J. Math. Computer Sci., 31 (2023), 448–460

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Numerical approximation of p-dimensional stochastic Volterra integral equation using Walsh function

Prit Pritam Paikaray, Sanghamitra Beuria*, Nigam Chandra Parida

Department of Mathematics, College of Basic Science and Humanities, OUAT, Bhubaneswar, Odisha,751003, India.

Abstract

In this paper, we propose a numerical approach for solving p-dimensional stochastic Volterra integral equations using the Walsh function approximation. The main goal is to transform integral equations into an algebraic system and solve this further to get an approximate solution to the integral equation. The convergence and error analysis of the proposed method are studied for integral equations having functions in the Lipschitz class. The computation of various examples for which analytical solutions are available shows that the proposed method is more accurate than the existing techniques for solving linear p-dimensional stochastic Volterra integral equations.

Keywords: Stochastic volterra integral equation, Brownian motion, Itô integral, Walsh approximation, Lipschitz condition.

2020 MSC: 60H05, 60H35, 65C30.

1. Introduction

In recent decades, stochastic integral equations [13, 14] have been widely used in different fields like financial mathematics [3], physics, biology, engineering, and many others. Since it is not always possible to have an exact solution to the problem, numerical approximation to the integral equation becomes vital. To approximate the stochastic integral equation, orthogonal functions like block pulse function, Haar wavelet, Legendre polynomial, and others have been applied to approximate the stochastic integral equation [7–12, 16–18]. The fact that a computer can accurately estimate any Walsh function's (which is a binary-valued function that takes values 1 and -1) current value at any given time gives it a significant edge over traditional trigonometric functions. Chen and Hsiao solved the variational problem using the Walsh function [1] in 1975. In 1979, they solved the integral equation using the same concept [6].

In this paper, we used the Walsh function to approximate the following p-dimensional stochastic Volterra integral equation (SVIE)

$$x(t) = f(t) + \int_0^t k(s,t)x(s)ds + \sum_{\gamma=1}^{p} \int_0^t k_\gamma(s,t)x(s)dB_\gamma(s), \quad s,t \in [0,T),$$

*Corresponding author

Email addresses: paikaraypritpritam@gmail.com (Prit Pritam Paikaray), sbeuria108@gmail.com (Sanghamitra Beuria), ncparida@gmail.com (Nigam Chandra Parida)

doi: 10.22436/jmcs.031.04.07

Received: 2023-03-24 *Revised:* 2023-04-18 *Accepted:* 2023-05-10

where, p is a positive integer and for $\gamma = 1, 2, \dots, p$, $x(t)$, $f(t)$, $k(s,t)$, and $k_\gamma(s,t)$ are the stochastic processes defined on the same probability space (Ω, F, P). $B_\gamma(t)$ are independent Brownian motions, $x(t)$ is an unknown stochastic process, and $\int_0^t k_\gamma(s,t)x(s)dB_\gamma(s)$ is Itô integral. The purpose of the work is to convert a p-dimensional SVIE into a system of an algebraic equation and then solve the system to get an approximate solution.

2. Walsh function and its properties

Definition 2.1 (Rademacher function)**.** Rademacher function $r_i(t)$, $i = 1, 2, \dots$, for $t \in [0, 1)$ is defined by [19]

$$r_i(t) = \begin{cases} 1, & i = 0, \\ \operatorname{sgn}(\sin(2^i \pi t)), & \text{otherwise}, \end{cases}$$

where,

$$\operatorname{sgn}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Definition 2.2 (Walsh function)**.** The n^{th} Walsh function for $n = 0, 1, 2, \dots$, denoted by $w_n(t)$, $t \in [0, 1)$ is defined [19] as

$$w_n(t) = (r_q(t))^{b_q}.(r_{q-1}(t))^{b_{q-1}}.(r_{q-2}(t))^{b_{q-2}} \cdots (r_1(t))^{b_1},$$

where $n = b_q 2^{q-1} + b_{q-1}2^{q-2} + b_{q-2}2^{q-3} + \cdots + b_1 2^0$ is the binary expression of n. Therefore, q, the number of digits present in the binary expression of n is calculated by $q = \left[\log_2 n\right] + 1$ in which $\left[\,\cdot\,\right]$ is the greatest integer less than or equal to $'\cdot'$.

The first m Walsh functions for $m \in \mathbb{N}$ can be written as an m-vector by

$$W(t) = \begin{bmatrix} w_0(t) & w_1(t) & w_2(t) \cdots w_{m-1}(t) \end{bmatrix}^T.$$

The Walsh functions satisfy the following properties.

Orthonormality: The set of Walsh functions is orthogonal, i.e.,

$$\int_0^1 w_i(t) w_j(t) dt = \begin{cases} 1, & \text{i=j}, \\ 0, & \text{otherwise}. \end{cases}$$

Completeness: For every $f \in L^2[0, 1)$,

$$\int_0^1 f^2(t)dt = \sum_{i=0}^{\infty} f_i^2 \|w_i(t)\|^2,$$

where $f_i = \int_0^1 f(t) w_i(t) dt$.

Walsh function approximation: Any real-valued function $f(t) \in L^2[0, 1)$ can be approximated as

$$f_m(t) = \sum_{i=0}^{m-1} c_i w_i(t)$$

where, $c_i = \int_0^1 f(t) w_i(t) dt$. The matrix form of the approximation is given by

$$f(t) = F^T T_W W(t) \tag{2.1}$$

where $F = \begin{bmatrix} f_0 & f_1 & f_2 \cdots f_{m-1} \end{bmatrix}^T$ and $f_i = \int_{ih}^{(i+1)h} f(s)ds$ and T_W is called the operational matrix for the Walsh function.

One can see from [2] that,

$$T_W T_W^T = mI \ \text{ and } \ T_W^T = T_W.$$

Similarly, $k(s,t) \in L^2([0,1) \times [0,1))$ can be approximated by

$$k_m(s,t) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{ij} w_i(s) w_j(t),$$

where, $c_{ij} = \int_0^1 \int_0^1 k(s,t) w_i(s) w_j(t) dt ds$, with the matrix form as

$$k(s,t) = W^T(s) T_W K T_W W(t) = W^T(t) T_W K^T T_W W(s), \tag{2.2}$$

where $K = [k_{ij}]_{m \times m}, k_{ij} = \int_{ih}^{(i+1)h} \int_{jh}^{(j+1)h} k(s,t) dt ds$.

In the next section, we will find a relation between the block pulse function and the Walsh function, which is later used to convert the SVIE to an algebraic equation.

3. Relationship between Walsh function and block pulse functions (BPFs)

Definition 3.1 (Block pulse functions)**.** For a fixed positive integer m, an m-set of BPFs $\phi_i(t), t \in [0,1)$ for $i = 0, 1, \ldots, m-1$ is defined as

$$\phi_i(t) = \begin{cases} 1, & \text{if } \frac{i}{m} \leqslant t < \frac{(i+1)}{m}, \\ 0, & \text{otherwise.} \end{cases}$$

ϕ_i is known as the i^{th} BPF.

The set of all m BPFs can be written concisely as an m-vector,

$$\Phi(t) = \begin{bmatrix} \phi_0(t) & \phi_1(t) & \phi_2(t) \cdots \phi_{m-1}(t) \end{bmatrix}^T, \ t \in [0,1).$$

The BPFs are disjoint, complete, and orthogonal [5]. The BPFs in vector form satisfy

$$\Phi(t)\Phi(t)^T X = \tilde{X}\Phi(t) \ \text{ and } \ \Phi^T(t) A \Phi(t) = \hat{A}\Phi(t),$$

where, $X \in \mathbb{R}^{m \times 1}, \tilde{X}$ is the $m \times m$ diagonal matrix with $\tilde{X}(i,i) = X(i)$ for $i = 1, 2, 3, \ldots, m, A \in \mathbb{R}^{m \times m}$ and $\hat{A} = \begin{bmatrix} a_{11} & a_{22} & \cdots & a_{mm} \end{bmatrix}^T$ is the m-vector with elements equal to the diagonal entries of A. The integration of BPF vector $\Phi(t)$, $t \in [0,1)$ can be performed by [5]

$$\int_0^t \Phi(\tau) d\tau = P\Phi(t), \ \ t \in [0,1),$$

where P is called the deterministic operational matrix of integration. Hence, the integral of every function $f(t) \in L^2[0,1)$ can be approximated as

$$\int_0^t f(s) ds = F^T P \Phi(t).$$

Similarly, Itô integral of the BPF vector $\Phi(t)$, $t \in [0,1)$ can be performed by [9]

$$\int_0^t \Phi(\tau) dB(\tau) = P_S \Phi(t), \ \ t \in [0,1),$$

where, P_S is called the stochastic operational matrix of integration. Hence, the Itô integral of every function $f(t) \in L^2[0,1)$ can be approximated as in [9] by

$$\int_0^t f(s) dB(s) = F^T P_S \Phi(t).$$

The following theorem describes a relationship between the Walsh function and the block pulse function.

Theorem 3.2. *Let the* m*-set of Walsh function and BPF vectors be* $W(t)$ *and* $\Phi(t)$*, respectively. Then the BPF vectors* $\Phi(t)$ *can be used to approximate* $W(t)$ *as* $W(t) = T_W\Phi(t)$*,* $m = 2^k$*, and* $k = 0, 1, \ldots$*, where* $T_W = \left[c_{ij}\right]_{m\times m}$*,* $c_{ij} = w_i(\eta_j)$*, for some* $\eta_j = \left(\frac{j}{m}, \frac{j+1}{m}\right)$ *and* $i, j = 0, 1, 2, \ldots, m-1$*.*

Proof. Let $w_i(t)$, $i = 0, 1, 2, \ldots, m-1$, where $m = 2^k$, be the i^{th} element of the Walsh function vector. By expanding $w_i(t)$ into an m-term vector of BPFs, we have $w_i(t) = \sum_{j=0}^{m-1} c_{ij}\phi_j(t) = C_i^T\Phi(t)$, $i = 0, 1, 2, \ldots, m-1$, where C_i^T is the i^{th} row and c_{ij} is the$(i, j)^{\text{th}}$ element of matrix T_W,

$$c_{ij} = \frac{1}{h}\int_0^1 w_i(t)\phi_j(t)dt = \frac{1}{h}\int_{jh}^{(j+1)h} w_i(t)dt.$$

Using the mean value theorem for integrals, we can write

$$c_{ij} = \frac{1}{h}\int_{jh}^{(j+1)h} w_i(t)dt = \frac{1}{h}\big((j+1)h - jh\big)w_i(\eta_j) = w_i(\eta_j),$$

where $\eta_j \in \left(\frac{j}{m}, \frac{j+1}{m}\right)$, $m = \frac{1}{h}$. Since $w_i(t)$ is constant in the interval $\left(\frac{j}{m}, \frac{j+1}{m}\right)$, we choose $c_{ij} = w_i(\frac{2j+1}{2m})$, $i, j = 0, 1, 2, \ldots, m-1$. Hence $W(t) = T_W\Phi(t)$. □

From the above theorem, it is easy to see $\Phi(t) = \frac{1}{m}T_W W(t)$. With the above condition, we prove the following theorem.

Lemma 3.3 (Integration of Walsh function)**.** *Suppose that* $W(t)$ *is a Walsh function vector, then the integral of* $W(t)$ *w.r.t.* t *is given by* $\int_0^t W(s)ds = \wedge W(t)$*, where* $\wedge = \frac{1}{m}T_W P T_W$ *and*

$$P = \frac{1}{h}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Proof. Let $W(t)$ be a Walsh function vector, and then the integral of $W(t)$ w.r.t. t is

$$\int_0^t W(s)ds = \int_0^t T_W\Phi(s)ds = T_W\int_0^t \Phi(s)ds = T_W P\Phi(t) = \frac{1}{m}\Big(T_W P T_W\Big)W(t) = \wedge W(t),$$

where $\wedge = \frac{1}{m}\Big(T_W P T_W\Big)$. □

Here, $\wedge$ is called the Walsh operational matrix of integration.

Lemma 3.4 (Stochastic integration of Walsh function)**.** *Suppose that* $W(t)$ *is a Walsh function vector, then the Itô integral of* $W(t)$ *is given by* $\int_0^t W(s)dB(s) = \wedge_S W(t)$*, where* $\wedge_S = \frac{1}{m}T_W P_S T_W$ *and*

$$P_S = \begin{bmatrix} B(\frac{h}{2}) & B(h) & \cdots & B(h) \\ 0 & B(\frac{3h}{2}) - B(h) & \cdots & B(2h) - B(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B(\frac{(2m-1)h}{2}) - B((m-1)h) \end{bmatrix}.$$

Proof. Let $W(t)$ be a Walsh function vector, and then the Itô integral of $W(t)$ is

$$\int_0^t W(s)dB(s) = \int_0^t T_W\Phi(s)dB(s) = T_W\int_0^t \Phi(s)dB(s) = T_W P_S\Phi(t) = \frac{1}{m}\Big(T_W P_S T_W\Big)W(t) = \wedge_S W(t),$$

where $\wedge_S = \frac{1}{m}\Big(T_W P_S T_W\Big)$. □

Here, $\wedge_S$ is called the Walsh operational matrix for Itô integral.

4. Numerical solution of stochastic Volterra integral equation

Consider the following p-dimensional linear stochastic Volterra integral equation

$$x(t) = f(t) + \int_0^t k(s,t)x(s)ds + \sum_{\gamma=1}^{p} \int_0^t k_\gamma(s,t)x(s)dB_\gamma(s), \quad s,t \in [0,T), \tag{4.1}$$

where, for $\gamma = 1,2,\ldots,p$, $x(t)$, $f(t)$, $k(s,t)$, and $k_\gamma(s,t)$ are the stochastic processes defined on the same probability space (Ω, F, P). $B_\gamma(t)$ are independent Brownian motions, $x(t)$ is the unknown stochastic process, and $\int_0^t k_\gamma(s,t)x(s)dB_\gamma(s)$ is Itô Integral.

As given in equations (2.1) and (2.2) we approximate $f(t)$, $k(s,t)$, $k_\gamma(s,t)$ for $s,t \in [0,T)$ as

$$f(t) = F^T T_W W(t), \tag{4.2}$$

where $F = \begin{bmatrix} f_0 & f_1 & f_2 \cdots f_{m-1} \end{bmatrix}^T$ and $f_i = \int_{ih}^{(i+1)h} f(s)ds$. Also,

$$k(s,t) = W^T(s)T_W K T_W W(t) = W^T(t)T_W K^T T_W W(s), \tag{4.3}$$

where $K = [k_{ij}]_{m\times m}, k_{ij} = \int_{ih}^{(i+1)h}\int_{jh}^{(j+1)h} k(s,t)dtds$. Similarly, for $\gamma = 1,2,\ldots,p$,

$$k_\gamma(s,t) = W^T(s)T_W K_\gamma T_W W(t) = W^T(t)T_W K_\gamma^T T_W W(s), \tag{4.4}$$

where $K_\gamma = [k_{ij}]_{m\times m}, k_{ij} = \int_{ih}^{(i+1)h}\int_{jh}^{(j+1)h} k_\gamma(s,t)dtds$. Assume that

$$x(t) = X^T T_W W(t), \tag{4.5}$$

where $X = \begin{bmatrix} x_0 & x_1 & x_2 \cdots x_{m-1} \end{bmatrix}^T$ and $x_i = \int_{ih}^{(i+1)h} x(s)ds$. Substituting (4.2), (4.3), (4.4), and (4.5) in (4.1) we have

$$\begin{aligned} X^T T_W W(t) &= F^T T_W W(t) + \int_0^t W^T(t)T_W K^T T_W W(s)W^T(s)T_W X ds \\ &\quad + \sum_{\gamma=1}^{p}\int_0^t W^T(t)T_W K_\gamma^T T_W W(s)W^T(s)T_W X dB_\gamma(s) \\ &= F^T T_W W(t) + W^T(t)T_W K^T T_W \int_0^t W(s)W^T(s)T_W X ds \\ &\quad + \sum_{\gamma=1}^{p} W^T(t)T_W K_\gamma^T T_W \int_0^t W(s)W^T(s)T_W X dB_\gamma(s). \end{aligned} \tag{4.6}$$

Now

$$\int_0^t W(s)W^T(s)T_W X ds = T_W \tilde{X} P T_W W(t). \tag{4.7}$$

Similarly,

$$\int_0^t W(s)W^T(s)T_W X dB_\gamma(s) = T_W \tilde{X} P_S^\gamma T_W W(t), \tag{4.8}$$

where

$$P_S^\gamma = \begin{bmatrix} B_\gamma(\frac{h}{2}) & B_\gamma(h) & \cdots & B_\gamma(h) \\ 0 & B_\gamma(\frac{3h}{2}) - B_\gamma(h) & \cdots & B_\gamma(2h) - B_\gamma(h) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_\gamma(\frac{(2m-1)h}{2}) - B_\gamma((m-1)h) \end{bmatrix}.$$

Substituting (4.7) and (4.8) in (4.6) we get

$$\begin{aligned}
X^T T_W W(t) &= F^T T_W W(t) + mW^T(t)T_W K^T \tilde{X} P T_W W(t) + m\sum_{\gamma=1}^{p} W^T(t) T_W K_\gamma^T \tilde{X} P_S^\gamma T_W W(t)\\
&= F^T T_W W(t) + W^T(t) T_W H T_W W(t) + \sum_{\gamma=1}^{p} W^T(t) T_W H_\gamma T_W W(t)\\
&= F^T T_W W(t) + m\hat{H}^T T_W W(t) + m\sum_{\gamma=1}^{p} \hat{H}_\gamma^T T_W W(t),
\end{aligned}$$

i.e.,

$$\Big(X^T - F^T - m\hat{H}^T - m\sum_{i=1}^{p} \hat{H}_\gamma^T\Big) T_W W(t) = 0,$$

where, $H = mK_1^T \tilde{X} P$ and $H_\gamma = mK_\gamma^T \tilde{X} P_S^\gamma$. Here, $\hat{H}$ and $\hat{H}_\gamma$ are m-vector with elements equal to the diagonal entries of H and H_γ, respectively. Hence,

$$\Big(X - F - m\hat{H} - m\sum_{\gamma=1}^{p} \hat{H}_\gamma\Big) = [0]_{m\times 1}$$

can be solved to yield a solution to the given linear SVIE (4.1).

5. Error analysis

In this section, we analyse the error between the approximate solution and the exact solution of the stochastic Volterra integral equation. Before we start the analysis, let us define $\|X\|_2 = E(|X|^2)^{\frac{1}{2}}$.

Theorem 5.1. *If $f \in L^2[0,1)$ satisfies the Lipschitz condition with Lipschitz constant C, then $\|e_m(t)\|_2 = O(h)$, where $e_m(t) = |f(t) - \sum_{i=0}^{m-1} c_i w_i(t)|$ and $c_i = \int_0^1 f(s) w_i(s) ds$.*

Proof. Let $f_m(t) = \sum_{i=0}^{m-1} c_i w_i(t)$, where $c_i = \int_0^1 f(s) w_i(s) ds$. Suppose f satisfies the Lipschitz condition. Now,

$$e_m(t) = |f(t) - f_m(t)| \leqslant \omega(\frac{1}{2^k}, f) \leqslant Ch.$$

Here $\omega(\frac{1}{2^k}, f)$ is called the modulus of continuity of the function f [4]. Therefore,

$$\|e_m(t)\|_2 \leqslant Ch = O(h).$$

□

Theorem 5.2. *Suppose $k \in L^2\big([0,1)\times[0,1)\big)$ satisfies the Lipschitz condition with the Lipschitz constant L. If $k_m(x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} w_i(x) w_j(y)$, $c_{ij} = \int_0^1\int_0^1 k(s,t) w_i(s) w_j(t) dt ds$, then $\|e_m(x,y)\|_2 = O(h)$, where $|e_m(x,y)| = |k(x,y) - k_m(x,y)|$.*

Proof. From [4], it is evident that,

$$\begin{aligned}
k_m(x,y) &= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\Big(\int_0^1\int_0^1 k(s,t) w_i(s) w_j(t) dt ds\Big) w_i(x) w_j(y)\\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}(\int_0^1\int_0^1 k(s,t) w_i(s) w_i(x) w_j(t) w_j(y) dt ds)
\end{aligned}$$

$$= \int_0^1 \int_0^1 k(s,t) D_m(t \oplus y) D_m(s \oplus x) dt ds = 2^k . 2^k \int_{\Delta_i^{(k)}} \int_{\Delta_j^{(k)}} k(s,t) dt ds,$$

where $D_m(t) = \sum_{i=0}^{m-1} w_i(t)$ is called the Dirichlet kernel and for $l = i, j$, $\Delta_l^{(k)} = \left[\frac{l}{2^k}, \frac{l+1}{2^k}\right), 0 \leqslant l < 2^k - 1$, [4]. Hence,

$$|k_m(X) - k(X)| \leqslant 2^{2k} \int_{\Delta_i^{(k)}} \int_{\Delta_j^{(k)}} |k(T) - k(X)| dT,$$

where $X = (x, y)$ and $T = (s, t)$. Also note that if k is uniformly Lipschitz with the Lipschitz constant L, then

$$|k_m(X) - k(X)| \leqslant 2^{2k} \int_{\Delta_i^{(k)}} \int_{\Delta_j^{(k)}} L|T - X| dT.$$

Therefore,

$$\|k_m(X) - k(X)\|_2 \leqslant \sqrt{2} L h = O(h).$$

□

Theorem 5.3. *Suppose $x_m(t)$ is the approximate solution of the linear p-dimensional SVIE (4.1). If*

1. $f \in L^2[0,1)$, $k(s,t) \in L^2\big([0,1) \times [0,1)\big)$, *and* $k_\gamma(s,t) \in L^2\big([0,1) \times [0,1)\big)$ *satisfies the Lipschitz condition with Lipschitz constants* C, L, *and* L_γ, *respectively, where for* $\gamma = 1, 2, \dots, p$;
2. $|x(t)| \leqslant \sigma$, $|k(s,t)| \leqslant \rho$ *and for* $\gamma = 1, 2, \dots, p$, $|k_\gamma(s,t)| \leqslant \rho_\gamma$,

then

$$\|x(t) - x_m(t)\|_2^2 = O(h^2).$$

Proof. Let (4.1) be the given SVIE and

$$x_m(t) = f_m(t) + \int_0^t k_m(s,t) x_m(s) ds + \sum_{\gamma=1}^{p} \int_0^t k_{\gamma m}(s,t) x_m(s) dB_\gamma(s)$$

be the approximation to the solution using the Walsh function. Then

$$x(t) - x_m(t) = f(t) - f_m(t) + \int_0^t \big(k(s,t)x(s) - k_m(s,t)x_m(s)\big) ds$$
$$+ \sum_{\gamma=1}^{p} \int_0^t \big(k_\gamma(s,t)x(s) - k_{\gamma m}(s,t)x_m(s)\big) dB_\gamma(s).$$

We know that $(a_1 + a_2 + \dots + a_{p+2})^2 \leqslant (2p+3)(a_1^2 + a_2^2 + \dots + a_{p+2}^2)$. Therefore

$$|x(t) - x_m(t)|^2 \leqslant (2p+3)|f(t) - f_m(t)|^2 + (2p+3)\left|\int_0^t \big(k(s,t)x(s) - k_m(s,t)x_m(s)\big) ds\right|^2$$
$$+ (2p+3) \sum_{\gamma=1}^{p} \left|\int_0^t \big(k_\gamma(s,t)x(s) - k_{\gamma m}(s,t)x_m(s)\big) dB_\gamma(s)\right|^2,$$

which implies that

$$E\big(|x(t) - x_m(t)|^2\big) \leqslant E\bigg((2p+3)|f(t) - f_m(t)|^2 + (2p+3)\left|\int_0^t \big(k(s,t)x(s) - k_m(s,t)x_m(s)\big) ds\right|^2$$
$$+ (2p+3) \sum_{\gamma=1}^{p} \left|\int_0^t \big(k_\gamma(s,t)x(s) - k_{\gamma m}(s,t)x_m(s)\big) dB_\gamma(s)\right|^2\bigg).$$

Hence,

$$E\left(|x(t)-x_m(t)|^2\right) \leqslant (2p+3)E\left(|f(t)-f_m(t)|^2\right) + (2p+3)I_0 + (2p+3)\sum_{\gamma=1}^{p} I_\gamma, \tag{5.1}$$

where

$$I_0 = E\left(\left|\int_0^t \big(k(s,t)x(s)-k_m(s,t)x_m(s)\big)\,ds\right|^2\right), \quad I_\gamma = E\left(\left|\int_0^t \big(k_\gamma(s,t)x(s)-k_{\gamma m}(s,t)x_m(s)\big)\,dB_\gamma(s)\right|^2\right).$$

Now

$$\begin{aligned}|k(s,t)x(s)-k_m(s,t)x_m(s)| \leqslant{}& |k(s,t)||x(s)-x_m(s)| + |k(s,t)-k_m(s,t)||x(s)| \\ &+ |k(s,t)-k_m(s,t)||x(s)-x_m(s)|.\end{aligned}$$

Let $|k(s,t)| \leqslant \rho$, $|x(s)| \leqslant \sigma$ and using Theorem 5.2, we get

$$|k(s,t)x(s)-k_m(s,t)x_m(s)| \leqslant \sqrt{2}Lh\sigma + (\rho+\sqrt{2}Lh)|x(s)-x_m(s)|,$$

which gives

$$I_0 \leqslant E\left(\left(\int_0^t \Big|k(s,t)x(s)-k_m(s,t)x_m(s)\Big|\,ds\right)^2\right), \quad \leqslant E\left(\left(\int_0^t \big(\sqrt{2}Lh\sigma + (\rho+\sqrt{2}Lh)|x(s)-x_m(s)|\big)\,ds\right)^2\right).$$

By Cauchy-Schwarz inequality, for $t > 0$ and $f \in L^2[0,1)$,

$$\left|\int_0^t f(s)\,ds\right|^2 \leqslant t\int_0^t |f|^2 ds.$$

Therefore,

$$I_0 \leqslant E\left(\int_0^t \big(\sqrt{2}Lh\sigma + (\rho+\sqrt{2}Lh)|x(s)-x_m(s)|\big)^2 ds\right),$$

this implies,

$$I_0 \leqslant 2(\sqrt{2}Lh\sigma)^2 + 2(\rho+\sqrt{2}Lh)^2\int_0^t E\left(|x(s)-x_m(s)|^2\right)ds. \tag{5.2}$$

As

$$E\left(\left(\int_0^t f(s)\,dB(s)\right)^2\right) = E\left(\int_0^t f^2(s)\,ds\right),$$

for $\gamma = 1,2,\dots,p$,

$$I_\gamma \leqslant E\left(\int_0^t \left(2(\sqrt{2}L_\gamma h\sigma)^2 + 2(\rho_\gamma+\sqrt{2}L_\gamma h)^2|x(s)-x_m(s)|^2\right)ds\right).$$

Hence,

$$I_\gamma \leqslant 2(\sqrt{2}L_\gamma h\sigma)^2 + 2(\rho_\gamma+\sqrt{2}L_\gamma h)^2\int_0^t E(|x(s)-x_m(s)|^2)\,ds. \tag{5.3}$$

Using Theorem 5.1 and equations (5.2) and (5.3) in (5.1), we get

$$E\left(|x(t)-x_m(t)|^2\right) \leqslant (2p+3)C^2h^2 + (2p+3)\left(2(\sqrt{2}Lh\sigma)^2 + 2(\rho+\sqrt{2}Lh)^2\int_0^t E\left(|x(s)-x_m(s)|^2\right)ds\right)$$

$$+ (2p+3)\sum_{\gamma=1}^{p}\left(2(\sqrt{2}L_\gamma h\sigma)^2 + 2(\rho_\gamma + \sqrt{2}L_\gamma h)^2\int_0^t E(|x(s)-x_m(s)|^2)ds\right),$$

which implies that,

$$E\big(|x(t)-x_m(t)|^2\big) \leqslant R_1 + R_2\int_0^t E\big(|x(s)-x_m(s)|^2\big)ds,$$

where

$$R_1 = (2p+3)\left(C^2h^2 + 2(\sqrt{2}Lh\sigma)^2 + 2\sum_{\gamma=1}^{p} 2(\sqrt{2}L_\gamma h\sigma)^2\right)$$

and

$$R_2 = (2p+3)\left(2(\rho+\sqrt{2}Lh) + \sum_{\gamma=1}^{p} 2(\rho_\gamma + \sqrt{2}L_\gamma h)^2\right).$$

Using Gronwall's inequality, we have

$$E\big(|x(t)-x_m(t)|^2\big) \leqslant R_1 \exp\left(\int_0^t R_2 ds\right).$$

Hence,

$$\|x(t)-x_m(t)\|_2^2 = E\big(|x(t)-x_m(t)|^2\big) \leqslant R_1 e^{R_2} = O(h^2).$$

□

6. Numerical examples

Example 6.1 ([11]). Consider the linear stochastic Volterra integral equation

$$x(t) = x_0 + \int_0^t r(s)x(s)ds + \sum_{\gamma=1}^{3}\int_0^t \alpha_\gamma(s)x(s)dB_\gamma(s), \ \ s,t \in [0,1),$$

where, $x_0 = \frac{1}{12}$, $r(s) = s^2$, $\alpha_1(s) = \sin(s)$, $\alpha_2(s) = \cos(s)$, and $\alpha_3(s) = s$, with the exact solution $x(t) = x_0 e^{\int_0^t (r(s) - \frac{1}{2}\sum_{\gamma=1}\alpha_\gamma(s)^2)ds + \sum_{\gamma=1}^{3}\int_0^t \alpha_\gamma(s)B_\gamma(s)ds}$ for $0 \leqslant t < 1$. $B(t) = (B_1(t), B_2(t), B_3(t))$ is a 3-dimensional Brownian motion, and $x(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P).

Table 1: Mean error, standard deviation of error, and interval of confidence for mean error in Example 6.1 with $n = 20$.

m	$\bar{x}_E$	s_E	95% interval of confidence for error mean	
			Lower	Upper
4	0.014005945459	0.019754394387	0.005348200393	0.022663690526
8	0.008937461602	0.016080255117	0.001889979068	0.015984944135
16	0.008982012760	0.008617089506	0.005205406771	0.012758618749
32	0.007189414115	0.012132261046	0.001872216273	0.012506611957
64	0.008312512269	0.008705582905	0.004497122338	0.012127902201
128	0.005024746531	0.004560862918	0.003025860224	0.007023632837

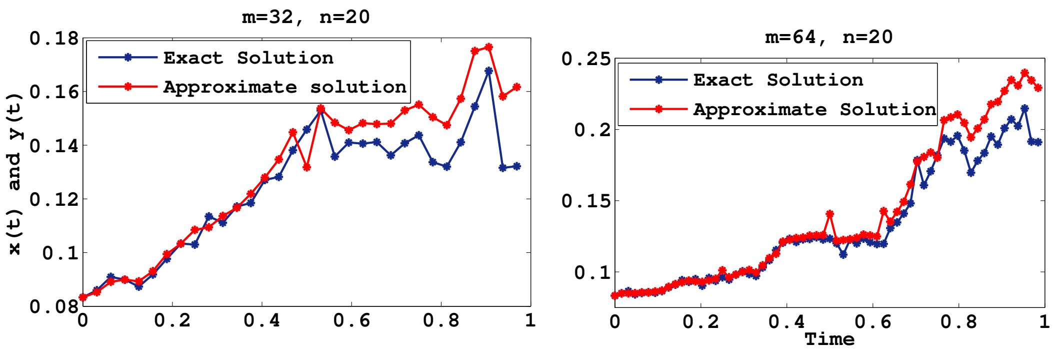

Figure 1: Example 6.1's approximate and exact solutions for $m = 32$ and $m = 64$.

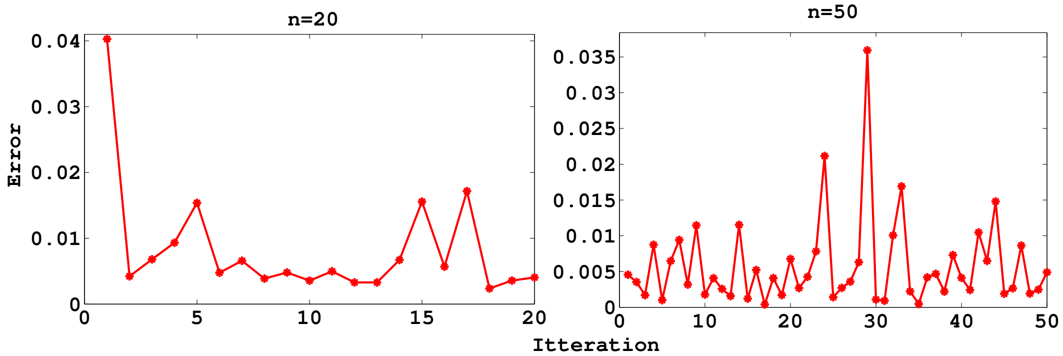

Figure 2: Example 6.1's error trend for $m = 64$, $n = 20$, and $n = 50$.

Example 6.2 ([8]). Consider the linear stochastic Volterra integral equation

$$x(t) = x_0 + \int_0^t rx(s)ds + \sum_{\gamma=1}^{4} \int_0^t \alpha_\gamma x(s)dB_\gamma(s), \;\; s, t \in [0, 1),$$

where, $x_0 = \frac{1}{200}$, $r = \frac{1}{20}$, $\alpha_1 = \frac{1}{50}$, $\alpha_2 = \frac{2}{50}$, $\alpha_3 = \frac{4}{50}$, and $\alpha_4 = \frac{9}{50}$, with the exact solution $x(t) = x_0 e^{(r-\frac{1}{2}\sum_{\gamma=1}^{4}\alpha_\gamma^2)t+\sum_{\gamma=1}^{4}\alpha_\gamma B_\gamma(t)}$ for $0 \leqslant t < 1$. $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))$ is a 4-dimensional Brownian motion, and $x(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P).

Table 2: Mean error, standard deviation of error, and interval of confidence for mean error in Example 6.2 with $n = 20$.

m	$\bar{x}_E$	s_E	95% interval of confidence for error mean	
			Lower	Upper
4	5.21192035e-05	8.41927264e-06	4.84292946e-05	5.58091124e-05
8	5.43804073e-05	5.26997898e-06	5.20707372e-05	5.66900774e-05
16	5.10944392e-05	5.50663806e-06	4.86810487e-05	5.35078298e-05
32	5.27546751e-05	6.30718751e-06	4.99904283e-05	5.55189219e-05
64	5.43255751e-05	1.03246528e-05	4.98005965e-05	5.88505537e-05
128	5.13975170e-05	6.74564509e-06	4.84411077e-05	5.43539264e-05

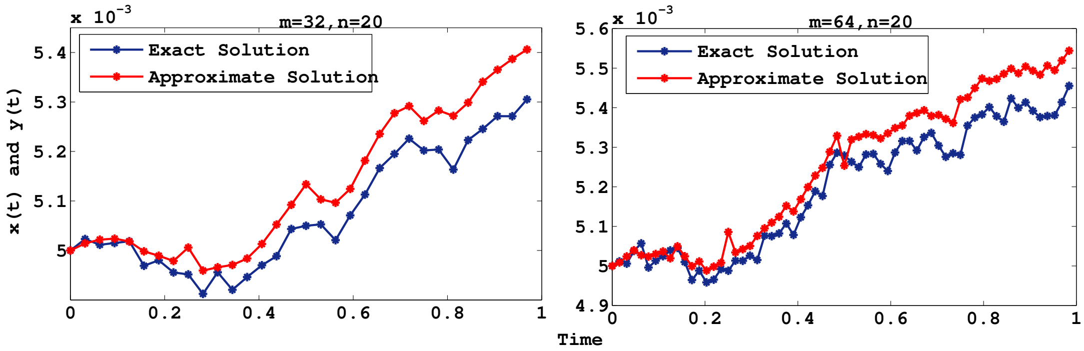

Figure 3: Example 6.2's approximate and exact solutions for $m = 32$ and $m = 64$.

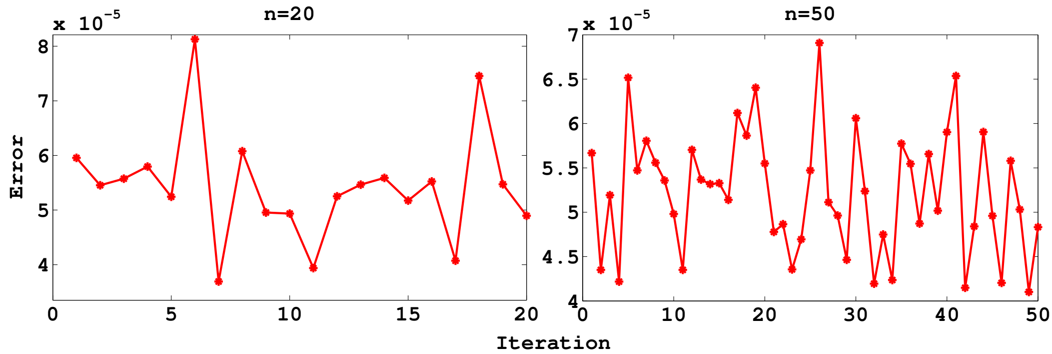

Figure 4: Example 6.2's error trend for $m = 64$, $n = 20$, and $n = 50$.

Example 6.3. Consider the linear stochastic Volterra integral equation

$$x(t) = x_0 + \int_0^t k(s+t)x(s)ds + \sum_{\gamma=1}^{3} \int_0^t k_\gamma(s,t)x(s)dB_\gamma(s),$$

where $s, t \in [0, 1)$ in which $x_0 = \frac{1}{12}$, $k(s,t) = (s+t)^2$, $k_1(s,t) = (s+t)$, $k_2(s,t) = \cos(s+t)$ and $k_3(s,t) = \sin(s+t)$. Here $B(t) = (B_1(t), B_2(t), B_3(t))$ is a 3-dimensional Brownian motion, and $x(t)$ is an unknown stochastic process defined on the probability space (Ω, F, P).

Table 3: Numerical result for $m = 32$, $m = 64$, and $m = 128$ with $n = 20$ in Example 6.3.

t	$m = 2^5$	$m = 2^6$	$m = 2^7$
0.1	0.09178598260	0.09869603885	0.08456237710
0.2	0.09657047975	0.09980363155	0.09166300290
0.3	0.09778185105	0.10929645010	0.12167983355
0.4	0.10744011755	0.13044535415	0.12254597820
0.5	0.26446428645	0.30381744415	0.25289232770
0.6	0.17720699205	0.19736396275	0.17564435724
0.7	0.29340060635	0.28647728650	0.24266162791
0.8	0.36339662485	0.45221749100	0.30225293181
0.9	0.69920166095	0.72602463780	0.61880977113

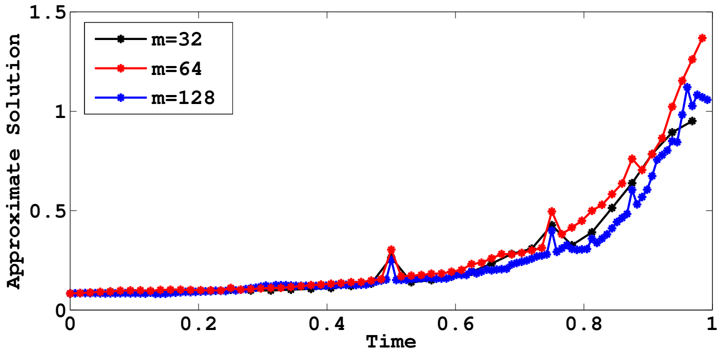

Figure 5: Example 6.3's approximate solution for $m = 32$, $m = 64$, and $m = 128$ with 20 iterations.

7. Conclusion

A new method to solve the linear p-dimensional SVIE has been proposed and investigated. The convergence and error analysis of the method have been carried out to confirm the validity of the method, which shows the method has a linear order of convergence. The method was illustrated by solving the problems given in the previous section. The numerical results of the problems show that the method has more accurate values in comparison with the earlier results. This method can be further developed to obtain the solution of the nonlinear stochastic Volterra integral equation and stochastic Volterra integral equation with singular kernel by using which many physical world problems can be solved. As the method has a linear order of convergence, we can modify the Walsh function to have a higher order of convergence.

References

[1] C. F. Chen, C. H. Hsiao, *A Walsh Series Direct Method for Solving Variational Problems*, J. Franklin Inst., **300** (1975), 265–280. 1
[2] C. F. Cheng, Y. T. Tsay, T. T. Wu. *Walsh operational matrices for fractional calculus and their application to distributed systems*, J. Franklin Inst., **303** (1977), 267–284. 2
[3] A. Etheridge, *A Course in Financial Calculus*, Cambridge University Press, Cambridge, (2002). 1
[4] B. Golubov, A. Efimov, V. Skvortsov, *Walsh series and transforms*, Kluwer Academic Publishers Group, Dordrecht, (1991). 5, 5
[5] S. Hatamzadeh-Varmazyar, Z. Masouri, E. Babolian, *Numerical method for solving arbitrary linear differential equations using a set of orthogonal basis functions and operational matrix*, Appl. Math. Model., **40** (2016), 233–253. 3
[6] C. H. Hsiao, C. F. Chen, *Solving integral equation via Walsh functions*, Comput. Electr. Engrg., **6** (1979), 279–292. 1
[7] K. Maleknejad, B. Basirat, E. Hashemizadeh, *Hybrid Legendre polynomials and block-pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations*, Comput. Math. Appl., **61** (2011), 2821–2828. 1
[8] K. Maleknejad, M. Khodabin, M. Rostami, *A numerical method for solving m-dimensional stochastic Itô-Volterra integral equations by stochastic operational matrix*, Comput. Math. Appl., **63** (2012), 133–143. 6.2
[9] K. Maleknejad, M. Khodabin, M. Rostami, *Numerical solution of stochastic Volterra integral equations by a stochastic operational matrix based on block pulse functions*, Math. Comput. Modelling, **55** (2012), 791–800. 3
[10] K. Maleknejad, S. Sohrabi, Y. Rostami, *Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials*, Appl. Math. Comput., **188** (2007), 123–128.
[11] F. Mohammadi, *Haar wavelets approach for solving multidimensional stochastic Itô-Volterra integral equations*, Appl. Math. E-Notes, **15** (2015), 80–96. 6.1
[12] F. Mohammadi, *Numerical Solution of Stochastic Ito-Volterra Integral Equations using Haar Wavelets*, Numer. Math. Theory Methods Appl., **9** (2016), 416–431. 1
[13] P. E. Kloeden, E. Platen, *Numerical solution of stochastic differential equations*, Springer-Verlag, Berlin, (1992). 1
[14] B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer-Verlag, New York, (1998). 1
[15] G. P. Rao, *Piecewise Constant Orthogonal Functions and Their Application to Systems and Control*, Springer-Verlag, Berlin, (1983).
[16] S. Saha Ray, S. Singh. *New stochastic operational matrix method for solving stochastic Itô-Volterra integral equations characterized by fractional Brownian motion*, Stoch. Anal. Appl., **39** (2021), 224–234. 1

[17] S. Singh, S. Saha Ray, *Stochastic operational matrix of Chebyshev wavelets for solving multi-dimensional stochastic Itô-Volterra integral equations*, Int. J. Wavelets Multiresolut. Inf. Process., **17** (2019), 16 pages.

[18] A. A. C. Tofigha, M. Khodabina, R. Ezzatia, *Numerical Solution of Linear Stochastic Volterra Integral Equations via New Basis Functions*, Filomat, **33** (2019), 5959–5966. 1

[19] J. L. Walsh, *A Closed Set of Normal Orthogonal Functions*, Amer. J. Math., **45** (1923), 5–24. 2.1, 2.2