcal{D}(\mu)$, $\forall \tau, \mu \in \mathfrak{R}^+$;
3. $\mathcal{D}(0) = I$, where I is the identity operator on $\mathbb{X}^n$.

Definition 2.2. A continuous function $\rho(\tau) : [0, \mathcal{T}] \to \mathbb{X}^n$ is said to be a mild solution of equation (1.1) if

$$\begin{aligned}
\rho(\tau) &= \mathcal{D}(\tau)[\psi(0) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0)] + \int_0^\tau \mathcal{D}(\tau - \mu)\mathcal{P}\Big(\mu, \rho_\mu, \int_0^\mu \mathcal{H}(\mu, \sigma, \rho_\sigma) d\sigma\Big) d\mu \\
&\quad + \sum_{0 < t_n < \tau} \mathcal{D}(\tau - t_n)\mathcal{I}_n \rho(t_n), \qquad \tau \in (0, \mathcal{T}], \\
\rho(\tau) &+ \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\tau) = \psi(\tau), \quad \tau \in [-\mathfrak{r}, 0].
\end{aligned}$$

Theorem 2.3 (Leary-Schauder alternative, [6]). *Assume that completely continuous operator is $\Lambda : \mathcal{S} \to \mathcal{S}$ and suppose that $0 \in \mathcal{S}$. Let $\mathcal{S}$ be a convex subset of a Banach space $\mathcal{E}$, and let*

$$\mathcal{V}(\Lambda) = \{\rho \in \mathcal{S} : \rho = \lambda\Lambda\rho, \textit{ for some } 0 < \lambda < 1\}.$$

Then either $\mathcal{V}(\Lambda)$ is unbounded or Λ has fixed point.

Lemma 2.4 ([1]). *Let for $\tau \geqslant \tau_0$, the following inequality holds*

$$\rho(\tau) \leqslant b(\tau) + \int_{\tau_0}^\tau c(\tau, \mu)\rho(\mu) d\mu + \int_{\tau_0}^\tau \Big(\int_{\tau 0}^\mu h(\tau, \mu, \sigma)\rho(\sigma) d\sigma\Big) d\mu + \sum_{\tau_0 < \sigma_n < \tau} \gamma_n(\tau)\rho(\tau_n),$$

where $\rho, b \in \mathcal{PC}([\tau_0, \infty), \mathbb{R}_+)$, b is nondecreasing, $c(\tau, \mu)$ and $h(\tau, \mu, \sigma)$ are continuous and nonnegative functions for $\tau, \mu, \sigma \geqslant \tau_0$ and are nondecreasing with respect to τ, $\gamma_n(\tau)(n \in \mathbb{N})$ are nondecreasing for $\tau \geqslant \tau_0$, then for $\tau \geqslant \tau_0$ the following inequality holds

$$\rho(\tau) \leqslant b(\tau) \prod_{\tau_0 < \sigma_n < \tau} (1 + \gamma_n(\tau)) \exp\Big(\int_{\tau_0}^\tau c(\tau, \mu) d\mu\Big) + \int_{\mu_0}^\mu \int_{\tau_0}^\mu h(\tau, \mu, \sigma) d\sigma) d\mu.$$

We introduce the following hypotheses.

($\mathcal{A}_0$) The linear and continuous operator $\mathcal{A}$ generates a $\mathcal{C}_0$ semigroup $(\mathcal{D}(\tau))_{\tau \geqslant 0}$ on $\mathbb{X}^n$ such that

$$\|\mathcal{D}(\tau)\|_{\mathbb{X}^n} \leqslant \mathcal{M} \;\; \forall \tau \geqslant 0 \text{ with } \mathcal{M} > 0.$$

($\mathcal{A}_1$) Let $\mathcal{P} : [0, \mathcal{T}] \times \mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$ and $\mathcal{H} : [0, \mathcal{T}] \times [0, \mathcal{T}] \times \mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \to \mathbb{X}^n$ such that for every $\eta \in \mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n)$, $\tau, \mu \in [0, \mathcal{T}]$, $\rho \in \mathbb{X}^n$ and there exists a continuous function $\mathfrak{p}, \mathfrak{q} : [0, \mathcal{T}] \to \mathfrak{R}_+$ such that:

$$d_\infty(\mathcal{P}(\tau, \eta, \rho), \hat{0}) \leqslant \mathfrak{p}(\tau)(d_\infty(\eta, \hat{0}) + d_\infty(\rho, \hat{0})), \qquad d_\infty(\mathcal{H}(\tau, \mu, \eta), \hat{0}) \leqslant \mathfrak{q}(\tau) d_\infty(\eta, \hat{0}).$$

($\mathcal{A}_2$) Let $\mathfrak{h} : (\mathcal{C}[-\mathfrak{r},0],\mathbb{X}^n)^q \to \mathbb{X}^n$ such that for every $\rho_{\sigma_n}, \nu_{\sigma_n} \in \mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n)$, $n = 1,2,\dots,q$ and there exists positive constants $\mathcal{Q}_n, \mathcal{Q}_1$, $n = 1,2,\dots,q$ such that:

$$d_\infty(\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(\tau),\mathfrak{h}(\nu_{\sigma_1},\nu_{\sigma_2},\dots,\nu_{\sigma_q})(\tau)) \leqslant \sum_{n=1}^{q} \mathcal{Q}_n d_\infty(\rho_{\sigma_n},\nu_{\sigma_n}),$$
$$\max\{d_\infty(\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(\tau),\hat{0})\} \leqslant \mathcal{Q}_1, \quad \forall \tau \in [-\mathfrak{r},0].$$

($\mathcal{A}_3$) Let $I_n : \mathbb{X}^n \to \mathbb{X}^n$ such that for every $\rho \in \mathbb{X}^n$, $\tau \in [0,\mathcal{T}]$, $n = 1,\dots,k$ and there exists a constant L_n such that

$$d_\infty(\mathcal{I}_n\rho(\tau_n),\hat{0}) \leqslant L_n d_\infty(\rho,\hat{0}).$$

($\mathcal{A}_4$) For each positive number $\mathfrak{r}$, there exists a function $\gamma_\mathfrak{r} \in L^1([0,T],\mathfrak{R}_+)$ such that:

$$\sup\{d_\infty(\mathcal{P}(\tau,\eta,\rho),\hat{0}) : d_\infty(\eta,\hat{0}) \leqslant \mathfrak{r}, d_\infty(\rho,\hat{0}) \leqslant \mathfrak{r}\} \leqslant \gamma_\mathfrak{r}(\tau)$$

for each $\tau \in [0,\mathcal{T}]$.

($\mathcal{A}_5$) The function $\mathcal{P}(\tau,.,.) : \mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$ is continuous, for each $\tau \in [0,\mathcal{T}]$ and the function $\mathcal{P}(.,\eta,\rho) : [0,\mathcal{T}] \to \mathbb{X}^n$ is strongly measurable, for each $(\eta,\rho) \in \mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n) \times \mathbb{X}^n$.

($\mathcal{A}_6$) The function $\mathcal{H}(\tau,.,.) : \mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n) \to \mathbb{X}^n$ is continuous, for each $t \in [0,T]$ and the function $\mathcal{H}(.,.,\eta) : [0,\mathcal{T}] \to \mathbb{X}^n$ is strongly measurable, for each $\eta \in \mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n)$.

3. Main result

Theorem 3.1. *Assume that the hypotheses* ($\mathcal{A}_0$)-($\mathcal{A}_6$) *hold. Then the equation* (1.1) *has a mild solution* ρ *on*$[-\mathfrak{r},\mathcal{T}]$.

Proof. To prove the existence of mild solution of the equation (1.1) first we establish the priori bounds on the solutions to the equation (1.1). For $\lambda \in (0,1)$,

$$\begin{aligned} &\rho'(\tau) = \mathcal{A}\rho(\tau) + \lambda\mathcal{P}(\tau,\rho_\tau,\int_0^\tau \mathcal{H}(\tau,\mu,\rho_\mu)d\mu), \quad \tau \in (0,\mathcal{T}],\\ &\Delta\rho(\tau_n) = \mathcal{I}_n\rho(\tau_n), \tau \neq t_n, \quad n = 1,2,\dots,k,\\ &\rho(\tau) + \mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(\tau) = \psi(\tau), \quad \tau \in [-\mathfrak{r},0]. \end{aligned} \tag{3.1}$$

Let $\rho(\tau)$ be a solution of equation (3.1). Then it satisfies equivalent integral equation

$$\begin{aligned} \rho(\tau) = {} & \mathcal{T}(\tau)[\psi(0) - \mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(0)] + \lambda\int_0^\tau \mathcal{T}(\tau-\mu)\mathcal{P}(\tau,\rho_\tau,\int_0^\mu \mathcal{H}(\mu,\sigma,\rho_\sigma)d\sigma)d\mu\\ & + \sum_{0<\sigma_n<\tau} \mathcal{T}(\tau-\sigma_n)\mathcal{I}_n\rho(\sigma_n) \quad ,\tau \in (0,\mathcal{T}]. \end{aligned}$$

Let $\|\psi\|_{\mathbb{X}^n} = \mathcal{G}$, using hypotheses ($\mathcal{A}_0$)-($\mathcal{A}_4$) and the truth that $\lambda \in (0,1)$, we have for $\tau \in [0,\mathcal{T}]$

$$\begin{aligned} d_\infty(\rho(\tau),\hat{0}) \leqslant {} & d_\infty(\mathcal{D}(\tau)\psi(0),\hat{0}) + d_\infty(\mathcal{D}(\tau)(\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(\tau),\hat{0}) + d_\infty(\lambda\int_0^\tau \mathcal{D}(\tau-\mu)\mathcal{P}(\tau,\rho_\tau\\ & ,\int_0^\mu \mathcal{H}(\mu,\sigma,\rho_\sigma)d\sigma),\hat{0})d\mu + d_\infty(\sum_{0<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(\rho(\sigma_n),\hat{0})\\ \leqslant {} & \mathcal{M}\|\psi\|_{\mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n)} + \mathcal{M}d_\infty(\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\cdots,\rho_{\sigma_q})(\tau),\hat{0}) + \mathcal{M}d_\infty(\int_0^\tau \mathcal{P}(\tau,\rho_\tau\\ & ,\int_0^\mu \mathcal{H}(\mu,\sigma,\rho_\sigma)d\sigma),\hat{0})d\mu + \mathcal{M}d_\infty(\sum_{0<\sigma_n<\tau} \mathcal{I}_n(\rho(\sigma_n),\hat{0})) \end{aligned}$$

$$\leqslant \mathcal{M}(\mathcal{G}+\mathcal{Q}_1)+\int_0^\tau \mathcal{M}\mathfrak{p}(\mu)[d_\infty(\rho_\mu,\hat{0})+\int_0^\mu \mathfrak{q}(\tau)d_\infty(\rho_\tau,\hat{0})d\tau]d\mu+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n d_\infty(\rho(\sigma_n),\hat{0}).$$

Then we get

$$\sup_{0\leqslant\tau\leqslant\mathcal{T}}(d_\infty(\rho(\tau),\hat{0}))\leqslant \sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big\{\mathcal{M}(\mathcal{G}+\mathcal{Q}_1)+\int_0^\tau \mathcal{M}\mathfrak{p}(\mu)[d_\infty(\rho_\mu,\hat{0})+\int_0^\mu \mathfrak{q}(\tau)d_\infty(\rho_\tau,\hat{0})d\tau]d\mu$$
$$+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n d_\infty(\rho(\sigma_n),\hat{0})\Big\}.$$

Therefore

$$\mathcal{H}_1(\rho(\tau),\hat{0})\leqslant \mathcal{M}(\mathcal{G}+\mathcal{Q}_1)+\int_0^\tau \mathcal{M}\mathfrak{p}(\mu)[\mathcal{H}_1(\rho_\mu,\hat{0})+\int_0^\mu \mathfrak{q}(\tau)\mathcal{H}_1(\rho_\tau,\hat{0})d\tau]d\mu+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n\mathcal{H}_1(\rho(\sigma_n),\hat{0}). \tag{3.2}$$

Let $\mathcal{R}(\tau)=\sup\{\mathfrak{p}(\tau),\mathfrak{q}(\tau)\}$ and $\mathcal{R}^*=\sup\{\mathcal{R}(\tau):\tau\in[-\mathfrak{r},\mathcal{T}]\}$. Define the function $\nu:[-\mathfrak{r},\mathcal{T}]\to\mathfrak{R}$ by $\nu(\tau)=\sup\{d_\infty(\rho(\mu),\hat{0}):-\mathfrak{r}\leqslant\mu\leqslant\tau\},\tau\in[0,\mathcal{T}]$. Let $\tau^*\in[-\mathfrak{r},\tau]$ be such that $\nu(\tau)=\mathcal{H}_1(\rho(\tau^*),\hat{0})$. If $\tau^*\in[0,\tau]$, then from (3.2) we have

$$\begin{aligned}
\nu(\tau)&=\mathcal{H}_1(\rho(\tau^*),\hat{0})\\
&\leqslant \mathcal{M}(\mathcal{G}+\mathcal{Q}_1)+\int_0^{\tau^*} \mathcal{M}\mathfrak{p}(\mu)[\mathcal{H}_1(\rho_\mu,\hat{0})+\int_0^\mu \mathfrak{q}(\tau)\mathcal{H}_1(\rho_\tau,\hat{0})d\tau]d\mu+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n\mathcal{H}_1(\rho(\sigma_n),\hat{0})\\
&\leqslant \mathcal{M}(\mathcal{G}+\mathcal{Q}_1)+\int_0^{\tau} \mathcal{M}\mathfrak{p}(\mu)[\mathcal{H}_1(\rho_\mu,\hat{0})+\int_0^\mu \mathfrak{q}(\tau)\mathcal{H}_1(\rho_\tau,\hat{0})d\tau]d\mu+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n\mathcal{H}_1(\rho(\sigma_n),\hat{0})\\
&\leqslant \mathcal{M}(\mathcal{G}+\mathcal{Q}_1)+\int_0^{\tau} \mathcal{M}\mathcal{R}(\mu)[\nu(\mu)+\int_0^\mu \mathcal{R}(\tau)\nu(\tau)d\tau]d\mu+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n\nu(\sigma_n)\\
&\leqslant \mathcal{M}(\mathcal{G}+\mathcal{Q}_1)+\int_0^{\tau} \mathcal{M}\mathcal{R}(\mu)\nu(\mu)d\mu+\int_0^\tau[\int_0^\mu \mathcal{M}\mathcal{R}(\mu)\mathcal{R}(\tau)\nu(\tau)d\tau]d\mu+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n\nu(\tau_n).
\end{aligned}\tag{3.3}$$

If $\tau^*\in[-\mathfrak{r},0]$, then

$$\nu(\tau)\leqslant\|\psi\|_{\mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n)}+\mathcal{Q}_1\leqslant\mathcal{G}+\mathcal{Q}_1. \tag{3.4}$$

In view of inequalities (3.3) and (3.4), we can say that for $\tau^*\in[-\mathfrak{r},\mathcal{T}]$, the inequality (3.3) holds good. Now applying inequality given in Lemma 2.4, we get

$$\begin{aligned}
\nu(\tau)&\leqslant\mathcal{M}(\mathcal{G}+\mathcal{Q}_1)\prod_{0<\sigma_n<\tau}(1+\mathcal{M}L_n)\exp\Big\{\int_0^\tau\mathcal{M}\mathcal{R}(\mu)d\mu+\int_0^\tau[\int_0^\mu\mathcal{M}\mathcal{R}(\mu)\mathcal{R}(\tau)d\tau]d\mu\Big\}\\
&\leqslant\mathcal{M}(\mathcal{G}+\mathcal{Q}_1)\prod_{0<\sigma_n<\tau}(1+\mathcal{M}L_n)\exp\Big\{\mathcal{M}\mathcal{R}^*\mathcal{T}+\mathcal{M}\mathcal{R}^{*2}\frac{\mathcal{T}^2}{2}\Big\}=\mathcal{K},
\end{aligned}$$

where $\mathcal{K}$ is constant. Therefore, we have $d_\infty(\rho,\hat{0})=\sup\{d_\infty(\rho(\tau),\hat{0}):\tau\in[-\mathfrak{r},\mathcal{T}]\}\leqslant\mathcal{K}$. Now we rewrite equation (1.1) as follows: For $\psi\in\mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n)$, define $\hat{\psi}\in\mathbb{X}^n$ by

$$\hat{\psi}(\tau)=\begin{cases}\psi(\tau)-\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(\tau), & \text{if } \tau\in[-\mathfrak{r},0],\\ \mathcal{D}(\tau)[\psi(0)-\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(0)], & \text{if } \tau\in[0,\mathcal{T}].\end{cases}$$

If $w \in \mathbb{X}^n$ and $\rho(\tau) = w(\tau) + \hat{\psi}(\tau)$, $\tau \in [-\mathfrak{r}, \mathcal{T}]$, then it is easy to see that w satisfies

$$w(\tau) = \begin{cases} 0, & \text{if } \tau \in [-\mathfrak{r}, 0], \\ \int_0^\tau \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, w_\mu + \hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu, \sigma, w_\sigma + \hat{\psi}_\tau)d\tau)d\mu & \\ \quad + \sum\limits_{0<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(w_\sigma + \hat{\psi}(\sigma_n)), & \text{if } \tau \in [0, \mathcal{T}], \end{cases}$$

if and only if $\rho(\tau)$ satisfies equivalent integro-differential to the (1.1),

$$\begin{aligned} \rho(\tau) &= \mathcal{D}(\tau)[\psi(0) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(0)] + \int_0^\tau \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, \rho_\mu, \int_0^\mu \mathcal{H}(\mu, \tau, \rho_\sigma)d\sigma)d\mu \\ &\quad + \sum_{0<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(\rho(\sigma_n)), \quad \tau \in [0, \mathcal{T}], \\ \rho(\tau) &= \psi(\tau) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(\tau), \quad \tau \in [-\mathfrak{r}, 0], \\ \Delta\rho(\tau_n) &= \mathcal{I}_n \rho(\tau_n), \quad n = 1, 2, \dots, k. \end{aligned} \tag{3.5}$$

We define the operator $\Lambda : \mathbb{X}_0^n \to \mathbb{X}_0^n$, $\mathbb{X}_0^n = \{w \in \mathbb{X}^n : w_0 = 0\}$ by

$$(\Lambda w)(\tau) = \begin{cases} 0, & \text{if } \tau \in [-\mathfrak{r}, 0], \\ \int_0^\tau \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, w_\mu + \hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu, \sigma, w_\sigma + \hat{\psi}_\sigma)d\sigma)d\mu & \\ \quad + \sum\limits_{0<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(w_\sigma + \hat{\psi}(\sigma_n)), & \text{if } \tau \in [0, \mathcal{T}]. \end{cases} \tag{3.6}$$

From the definition of an operator Λ defined by the equation (3.6), it is to be noted that the equation (3.5) can be written as $w = \Lambda w$ and the integral equation (3.1) can be written as $w = \lambda \Lambda w$.

Now we prove that Λ is completely continuous. First, we show that $\Lambda : \mathbb{X}_0^n \to \mathbb{X}_0^n$ is continuous. Let $\{\rho_m\}$ be a sequence of elements of $\mathbb{X}_0^n$ converging to ρ in $\mathbb{X}_0^n$. Then there exists an integer $\mathcal{L}$ such that $d_\infty(\rho_m(\tau), \hat{0}) \leqslant \mathcal{L}$, $\forall m$, $\tau \in [0, \mathcal{T}]$. So $\rho_m \in \mathbb{X}_0^n$ and $\rho \in \mathbb{X}_0^n$. Then by using hypothesis $(\mathcal{A}_5)$-$(\mathcal{A}_6)$ we have

$$\mathcal{P}(\tau, \rho_{m\tau} + \hat{\psi}_\tau, \int_0^\tau \mathcal{H}(\tau, \mu, \rho_{m\mu} + \hat{\psi}_\mu)d\mu) \to \mathcal{P}(\tau, \rho_\tau + \hat{\psi}_\tau, \int_0^\tau \mathcal{H}(\tau, \mu, \rho_\mu + \hat{\psi}_\mu)d\mu)$$

and since $\mathcal{I}_n$ are continuous, we get

$$\sum_{0<\sigma_n<\tau} \mathcal{I}_n(\rho_m(\sigma_n)) \to \sum_{0<\sigma_n<\tau} \mathcal{I}_n(\rho(\sigma_n)),$$

for each $\tau \in [0, \mathcal{T}]$. Since

$$d_\infty \mathcal{P}\left(\tau, \rho_{m\tau}, \hat{\psi}_\tau, \int_0^\tau \mathcal{H}(\tau, \mu, \rho_{m\mu} + \hat{\psi}_\mu)d\mu, \hat{0}\right) + d_\infty \mathcal{P}\left(\tau, \rho_\tau, \hat{\psi}_\tau, \int_0^\tau \mathcal{H}(\tau, \mu, \rho_\mu + \hat{\psi}_\mu)d\mu, \hat{0}\right) \leqslant 2\gamma_{\mathcal{L}'}$$

where $\mathcal{L}' = \max\{\mathcal{L} + d_\infty(\hat{\psi}, \hat{0}), \mathcal{T}\mathcal{R}^*[\mathcal{L} + d_\infty(\hat{\psi}, \hat{0})]\}$. Then by dominated convergence theorem we have

$$\begin{aligned} d_\infty((\Lambda\rho_m)(\tau), (\Lambda\rho(\tau))) \leqslant d_\infty \Bigg(& \int_0^\tau \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, \rho_{m\mu} + \hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu, \tau, \rho_{m\sigma} + \hat{\psi}_\sigma)d\sigma)d\mu \\ & + \sum_{0<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(\rho_{m\sigma} + \hat{\psi}(\sigma_n)), \int_0^\tau \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, \rho_\mu + \hat{\psi}_\mu \\ & , \int_0^\mu \mathcal{H}(\mu, \sigma, \rho_\sigma + \hat{\psi}_\sigma)d\sigma)d\mu + \sum_{0<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(\rho_\sigma + \hat{\psi}(\sigma_n))\Bigg) \\ \leqslant & \int_0^\tau \|\mathcal{D}(\tau-\mu)\|_{\mathbb{X}^n} d_\infty \Big(\mathcal{P}(\mu, \rho_{m\mu} + \hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu, \tau, \rho_{m\sigma} + \hat{\psi}_\sigma)d\sigma) \end{aligned}$$

$$,\mathcal{P}(\mu,\rho_\mu+\hat{\psi}_\mu,\int_0^\mu \mathcal{H}(\mu,\tau,\rho_\sigma+\hat{\psi}_\sigma)d\sigma)\Big)d\mu$$
$$+\sum_{0<\sigma_n<\tau}\|\mathcal{D}(\tau-\sigma_n)\|_{\mathbb{X}^n}d_\infty\Big(\mathcal{I}_n(\rho_{m\sigma}+\hat{\psi}(\sigma_n)),\mathcal{I}_n(\rho_\sigma+\hat{\psi}(\sigma_n))\Big).$$

Then we get

$$\begin{aligned}\sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big((\Lambda\rho_m)(\tau),(\Lambda\rho(\tau))\Big)\leqslant\Big\{\int_0^\tau\|\mathcal{D}(\tau-\mu)\|_{\mathbb{X}^n}d_\infty\Big(&\mathcal{P}(\mu,\rho_{m\mu}+\hat{\psi}_\mu,\int_0^\mu\mathcal{H}(\mu,\tau,\rho_{m\sigma}+\hat{\psi}_\sigma)d\sigma)\\ &,\mathcal{P}(\mu,\rho_\mu+\hat{\psi}_\mu,\int_0^\mu\mathcal{H}(\mu,\tau,\rho_\sigma+\hat{\psi}_\sigma)d\sigma)\Big)d\mu\\ &+\sum_{0<\sigma_n<t}\|\mathcal{D}(\tau-\sigma_n)\|_{\mathbb{X}^n}d_\infty\Big(\mathcal{I}_n(\rho_{m\sigma}+\hat{\psi}(\sigma_n)),\mathcal{I}_n(\rho_\sigma+\hat{\psi}(\sigma_n))\Big)\Big\}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{H}_1\Big((\Lambda\rho_m)(\tau),(\Lambda\rho(\tau))\Big)\leqslant\int_0^\tau\|\mathcal{D}(\tau-\mu)\|_{\mathbb{X}^n}d_\infty\Big(&\mathcal{P}(\mu,\rho_{m\mu}+\hat{\psi}_\mu,\int_0^\mu\mathcal{H}(\mu,\tau,\rho_{m\sigma}+\hat{\psi}_\sigma)d\sigma)\\ &,\mathcal{P}(\mu,\rho_\mu+\hat{\psi}_\mu,\int_0^\mu\mathcal{H}(\mu,\tau,\rho_\sigma+\hat{\psi}_\sigma)d\sigma)\Big)d\mu\\ &+\sum_{0<\sigma_n<t}\|\mathcal{D}(\tau-\sigma_n)\|_{\mathbb{X}^n}d_\infty\Big(\mathcal{I}_n(\rho_{m\sigma}+\hat{\psi}(\sigma_n)),\mathcal{I}_n(\rho_\sigma+\hat{\psi}(\sigma_n))\Big)\\ &\to 0 \text{ as } m\to\infty,\forall\,\tau\in[0,\mathcal{T}].\end{aligned}$$

Since $d_\infty(\Lambda\rho_m,\Lambda\rho)=\sup_{\tau\in[0,\mathcal{T}]}d_\infty(\Lambda\rho_m(\tau),\Lambda\rho(\tau))$, it follows that $d_\infty(\Lambda\rho_m,\Lambda\rho)\to 0$ as $m\to\infty$, which implies $\Lambda\rho_m\to\Lambda\rho$ in $\mathbb{X}_0^n$. Therefore Λ is continuous.

Now, we show that Λ is completely continuous, i.e., Λ maps a bounded set of $\mathbb{X}_0^n$ into a relatively compact set of $\mathbb{X}_0^n$. Let $\Omega_\beta=\{w\in\mathbb{X}_0^n : d_\infty(w,\hat{0})\leqslant\beta\}$ for $\beta\geqslant 1$, we prove that Λ_{Ω_β} is uniformly bounded. Let $\mathcal{R}^*=\sup\{\mathcal{R}(\tau):\tau\in[0,\mathcal{T}]\}$ and $\|\psi\|_{\mathcal{C}([-\mathfrak{r},0],\mathbb{X}^n)}\leqslant\mathcal{G}+\mathcal{Q}_1$. Then from the equation (3.5) and using hypotheses $(\mathcal{A}_0)$-$(\mathcal{A}_4)$ and the fact that $d_\infty(w,\hat{0})\leqslant\beta$, $w\in\Omega_\beta$ implies $d_\infty(w_\tau,\hat{0})\leqslant\beta$, $\tau\in[0,\mathcal{T}]$. Let $\beta+\mathcal{G}+\mathcal{Q}_1=\mathcal{B}$, we obtain

$$\begin{aligned}d_\infty(\Lambda w(\tau),\hat{0})\leqslant&\int_0^\tau\|\mathcal{D}(\tau-\mu)\|_{\mathbb{X}^n}d_\infty\Big(\mathcal{P}(\mu,\rho_\mu+\hat{\psi}_\mu,\int_0^\mu\mathcal{H}(\mu,\tau,\rho_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu,\hat{0}\Big)\\ &+\sum_{0<\tau_n<t}\|\mathcal{D}(\tau-\sigma_n)\|_{\mathbb{X}^n}d_\infty\Big(\mathcal{I}_n(\rho_{\sigma_n}+\hat{\psi}_{\sigma_n}),\hat{0}\Big)\leqslant\mathcal{M}\mathcal{T}\mathcal{R}^*[\mathcal{B}+\frac{\mathcal{T}\mathcal{R}^*}{2}\mathcal{B}]+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n\mathcal{B}.\end{aligned}$$

Then we get

$$\sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big(d_\infty(\Lambda w(\tau),\hat{0})\Big)\leqslant\sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big\{\mathcal{M}\mathcal{T}\mathcal{R}^*[\mathcal{B}+\frac{T\mathcal{R}^*}{2}\mathcal{B}]+\sum_{0<\sigma_n<t}\mathcal{M}L_n\mathcal{B}\Big\}.$$

Therefore

$$\mathcal{H}_1\Big(\Lambda w(\tau),\hat{0}\Big)\leqslant\mathcal{M}\mathcal{T}\mathcal{R}^*[\mathcal{B}+\frac{\mathcal{T}\mathcal{R}^*}{2}\mathcal{B}]+\sum_{0<\sigma_n<\tau}\mathcal{M}L_n\mathcal{B}.$$

This implies that the set $\{(\Lambda w)(\tau):d_\infty(w,\hat{0})\leqslant\beta,-\mathfrak{r}\leqslant\tau\leqslant\mathcal{M}\}$ is uniformly bounded in $\mathbb{X}^n$ and hence Λ_{Ω_β} is uniformly bounded. Now we will discuss that Λ maps Ω_β into an equicontinuous family of functions with values in $\mathbb{X}^n$. Let $w\in\Omega_\beta$ and $\tau_1,\tau_2\in[-\mathfrak{r},\mathcal{T}]$. Then from the equation (3.5) and using the hypotheses $(\mathcal{A}_0)$-$(\mathcal{A}_4)$, we have following.

i. Suppose $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant \mathcal{T}$,

$$\begin{aligned}
&d_\infty((\Lambda w)(\tau_2), (\Lambda w)(\tau_1)) \\
&\leqslant d_\infty\Bigg(\int_0^{\tau_2} \mathcal{D}(\tau_2-\mu)\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu \\
&\quad + \sum_{0<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n)), \int_0^{\tau_1} \mathcal{D}(\tau_1-\mu)\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu \\
&\quad , \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)ds + \sum_{0<\sigma_n<\tau_1} \mathcal{D}(\tau_1-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n))\Bigg) \\
&\leqslant \int_{\tau_1}^{\tau_2} \|\mathcal{D}(\tau_2-\mu)+\mathcal{D}(\tau_1-\mu)\|_{\mathbb{X}^n} d_\infty\Bigg(\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\tau)d\sigma), \hat{0}\Bigg)d\mu \\
&\quad + \|\sum_{0<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n) + \sum_{0<\sigma_n<\tau_1} \mathcal{D}(\tau_1-\sigma_n)\|_{\mathbb{X}^n} d_\infty\Bigg(\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n), \hat{0}\Bigg) \\
&\leqslant \int_{\tau_1}^{\tau_2} \|\mathcal{D}(\tau_2-\mu)+\mathcal{D}(\tau_1-\mu)\|_{\mathbb{X}^n} \mathfrak{p}(\mu)\Bigg[d_\infty(w_\mu+\hat{\psi}_\mu, \hat{0}) + \int_0^\mu \mathfrak{q}(\sigma) d_\infty(w_\sigma+\hat{\psi}_\sigma, \hat{0})d\sigma\Bigg]d\mu \\
&\quad + \|\sum_{0<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n) + \sum_{0<\sigma_n<\tau_1} \mathcal{D}(\tau_1-\sigma_n)\|_{\mathbb{X}^n} L_n d_\infty\Bigg(w(\sigma_n)+\hat{\psi}(\sigma_n), \hat{0}\Bigg) \\
&\leqslant \int_{\tau_1}^{\tau_2} \|\mathcal{D}(\tau_2-\mu)+\mathcal{D}(\tau_1-\mu)\|_{\mathbb{X}^n} \mathcal{R}(\mu)\Bigg[\mathcal{B} + \int_0^\mu \mathcal{R}(\tau)\mathcal{B}d\tau\Bigg]d\mu \\
&\quad + \|\sum_{0<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n) + \sum_{0<\sigma_n<\tau_1} \mathcal{D}(\tau_1-\sigma_n)\|_{\mathbb{X}^n} L_n \mathcal{B} \\
&\leqslant \int_{\tau_1}^{\tau_2} \|\mathcal{D}(\tau_2-\mu)+\mathcal{D}(\tau_1-\mu)\|_{\mathbb{X}^n} \mathcal{R}^*\Bigg[\mathcal{B} + \mathcal{T}\mathcal{R}^*\mathcal{B}\Bigg]d\mu + \Bigg[\|\sum_{0<\sigma_n<\tau_1} [\mathcal{D}(\tau_2-\sigma_n) \\
&\quad + \mathcal{D}(\tau_1-\sigma_n)]\|_{\mathbb{X}^n} + \|\sum_{\tau_1<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n)\|_{\mathbb{X}^n}\Bigg] L_n \mathcal{B},
\end{aligned}$$

$$\begin{aligned}
&\sup_{0\leqslant\tau\leqslant\mathcal{T}}\Bigg((\Lambda w)(\tau_1), (\Lambda w)(\tau_2)\Bigg) \\
&\leqslant \sup_{0\leqslant\tau\leqslant\mathcal{T}}\Bigg\{\int_{\tau_1}^{\tau_2} \|\mathcal{D}(\tau_2-\mu)+\mathcal{D}(\tau_1-\mu)\|_{\mathbb{X}^n} \mathcal{R}^*\Bigg[\mathcal{B} + \mathcal{T}\mathcal{R}^*\mathcal{B}\Bigg]d\mu + \Bigg[\|\sum_{0<\sigma_n<\tau_1} \\
&\quad [\mathcal{D}(\tau_2-\sigma_n)+\mathcal{D}(\tau_1-\sigma_n)]\|_{\mathbb{X}^n} + \|\sum_{\tau_1<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n)\|_{\mathbb{X}^n}\Bigg] L_n \mathcal{B}\Bigg\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{H}_1\Bigg((\Lambda w)(\tau_1), (\Lambda w)(\tau_2)\Bigg) \\
&\leqslant \int_{\tau_1}^{\tau_2} \|\mathcal{D}(\tau_2-\mu)+\mathcal{D}(\tau_1-\mu)\|_{\mathbb{X}^n} \mathcal{R}^*\Bigg[\mathcal{B} + \mathcal{T}\mathcal{R}^*\mathcal{B}\Bigg]d\mu + \Bigg[\|\sum_{0<\sigma_n<\tau_1} [\mathcal{D}(\tau_2-\sigma_n) \\
&\quad + \mathcal{D}(\tau_1-\sigma_n)]\|_{\mathbb{X}^n} + \|\sum_{\tau_1<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n)\|_{\mathbb{X}^n}\Bigg] L_n \mathcal{B}.
\end{aligned}$$

ii. Suppose $-\mathfrak{r} \leqslant \tau_1 \leqslant 0 \leqslant \tau_2 \leqslant \mathcal{T}$ calculating as in (i), we get

$$d_\infty((\Lambda w)(\tau_2), (\Lambda w)(\tau_1)) \leqslant \int_0^{\tau_2} d_\infty\Bigg(\mathcal{D}(\tau_2-\mu)\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma), \hat{0}\Bigg)d\mu$$

$$
\begin{aligned}
&+ d_\infty\Big(\sum_{\tau_1<\sigma_n<\tau_2} \mathcal{D}(\tau_2-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n)),\hat{0}\Big)\\
&\leqslant \int_0^{\tau_2} \|\mathcal{D}(\tau_2-\mu)\|_{\mathbb{X}^n} d_\infty\Big(\mathcal{D}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma),\hat{0}\Big)d\mu\\
&+ \sum_{0<\sigma_n<\tau_2} \|\mathcal{D}(\tau_2-\sigma_n)\|_{\mathbb{X}^n} d_\infty\Big(\mathcal{I}_n(w(\tau_n)+\hat{\psi}(\sigma_n)),\hat{0}\Big)\\
&\leqslant \int_0^{\tau_2} \|\mathcal{D}(\tau_2-\mu)\|_{\mathbb{X}^n}\mathcal{R}^*\Big[\mathcal{B}+\mathcal{T}\mathcal{R}^*\mathcal{B}\Big]d\mu + \sum_{0<\sigma_n<\tau_2}\|\mathcal{D}(\tau_2-\sigma_n)\|_{\mathbb{X}^n} L_n\mathcal{B}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big(d_\infty((\Lambda w)(\tau_2),(\Lambda w)(\tau_1))\Big) \leqslant \sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big\{\int_0^{\tau_2} \|\mathcal{D}(\tau_2-\mu)\|_{\mathbb{X}^n}\mathcal{R}^*\Big[\mathcal{B}+\mathcal{T}\mathcal{R}^*\mathcal{B}\Big]d\mu\\
+ \sum_{0<\sigma_n<\tau_2}\|\mathcal{D}(\tau_2-\sigma_n)\|_{\mathbb{X}^n} L_n\mathcal{B}\Big\}.
\end{aligned}
$$

Therefore

$$
\mathcal{H}_1\Big((\Lambda w)(\tau_2),(\Lambda w)(\tau_1)\Big) \leqslant \int_0^{\tau_2} \|\mathcal{D}(\tau_2-\mu)\|_{\mathbb{X}^n}\mathcal{R}^*\Big[\mathcal{B}+\mathcal{T}\mathcal{R}^*\mathcal{B}\Big]d\mu + \sum_{0<\sigma_n<\tau_2}\|\mathcal{D}(\tau_2-\sigma_n)\|_{\mathbb{X}^n} L_n\mathcal{B}.
$$

iii. Suppose $-\mathfrak{r} \leqslant \tau_1 \leqslant \tau_2 \leqslant 0$, then

$$
d_\infty((\Lambda w)(\tau_2),(\Lambda w)(\tau_1)) = 0,\ \sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big(d_\infty((\Lambda w)(\tau_2),(\Lambda w)(\tau_1))\Big)=0,\ \mathcal{H}_1\Big(d_\infty((\Lambda w)(\tau_2),(\Lambda w)(\tau_1))\Big)=0.
$$

The right hand side in (i)-(iii) are independent of $w \in \Omega_\beta$ and tend to zero as $\tau_2-\tau_1 \to 0$, since the compactness of $\mathcal{D}(\tau)$ for $\tau > 0$ implies continuity in uniform operator topology, thus Λ maps Ω_β into equicontinuous family of functions with value $\mathbb{X}^n$. We have already proved that Λ_{Ω_β} is an equicontinuous and uniformly bounded collection. To show the set Λ_{Ω_β} is relatively compact in $\mathbb{X}^n$, it is sufficient by Arzela-Ascoli's argument, to show that Λ maps Ω_β into a relatively compact set in $\mathbb{X}^n$. Let $0 < \tau \leqslant \mathcal{T}$ be fixed and ϵ a real number satisfying $0 < \epsilon < \tau$. Moreover for $w \in \Omega_\beta$, we define

$$
\begin{aligned}
(\Lambda_\epsilon w)(\tau) = \int_0^{\tau-\epsilon} \mathcal{D}(\tau-\mu)\mathcal{H}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu\\
+ \sum_{0<\sigma_n<\tau-\epsilon}\mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n)).
\end{aligned}
$$

Since $\mathcal{D}(\tau)$ is the compact operator, the set $\Upsilon_\epsilon(\tau) = \{(\Lambda_\epsilon w)(\tau) : w \in \Omega_\beta\}$ is relatively compact in $\mathbb{X}^n$ for every ϵ, $0 < \epsilon < \tau$. Also by making use of hypotheses $(\mathcal{A}_0)$-$(\mathcal{A}_4)$ and the fact that $d_\infty(w,\hat{0}) \leqslant \beta, w \in \Omega_\beta$ implies $d_\infty(w(\tau),\hat{0}) \leqslant \beta, \tau \in [0,\mathcal{T}]$, we have

$$
\begin{aligned}
d_\infty((\Lambda w)(\tau),(\Lambda_\epsilon)w(\tau)) = d_\infty\Big(&\int_0^{\tau} \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu\\
&+ \sum_{0<\sigma_n<\tau}\mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n))\\
&, \int_0^{\tau-\epsilon} \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu\\
&+ \sum_{0<\sigma_n<\tau}\mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= d_\infty\bigg(\int_{\tau-\epsilon}^{\tau} \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu \\
&\quad + \sum_{\tau-\epsilon<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n)), \hat{0}\bigg) \\
&\leqslant d_\infty\bigg(\int_{\tau-\epsilon}^{\tau} \mathcal{D}(\tau-\mu)\mathcal{P}(\mu, w_\mu+\hat{\psi}_\mu, \int_0^\mu \mathcal{H}(\mu,\sigma,w_\sigma+\hat{\psi}_\sigma)d\sigma)d\mu, \hat{0}\bigg) \\
&\quad + d_\infty\bigg(\sum_{\tau-\epsilon<\sigma_n<\tau} \mathcal{D}(\tau-\sigma_n)\mathcal{I}_n(w(\sigma_n)+\hat{\psi}(\sigma_n)), \hat{0}\bigg) \\
&\leqslant \int_{\tau-\epsilon}^{\tau} \|\mathcal{D}(\tau-\mu)\|_{\mathbb{X}^n}\mathcal{R}(\mu)\left[\mathcal{B}+\int_0^\mu \mathcal{R}(\sigma)\mathcal{B}d\sigma\right]d\mu + \sum_{\tau-\epsilon<\sigma_n<\tau}\|\mathcal{D}(\tau-\sigma_n)\|_{\mathbb{X}^n}L_n\mathcal{B} \\
&\leqslant \int_{\tau-\epsilon}^{\tau} \|\mathcal{D}(\tau-\mu)\|_{\mathbb{X}^n}\mathcal{R}^*\left[\mathcal{B}+\int_0^\mu \mathcal{R}^*\mathcal{B}d\tau\right]d\mu + \sum_{\tau-\epsilon<\sigma_n<\tau}\|\mathcal{D}(\tau-\sigma_n)\|_{\mathbb{X}^n}L_n\mathcal{B} \\
&\leqslant \int_{\tau-\epsilon}^{\tau} \|\mathcal{D}(\tau-\mu)\|_{\mathbb{X}^n}\mathcal{R}^*\left[\mathcal{B}+\mathcal{T}\mathcal{R}^*\mathcal{B}\right] + \sum_{\tau-\epsilon<\sigma_n<\tau}\|\mathcal{D}(\tau-\sigma_n)\|_{\mathbb{X}^n}L_n\mathcal{B} \\
&\leqslant M\mathcal{R}^*\left[\mathcal{B}+\mathcal{T}\mathcal{R}^*\mathcal{B}\right]\epsilon + \sum_{\tau-\epsilon<\sigma_n<\tau}\mathcal{M}L_n\mathcal{B}.
\end{aligned}
$$

Then we get

$$
\sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big(d_\infty((\Lambda w)(\tau),(\Lambda_\epsilon)w(\tau))\Big) \leqslant \sup_{0\leqslant\tau\leqslant\mathcal{T}}\Big\{\mathcal{M}\mathcal{R}^*\left[\mathcal{B}+\mathcal{T}\mathcal{R}^*\mathcal{B}\right]\epsilon + \sum_{\tau-\epsilon<\sigma_n<\tau}\mathcal{M}L_n\mathcal{B}\Big\}.
$$

Therefore

$$
\mathcal{H}_1\Big((\Lambda w)(\tau),(\Lambda_\epsilon)w(\tau))\Big) \leqslant \mathcal{M}\mathcal{R}^*\left[\mathcal{B}+\mathcal{T}\mathcal{R}^*\mathcal{B}\right]\epsilon + \sum_{\tau-\epsilon<\sigma_n<\tau}\mathcal{M}L_n\mathcal{B}.
$$

This proves that there exist relatively compact sets arbitrarily close to the set $\{(\Lambda w)(\tau) : w \in \Omega_\beta\}$. Hence the set $\{(\Lambda w)(\tau) : w \in \Omega_\beta\}$ is relatively compact in $\mathbb{X}^n$. This completes the proof that Λ is completely continuous operator. Moreover, the set

$$
\mathcal{V}(\Lambda) = \{w \in \mathbb{X}_0^n : w = \lambda\Lambda w,\ 0 < \lambda < 1\},
$$

is bounded in $\mathbb{X}^n$, since for every w in $\mathcal{V}(\Lambda)$, the function $\rho(\tau) = w(\tau) + \hat{\psi}(\tau)$ is a solution of equation (1.1) for which we have proved that $d_\infty(\rho,\hat{0}) \leqslant \mathcal{J}$ and hence $d_\infty(\rho,\hat{0}) \leqslant \mathcal{J}+\mathcal{G}+\mathcal{Q}_1$. Now by virtue of Theorem 3.1, the operator Λ has a fixed point $\tilde{\tau} = \tilde{w} + \hat{\psi}$ that is a solution of the equation (1.1). □

4. Application

To illustrate the application of our result proved in Section 3, consider the following nonlinear fuzzy partial functional differential equation of the form

$$
\frac{\partial}{\partial\tau}\nu(v,\tau) = \frac{\partial^2}{\partial v^2}\nu(v,\tau) + \mathcal{Q}\Big(\tau, \nu(v,\tau-r), \int_0^\tau \mathcal{W}(\tau,\mu,\nu(\mu-r))d\mu\Big), \tag{4.1}
$$

$v \in [0,\pi], \tau \in [0,\mathcal{T}]$,

$$
\nu(0,\tau) = \nu(\pi,\tau) = 0, \quad 0 \leqslant \tau \leqslant \mathcal{T}, \tag{4.2}
$$

$$
\nu(v,\tau) + \sum_{n=1}^{q} \nu(v,\tau_n+\tau) = \psi(v,\tau), \quad 0 \leqslant v \leqslant \pi, \quad -\mathfrak{r} \leqslant \tau \leqslant 0, \tag{4.3}
$$

$$\Delta\nu(\nu, t_n) = \mathcal{J}_n(\nu(\nu, t_n)), \quad n = 1, 2, \dots, k, \tag{4.4}$$

where $\mathcal{Q} : [0, \mathcal{T}] \times \mathbb{X}^n \times \mathbb{X}^n \to \mathbb{X}^n, \mathcal{W} : [0, \mathcal{T}] \times \mathbb{X}^n \to \mathbb{X}^n, \mathcal{J}_n : \mathbb{X}^n \to \mathbb{X}^n$ are continuous. We assume that the functions $\mathcal{Q}, \mathcal{W}$, and $\mathcal{J}_n$ satisfy the following conditions.

i. For every $\tau \in [0, \mathcal{T}]$ and $\nu, \nu \in \mathbb{X}^n$, there exists nondecreasing continuous functions $\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}} : [0, \mathcal{T}] \to \mathcal{R}_+$ such that:

$$d_\infty(\mathcal{Q}(\tau, \nu, \rho), \hat{0}) \leqslant \tilde{\mathfrak{p}}(\tau)(d_\infty(\nu, \hat{0}) + d_\infty(\rho, \hat{0})), \qquad d_\infty(\mathcal{W}(\tau, \mu, \nu), \hat{0}) \leqslant \tilde{\mathfrak{q}}(\tau)(d_\infty(\nu, \hat{0})).$$

ii. There exists constants L_n and $\mathcal{G}_n$ such that:

$$d_\infty(\mathcal{J}_n(\rho), \hat{0}) \leqslant L_n d_\infty(\rho, \hat{0}), \qquad \sum_{n=1}^{q} d_\infty(\nu(\nu, \tau_n + \tau), \hat{0}) \leqslant \sum_{n=1}^{q} \mathcal{G}_n.$$

iii. For every positive integer $\mathfrak{r}$, there exists $\gamma_\mathfrak{r} \in \mathcal{L}^1(0, \mathcal{T})$:

$$\sup\{d_\infty(\mathcal{Q}(\tau, \eta, \rho), \hat{0}) : d_\infty(\eta, \hat{0}) \leqslant \mathfrak{r}, d_\infty(\rho, \hat{0}) \leqslant \mathfrak{r}\} \leqslant \gamma_\mathfrak{r}(\tau) \ \text{ a.e. for } \ \tau \in [0, \mathcal{T}].$$

We define the operator $\mathcal{A} : \mathbb{X}^n \to \mathbb{X}^n$ by $\mathcal{A}w = w''$ with domain $D(\mathcal{A}) = \{w \in \mathbb{X}^n : w$ and w' are absolutely continuous, $w'' \in \mathbb{X}^n$, and $w(0) = w(\pi) = 0\}$. Then the operator $\mathcal{A}$ can be written as

$$\mathcal{A}w = \sum_{m=1}^{\infty} -m^2(w, w_m)w_m, \quad w \in D(\mathcal{A}),$$

where $w_m(\nu) = (\sqrt{\frac{2}{\pi}})\sin(m\nu)$, $m = 1, 2, \dots$ is the orthogonal set of eigenvectors of $\mathcal{A}$ and $\mathcal{A}$ is the infinitesimal generator of an analytic semigroup $\mathcal{D}(\tau)$, $\tau \geqslant 0$ and is given by

$$\mathcal{D}(\tau)w = \sum_{m=1}^{\infty} \exp(-m^2\tau)(w, w_m)w_m, \ w \in \mathbb{X}^n.$$

Now, the analytic semigroup $\mathcal{D}(\tau)$ being compact, there exists constant $\mathcal{M}$ such that $|\mathcal{D}(\tau)| \leqslant \mathcal{M}$ for each $\tau \in [0, \mathcal{T}]$. Define the functions $\mathcal{P} : [0, \mathcal{T}] \times \mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \times \mathbb{X}^n \to \mathbb{X}^n$, $\mathcal{H} : [0, \mathcal{T}] \times [0, \mathcal{T}] \times \mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n) \to \mathbb{X}^n$, $\mathcal{J}_n : \mathbb{X}^n \to \mathbb{X}^n$ as follows

$$\mathcal{P}(\tau, \eta, \rho)(\nu) = \mathcal{Q}(\tau, \eta(-\mathfrak{r})\nu, \rho(\nu)), \qquad \mathcal{H}(\tau, \mu, \zeta) = \mathcal{W}(\tau, \mu, \zeta(-\mathfrak{r})\nu),$$

where $\tau \in [0, \mathcal{T}]$, $\eta, \zeta \in \mathcal{C}([-\mathfrak{r}, 0], \mathbb{X}^n)$, $\rho \in \mathbb{X}^n$, and $0 \leqslant \nu \leqslant \pi$. With these choices of the functions the equations (4.1)-(4.4) can be formulated as an fuzzy integro-differential equations in $\mathbb{X}^n$,

$$\begin{aligned}
&\rho'(\tau) = \mathcal{A}\rho(\tau) + \mathcal{P}(\tau, \rho_\tau, \int_0^\tau \mathcal{H}(\tau, \mu, \rho_\mu)d\mu), \quad \tau \in (0, \mathcal{T}],\\
&\Delta\rho(\tau_n) = \mathcal{J}_n\rho(\tau_n), \tau \neq t_n, \quad n = 1, 2, \dots, k,\\
&\rho(\tau) + \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(\tau) = \psi(\tau), \quad \tau \in [-\mathfrak{r}, 0].
\end{aligned}$$

Since all the hypotheses of the Theorem 3.1 are satisfied, Theorem 3.1 can be applied to guarantee the existence of mild solution $\nu(\nu, \tau) = \rho(\tau)\nu$, $\tau \in [0, \mathcal{T}]$, $\nu \in [0.\pi]$, of the nonlinear fuzzy partial integro-differential equations (4.1)-(4.4).

5. Conclusions

In this paper, the Leray-Schauder alternative fixed point theorem was employed to get the existence results for impulsive fuzzy solutions for nonlinear integro-differential equations with the nonlocal condition and the generalization of Grownwall-type inequality to draw the result. As an application, an example was given to prove the validity of our result.

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