J. Math. Computer Sci., 31 (2023), 338–344

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Intutionstic fuzzy permeable values and energetic subsets with applications in Hilbert algebras

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Abstract

In this paper, the concept of permeable values and energetic sets are introduced and studied in intutionstic fuzzy Hilbert algebras. Some properties and relevant examples are given. Moreover, the relations between intutionstic fuzzy level sets and energetic subsets are also discussed.

Keywords: Hilbert algebras, subalgebras, ideals, permeable value, energetic subset.

2020 MSC: 06F35, 03G25, 03B52.

1. Introduction

The concept of fuzzy sets was proposed by Zadeh [3]. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The idea of intuitionistic fuzzy sets suggested by Atanassov [1] is one of the extensions of fuzzy sets with better applicability. Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and so on. Hilbert algebras are a class of logical algebras which is introduced by Diego [2]. There is a deep relation between Hilbert algebras and posets. Today Hilbert algebras have been studied by many authors and they have been applied to many branches of mathematics, such as group, functional analysis, probability theory, topology, fuzzy set theory, and so on. In this paper, the concept of permeable values and energetic sets are introduced and studied in intutionstic fuzzy Hilbert algebras. Some properties and relevant examples are given. Moreover, the relations between intutionstic fuzzy level sets and energetic subsets are also discussed.

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doi: 10.22436/jmcs.031.03.08

Received: 2023-01-28 Revised: 2023-02-27 Accepted: 2023-04-19

2. Preliminaries

Definition 2.1 ([2]). A Hilbert algebra is a triplet $H = (H, *, 1)$, where H is a nonempty set, $*$ is a binary operation and 1 is fixed element of H such that the following axioms hold for each $x, y, z \in H$.

1. $x * (y * x) = 1$,
2. $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
3. $x * y = 1$ and $y * x = 1$ imply $x = y$.

The following result was proved in [2].

Lemma 2.2. *Let $H = (H, *, 1)$ be a Hilbert algebra and $x, y, z \in H$. Then*

1. $x * x = 1$,
2. $1 * x = x$,
3. $x * 1 = 1$,
4. $x * (y * z) = y * (x * z)$.

It is easily checked that in a Hilbert algebra H the relation $\leqslant y$ defined by $x \leqslant y \;\Leftrightarrow\; x * y = 1$ is a partial order on H with 1 as the largest element.

Definition 2.3 ([2]). A nonempty subset S of a Hilbert algebra $H = (H, *, 1)$ is called a subalgebra of H if $(\forall x, y \in H)(x \in S, y \in S \Rightarrow x * y \in S)$.

We let X be a nonempty set. An intutionstic fuzzy set in X (see [1]) is a structure of the form $A = \{\langle x; \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\mu_A : X \to [0, 1]$ is a membership function and $\gamma_A : X \to [0, 1]$ is a non-membership function. For the sake of simplicity, we use the symbol $A = (\mu_A, \gamma_A)$ for the intutionstic fuzzy set $A = \{\langle x; \mu_A(x), \gamma_A(x)\rangle : x \in X\}$.

3. Intutionistic fuzzy permeable values

Definition 3.1. A subset A of a Hilbert algebra X is said to be S-energetic if it satisfies

$$(\forall x, y \in X)(x * y \in A \Rightarrow \{x, y\} \cap A \neq \emptyset). \tag{3.1}$$

A subset A of a Hilbert algebra X is said to be I-energetic if it satisfies

$$(\forall x, y_1, y_2 \in X)((y_1 * (y_2 * x)) * x \in A \Rightarrow \{y_1, y_2\} \cap A \neq \emptyset). \tag{3.2}$$

Given an intutionstic fuzzy set $A = (\mu_A, \gamma_A)$ in a set X, $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, we consider the following sets:

$$U^{\in}(\mu_A, \alpha) = \{x \in X : \mu_A(x) \geqslant \alpha\}, U^{\in}(\mu_A, \alpha)^* = \{x \in X : \mu_A(x) > \alpha\},$$
$$U^{\in}(\gamma_A, \beta) = \{x \in X : \gamma_A(x) \leqslant \beta\}, U^{\in}(\gamma_A, \beta)^* = \{x \in X : \gamma_A(x) < \beta\},$$
$$L^{\in}(\mu_A, \alpha) = \{x \in X : \mu_A(x) \leqslant \alpha\}, L^{\in}(\mu_A, \alpha)^* = \{x \in X : \mu_A(x) < \alpha\},$$
$$L^{\in}(\gamma_A, \beta) = \{x \in X : \gamma_A(x) \geqslant \beta\}, L^{\in}(\gamma_A, \beta)^* = \{x \in X : \gamma_A(x) > \beta\}.$$

Definition 3.2. An intutionstic fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra X is called an $(\in, \in)$-intutionstic fuzzy subalgebra of X if the following assertions are valid:

$$\begin{aligned} x \in U^{\in}(\mu_A, \alpha_x), y \in U^{\in}(\mu_A, \alpha_y) \Rightarrow x * y \in U^{\in}(\mu_A, \alpha_x \wedge \alpha_y), \\ x \in U^{\in}(\gamma_A, \beta_x), y \in U^{\in}(\gamma_A, \beta_y) \Rightarrow x * y \in U^{\in}(\gamma_A, \beta_x \vee \beta_y), \end{aligned} \tag{3.3}$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0, 1]$ and $\beta_x, \beta_y \in [0, 1)$.

Lemma 3.3. *An intutionstic fuzzy set* $A = (\mu_A, \gamma_A)$ *in a Hilbert algebra* X *is an* $(\in, \in)$*-intutionstic fuzzy subalgebra of* X *if and only if* $A = (\mu_A, \gamma_A)$ *satisfies*

$$(\forall x, y \in X) \left(\begin{array}{c} \mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y) \\ \gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y) \end{array} \right). \tag{3.4}$$

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an $(\in, \in)$-intutionstic fuzzy subalgebra of X. If there exist $x, y \in X$ such that $\mu_A(x * y) < \mu_A(x) \wedge \mu_A(y)$, then $\mu_A(x * y) < \alpha \leqslant \mu_A(x) \wedge \mu_A(y)$ for some $\alpha \in (0, 1]$. It follows that $x, y \in U^{\in}(\mu_A, \alpha)$ but $x * y \notin U^{\in}(\mu_A, \alpha)$. Hence $\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y)$ for all $x, y \in X$. Suppose that there exist $a, b \in X$ such that $\gamma_A(a * b) > \gamma_A(a) \vee \gamma_A(b)$, then $\gamma_A(a * b) > \beta \geqslant \gamma_A(a) \vee \gamma_A(b)$ for some $\beta \in [0, 1)$. It follows that $a, b \in U^{\in}(\gamma_A, \beta)$ but $a * b \notin U^{\in}(\gamma_A, \beta)$, which is a contradiction. Hence $\gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y)$ for all $x, y \in X$. Conversely, let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in X which satisfies the condition (3.4). Let $x, y \in X$ be such that $x \in U^{\in}(\mu_A, \alpha_x)$ and $y \in U^{\in}(\mu_A, \alpha_y)$. Then $\mu_A(x) \geqslant \alpha_x$ and $\mu_A(y) \geqslant \alpha_y$, which imply that $\mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y) \geqslant \alpha_x \wedge \alpha_y$, that is, $x * y \in U^{\in}(\mu_A, \alpha_x \wedge \alpha_y)$. Now, let $x \in U^{\in}(\gamma_A, \beta_x)$ and $y \in U^{\in}(\gamma_A, \beta_y)$ for $x, y \in X$. Then $\gamma_A(x) \leqslant \beta_x$ and $\gamma_A(y) \leqslant \beta_y$, and so $\gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y) \leqslant \beta_x \vee \beta_y$. Hence $x * y \in U^{\in}(\gamma_A, \beta_x \vee \beta_y)$. Therefore $A = (\mu_A, \gamma_A)$ is an $(\in, \in)$-intutionstic fuzzy subalgebra of X. □

Proposition 3.4. *Every* $(\in, \in)$*-intutionstic fuzzy subalgebra* $A = (\mu_A, \gamma_A)$ *of a Hilbert algebra* X *satisfies*

$$(\forall x \in X) \left(\begin{array}{c} \mu_A(1) \geqslant \mu_A(x) \\ \gamma_A(1) \leqslant \gamma_A(x) \end{array} \right). \tag{3.5}$$

Proof. Let $x \in X$. Then

$$\mu_A(1) = \mu_A(x * x) \geqslant \mu_A(x) \wedge \mu_A(x) = \mu_A(x),$$
$$\gamma_A(1) = \gamma_A(x * x) \leqslant \gamma_A(x) \vee \gamma_A(x) = \gamma_A(x).$$
□

Theorem 3.5. *If* $A = (\mu_A, \gamma_A)$ *is an* $(\in, \in)$*-intutionstic fuzzy subalgebra of a Hilbert algebra* X*, then the subsets* $L^{\in}(\mu_A, \alpha)$ *and* $L^{\in}(\gamma_A, \beta)$ *of* X *are S-energetic subsets of* X.

Proof. Let $x, y \in X$ and $\alpha \in (0, 1]$ be such that $x * y \in L^{\in}(\mu_A, \alpha)$. Then $\alpha \geqslant \mu_A(x * y) \geqslant \mu_A(x) \wedge \mu_A(y)$, and thus $\mu_A(x) \leqslant \alpha$ or $\mu_A(y) \leqslant \alpha$; that is, $x \in L^{\in}(\mu_A, \alpha)$ or $y \in L^{\in}(\mu_A, \alpha)$. Thus $\{x, y\} \cap L^{\in}(\mu_A, \alpha) \neq \emptyset$. Therefore $L^{\in}(\mu_A, \alpha)$ is an S-energetic subset of X. We let $x, y \in X$ and $\beta \in [0, 1)$ be such that $x * y \in L^{\in}(\gamma_A, \beta)$. Then $\beta \leqslant \gamma_A(x * y) \leqslant \gamma_A(x) \vee \gamma_A(y)$. It follows that $\gamma_A(x) \geqslant \beta$ or $\gamma_A(y) \geqslant \beta$; that is, $x \in L^{\in}(\gamma_A, \beta)$ or $y \in L^{\in}(\gamma_A, \beta)$. Hence $\{x, y\} \cap L^{\in}(\gamma_A, \beta) \neq \emptyset$, and hence $L^{\in}(\gamma_A, \beta)$ is an S-energetic subset of X. □

The converse of Theorem 3.5 is not true, as seen in the following example.

Example 3.6. Consider a Hilbert algebra $X = \{1, a, b, c\}$ with the following Cayley table:

Table 1: Cayley table for the binary operation $*$.

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in X that is given in Table 2.

Table 2: Tabular representation of $A = (\mu_A, \gamma_A)$.

X	1	a	b	c
μ_A	0.6	0.2	0.3	0.3
γ_A	0.3	0.7	0.8	0.7

If $\alpha \in [0.4, 0.5]$ and $\beta \in [0.5, 0.6]$, then $L^{\in}(\mu_A, \alpha) = \{a, b, c\}$ and $L^{\in}(\gamma_A, \beta) = \{a, b, c\}$ are S-energetic subsets of X. Hence $L^{\in}(\mu_A, \alpha)$ and $L^{\in}(\gamma_A, \beta)$ of X are S-energetic subsets of X but A is not an $(\in, \in)$-intutionstic fuzzy subalgebra of X.

Definition 3.7. Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. Then (α, β) is called an intutionstic fuzzy permeable S-value for $A = (\mu_A, \gamma_A)$ if the following assertion is valid:

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in U^{\in}(\mu_A, \alpha) \Rightarrow \mu_A(x) \vee \mu_A(y) \geqslant \alpha \\ x * y \in U^{\in}(\gamma_A, \beta) \Rightarrow \gamma_A(x) \wedge \gamma_A(y) \leqslant \beta \end{array} \right). \tag{3.6}$$

Theorem 3.8. *Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. If $A = (\mu_A, \gamma_A)$ satisfies the following condition:*

$$(\forall x, y \in X) \left(\begin{array}{l} \mu_A(x * y) \leqslant \mu_A(x) \vee \mu_A(y) \\ \gamma_A(x * y) \geqslant \gamma_A(x) \wedge \gamma_A(y) \end{array} \right), \tag{3.7}$$

then (α, β) is an intutionstic fuzzy permeable S-value for $A = (\mu_A, \gamma_A)$.

Proof. Let $x, y \in X$ be such that $x * y \in U^{\in}(\mu_A, \alpha)$. Then $\alpha \leqslant \mu_A(x * y) \leqslant \mu_A(x) \vee \mu_A(y)$. Now, let $a, b \in X$ be such that $a * b \in U^{\in}(\gamma_A, \beta)$. Then $\beta \geqslant \gamma_A(a * b) \geqslant \gamma_A(a) \wedge \gamma_A(b)$. Therefore (α, β) is an intutionstic fuzzy permeable S-value for $A = (\mu_A, \gamma_A)$. □

Theorem 3.9. *Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. If (α, β) is an intutionstic fuzzy permeable S-value for $A = (\mu_A, \gamma_A)$, then the subsets $U^{\in}(\mu_A, \alpha)$ and $U^{\in}(\gamma_A, \beta)$ of X are S-energetic subsets of X.*

Proof. Let $x, y, u, v \in X$ be such that $x * y \in U^{\in}(\mu_A, \alpha)$ and $u * v \in U^{\in}(\gamma_A, \beta)$. Using (3.6), we have $\mu_A(x) \vee \mu_A(y) \geqslant \alpha$ and $\gamma_A(u) \wedge \gamma_A(v) \leqslant \beta$. It follows that $\mu_A(x) \geqslant \alpha$ or $\mu_A(y) \geqslant \alpha$, that is, $x \in U^{\in}(\mu_A, \alpha)$ or $y \in U^{\in}(\mu_A, \alpha)$ and $\gamma_A(u) \leqslant \beta$ or $\gamma_A(v) \leqslant \beta$, that is, $u \in U^{\in}(\gamma_A, \beta)$ or $v \in U^{\in}(\gamma_A, \beta)$. Hence $\{x, y\} \cap U^{\in}(\mu_A, \alpha) \neq \emptyset$ and $\{u, v\} \cap U^{\in}(\gamma_A, \beta) \neq \emptyset$. Therefore $U^{\in}(\mu_A, \alpha)$ and $U^{\in}(\gamma_A, \beta)$ are S-energetic subsets of X. □

Theorem 3.10. *If $A = (\mu_A, \gamma_A)$ is an intutionstic fuzzy subalgebra of a Hilbert algebra X, then the subsets $X \backslash U^{\in}(\mu_A, \alpha)$ and $X \backslash U^{\in}(\gamma_A, \beta)$ of X are S-energetic subsets of X whenever they are nonempty.*

Proof. Let $x, y \in X$ be such that $x * y \in X \backslash U^{\in}(\mu_A, \alpha)$ and $x * y \in X \backslash U^{\in}(\gamma_A, \beta)$ for $\alpha, \beta \in [0, 1]$. If $\{x, y\} \cap (X \backslash U^{\in}(\mu_A, \alpha)) = \emptyset$, then $x, y \in U^{\in}(\mu_A, \alpha)$ and so $x * y \in U^{\in}(\mu_A, \alpha)$ since $U^{\in}(\mu_A, \alpha)$ is a subalgebra of X. Assume that $\{x, y\} \cap (X \backslash U^{\in}(\gamma_A, \beta)) = \emptyset$. Then $x, y \in U^{\in}(\gamma_A, \beta)$ and so $x * y \in U^{\in}(\gamma_A, \beta)$ since $U^{\in}(\gamma_A, \beta)$ is a subalgebra of X. This is a contradiction. Hence $X \backslash U^{\in}(\mu_A, \alpha)$ and $X \backslash U^{\in}(\gamma_A, \beta)$ are S-energetic subsets of X for all $t \in [0, 1]$. □

Definition 3.11. Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. Then (α, β) is called an intutionstic fuzzy anti-permeable S-value for $A = (\mu_A, \gamma_A)$ if the following assertion is valid:

$$(\forall x, y \in X) \left(\begin{array}{l} x * y \in L_T^{\in}(A, \alpha) \Rightarrow \mu_A(x) \wedge \mu_A(y) \leqslant \alpha, \\ x * y \in L_F^{\in}(A, \beta) \Rightarrow \gamma_A(x) \vee \gamma_A(y) \geqslant \beta \end{array} \right). \tag{3.8}$$

Theorem 3.12. *Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. If $A = (\mu_A, \gamma_A)$ is an $(\in, \in)$-intutionstic fuzzy subalgebra of X, then (α, β) is an intutionstic fuzzy anti-permeable S-value for $A = (\mu_A, \gamma_A)$.*

Proof. Let $x, y, u, v \in X$ be such that $x * y \in L^{\in}(\mu_A, \alpha)$ and $u * v \in L^{\in}(\gamma_A, \beta)$. Using Lemma 3.3, we have $\mu_A(x) \wedge \mu_A(y) \leqslant \mu_A(x * y) \leqslant \alpha$, $\gamma_A(u) \vee \gamma_A(v) \geqslant \gamma_A(u * v) \geqslant \beta$, and thus (α, β) is an intutionstic fuzzy anti-permeable S-value for $A = (\mu_A, \gamma_A)$. □

Theorem 3.13. *Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. If (α, β) is an intutionstic fuzzy anti-permeable S-value for $A = (\mu_A, \gamma_A)$, then the subsets $L^{\in}(\mu_A, \alpha)$ and $L^{\in}(\gamma_A, \beta)$ of X are S-energetic subsets of X.*

Proof. Let $x, y, u, v \in X$ be such that $x * y \in L^{\in}_T(\mu_A, \alpha)$ and $u * v \in L^{\in}(\gamma_A, \beta)$. Using (3.8), we have $\mu_A(x) \wedge \mu_A(y) \leqslant \alpha$ and $\gamma_A(u) \vee \gamma_A(v) \geqslant \beta$, which imply that $\mu_A(x) \leqslant \alpha$ or $\mu_A(y) \leqslant \alpha$, that is, $x \in L^{\in}(\mu_A, \alpha)$ or $y \in L^{\in}(\mu_A, \alpha)$; and $\gamma_A(u) \geqslant \beta$ or $\gamma_A(v) \geqslant \beta$, that is, $u \in L^{\in}(\gamma_A, \beta)$ or $v \in L^{\in}(\gamma_A, \beta)$. Hence $\{x, y\} \cap L^{\in}(\mu_A, \alpha) \neq \emptyset$ and $\{u, v\} \cap L^{\in}(\gamma_A, \beta) \neq \emptyset$. Therefore $L^{\in}(\mu_A, \alpha)$ and $L^{\in}(\gamma_A, \beta)$ are S-energetic subsets of X. □

Definition 3.14. An intutionstic fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra X is called an $(\in, \in)$-intutionstic fuzzy ideal of X if the following assertions are valid:

$$(\forall x \in X)\begin{pmatrix} x \in U^{\in}(\mu_A, \alpha) \Rightarrow 1 \in U^{\in}(\mu_A, \alpha) \\ x \in U^{\in}(\gamma_A, \beta) \Rightarrow 1 \in U^{\in}(\gamma_A, \beta) \end{pmatrix} \tag{3.9}$$

$$(\forall x, y \in X)\begin{pmatrix} y \in U^{\in}(\mu_A, \alpha) \Rightarrow x * y \in U^{\in}(\mu_A, \alpha) \\ y \in U^{\in}(\gamma_A, \beta) \Rightarrow x * y \in U^{\in}(\gamma_A, \beta) \end{pmatrix} \tag{3.10}$$

$$(\forall x, y_1, y_2 \in X)\begin{pmatrix} y_1 \in U^{\in}(\mu_A, \alpha_x), y_2 \in U^{\in}(\mu_A, \alpha_y) \Rightarrow (y_1 * (y_2 * x)) * x \in U^{\in}(\mu_A, \alpha_x \wedge \alpha_y) \\ y_1 \in U^{\in}(\gamma_A, \beta_x), y_2 \in U^{\in}(\gamma_A, \beta_y) \Rightarrow (y_1 * (y_2 * x)) * x \in U^{\in}(\gamma_A, \beta_x \vee \beta_y) \end{pmatrix}, \tag{3.11}$$

for all $\alpha, \alpha_x, \alpha_y \in (0, 1]$ and $\beta, \beta_x, \beta_y \in [0, 1)$.

Theorem 3.15. *An intutionstic fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra X is an $(\in, \in)$-intutionstic fuzzy ideal of X if and only if $A = (\mu_A, \gamma_A)$ satisfies*

$$(\forall x \in X)\begin{pmatrix} \mu_A(1) \geqslant \mu_A(x) \\ \gamma_A(1) \leqslant \gamma_A(x) \end{pmatrix}, \tag{3.12}$$

$$(\forall x, y \in X)\begin{pmatrix} \mu_A(x * y) \geqslant \mu_A(y) \\ \gamma_A(x * y) \leqslant \gamma_A(y) \end{pmatrix}, \tag{3.13}$$

$$(\forall x, y_1, y_2 \in X)\begin{pmatrix} \mu_A((y_1 * (y_2 * x)) * x) \geqslant \mu_A(y_1) \wedge \mu_A(y_2) \\ \gamma_A((y_1 * (y_2 * x)) * x) \leqslant \gamma_A(y_1) \vee \gamma_A(y_2) \end{pmatrix}. \tag{3.14}$$

Proof. Assume that (3.12) is valid, and let $x \in U^{\in}(\mu_A, \alpha)$ and $u \in U^{\in}(\gamma_A, \beta)$ for any $x, u \in X$, $\alpha \in (0, 1]$ and $\beta \in [0, 1)$. Then $\mu_A(1) \geqslant \mu_A(x) \geqslant \alpha$ and $\gamma_A(1) \leqslant \gamma_A(u) \leqslant \beta$. Hence $1 \in U^{\in}(\mu_A, \alpha)$ and $1 \in U^{\in}(\gamma_A, \beta)$, and thus (3.9) is valid. Assume that (3.13) is valid, and let $x, y, u, v \in X$ such that $y \in U^{\in}(\mu_A, \alpha)$ and $v \in U^{\in}(\gamma_A, \beta)$ for any $\alpha \in (0, 1]$ and $\beta \in [0, 1)$. Then $\mu_A(x * y) \geqslant \mu_A(y) \geqslant \alpha$ and $\gamma_A(u * v) \leqslant \gamma_A(v) \leqslant \beta$. Hence $x * y \in U^{\in}(\mu_A, \alpha)$ and $u * v \in U^{\in}(\gamma_A, \beta)$, and thus (3.10) is valid. Let $x, y_1, y_2, u, v_1, v_2 \in X$ be such that $y_1 \in U^{\in}(\mu_A, \alpha_x)$, $y_2 \in U^{\in}(\mu_A, \alpha_y)$, $v_1 \in U^{\in}(\gamma_A, \beta_u)$, and $v_2 \in U^{\in}(\gamma_A, \beta_v)$ for all $\alpha_x, \alpha_y \in (0, 1]$ and $\beta_u, \beta_v \in [0, 1)$. Then $\mu_A(y_1) \geqslant \alpha_x$, $\mu_A(y_2) \geqslant \alpha_y$, $\gamma_A(v_1) \leqslant \beta_u$, and $\gamma_A(v_2) \leqslant \beta_v$. It follows from (3.14) that $\mu_A((y_1 * (y_2 * x)) * x) \geqslant \mu_A(y_1) \wedge \mu_A(y_2) \geqslant \alpha_x \wedge \alpha_y$, $\gamma_A((v_1 * (v_2 * u)) * u) \leqslant \gamma_A(v_1) \vee \gamma_A(v_2) \leqslant \beta_u \vee \beta_v$. Hence $(y_1 * (y_2 * x)) * x \in U^{\in}(\mu_A, \alpha_x \wedge \alpha_y)$ and $(v_1 * (v_2 * u)) * u \in U^{\in}(\gamma_A, \beta_u \vee \beta_v)$. Therefore $A = (\mu_A, \gamma_A)$ is an $(\in, \in)$-intutionstic fuzzy ideal of X. Conversely, let $A = (\mu_A, \gamma_A)$ be an $(\in, \in)$-intutionstic fuzzy ideal of X. If there exists $x_0 \in X$ such that $\mu_A(1) < \mu_A(x_0)$, then $x_0 \in U^{\in}(\mu_A, \alpha)$ and $1 \notin U^{\in}(\mu_A, \alpha)$, where $\alpha = \mu_A(x_0)$. This is a contradiction, and thus $\mu_A(1) \geqslant \mu_A(x)$ for all $x \in X$. If there exists $x_0 \in X$ such that $\gamma_A(1) > \gamma_A(x_0)$, then $x_0 \in U^{\in}(\gamma_A, \beta)$ and $1 \notin U^{\in}(\gamma_A, \beta)$, where $\beta = \gamma_A(x_0)$. This is a contradiction, and thus $\gamma_A(1) \leqslant \gamma_A(x)$ for all $x \in X$. If there exists $x_0, y_0 \in X$ such that $\mu_A(x_0 * y_0) < \mu_A(y_0)$, then $y_0 \in U^{\in}(\mu_A, \alpha)$ and $x_0 * y_0 \notin U^{\in}(\mu_A, \alpha)$, where $\alpha = \mu_A(y_0)$. This is a contradiction, and thus $\mu_A(x * y) \geqslant \mu_A(y)$ for all $x, y \in X$. If there exists $x_0, y_0 \in X$ such that $\gamma_A(x_0 * y_0) > \gamma_A(y_0)$, then $y_0 \in U^{\in}(\gamma_A, \beta)$ and $x_0 * y_0 \notin U^{\in}(\gamma_A, \beta)$, where $\beta = \gamma_A(y_0)$. This is a contradiction, and thus $\gamma_A(x * y) \leqslant \gamma_A(y)$ for all $x, y \in X$. Assume that $\mu_A((y_1' * (y_2' * x')) * x') < \mu_A(y_1') \wedge \mu_A(y_2')$ for some $x', y_1', y_2' \in X$. Taking $\alpha = \mu_A(y_1') \wedge \mu_A(y_2')$ implies that $y_1' \in U^{\in}(\mu_A, \alpha)$ and $y_2' \in U^{\in}(\mu_A, \alpha)$; but $(y_1' * (y_2' * x')) * x' \notin U^{\in}(\mu_A, \alpha)$. This is a contradiction, and thus $\mu_A((y_1 * (y_2 * x)) * x) \geqslant \mu_A(y_1) \wedge \mu_A(y_2)$ for all $x, y_1, y_2 \in X$. Suppose there exist $a', b_1', b_2' \in X$ such that $\gamma_A((b_1' * (b_2' * a')) * a') > \gamma_A(b_1') \vee \gamma_A(b_2')$, and take $\beta = \gamma_A(b_1') \vee \gamma_A(b_2')$. Then $b_1' \in U^{\in}(\gamma_A, \beta)$, $b_2' \in U^{\in}(\gamma_A, \beta)$, and $(b_1' * (b_2' * a')) * a' \notin U^{\in}(\gamma_A, \beta)$, which is a contradiction. Thus $\gamma_A((y_1 * (y_2 * x)) * x) \leqslant \gamma_A(y_1) \vee \gamma_A(y_2)$ for all $x, y_1, y_2 \in X$. Hence $A = (\mu_A, \gamma_A)$ satisfies (3.14). □

Lemma 3.16. *Every $(\in, \in)$-intutionstic fuzzy ideal $A = (\mu_A, \gamma_A)$ of a Hilbert algebra X satisfies*

$$(\forall x, y \in X)\left(x \leqslant y \Rightarrow \begin{cases} \mu_A(x) \leqslant \mu_A(y) \\ \gamma_A(x) \geqslant \gamma_A(y) \end{cases} \right). \tag{3.15}$$

Proof. Let $x, y \in X$ be such that $x \leqslant y$. Then $x * y = 1$, and thus

$$\begin{aligned} \mu_A(y) &= \mu_A(1 * y) \\ &= \mu_A(((x * y) * (x * y)) * y) \\ &\geqslant \mu_A(x * y) \wedge \mu_A(x) \\ &= \mu_A(1) \wedge \mu_A(x) \\ &= \mu_A(x), \end{aligned}$$

$$\begin{aligned} \gamma_A(y) &= \gamma_A(1 * y) \\ &= \gamma_A(((x * y) * (x * y)) * y) \\ &\leqslant \gamma_A(x * y) \vee \gamma_A(x) \\ &= \gamma_A(1) \vee \gamma_A(x) \\ &= \gamma_A(x). \end{aligned}$$

□

Theorem 3.17. *If $A = (\mu_A, \gamma_A)$ is an $(\in, \in)$-intutionstic fuzzy ideal of a Hilbert algebra X, then the subsets $L^{\in}(\mu_A, \alpha)$ and $L^{\in}(\gamma_A, \beta)$ of X are I-energetic subsets of X.*

Proof. Let $x, y_1, y_2, u, v_1, v_2 \in X$, $\alpha \in (0, 1]$ and $\beta \in [0, 1)$ be such that $(y_1 * (y_2 * x)) * x \in L^{\in}(\mu_A, \alpha)$ and $(v_1 * (v_2 * u)) * u \in L^{\in}(\gamma_A, \beta)$. Using Theorem 3.15, we have $\alpha \geqslant \mu_A((y_1 * (y_2 * x)) * x) \geqslant \mu_A(y_1) \wedge \mu_A(y_2)$, $\beta \leqslant \gamma_A((v_1 * (v_2 * u)) * u) \leqslant \gamma_A(v_1) \vee \gamma_A(v_2)$. It follows that $\mu_A(y_1) \leqslant \alpha$ or $\mu_A(y_2) \leqslant \alpha$, that is, $y_1 \in L^{\in}(\mu_A, \alpha)$ or $y_2 \in L^{\in}(\mu_A, \alpha)$ and $\gamma_A(v_1) \geqslant \beta$ or $\gamma_A(v_2) \geqslant \beta$, that is, $v_1 \in L^{\in}(\mu_A, \gamma)$ or $v_2 \in L^{\in}(\gamma_A, \beta)$. Hence $\{y_1, y_2\} \cap L^{\in}(\mu_A, \alpha) \neq \emptyset$ and $\{v_1, v_2\} \cap L^{\in}(\gamma_A, \beta) \neq \emptyset$, and hence $L^{\in}(\mu_A, \alpha)$ and $L^{\in}(\gamma_A, \beta)$ are I-energetic subsets of X. □

Theorem 3.18. *If $A = (\mu_A, \gamma_A)$ is an intutionstic fuzzy ideal of a Hilbert algebra X, then the subsets $X \backslash U^{\in}(\mu_A, \alpha)$ and $X \backslash U^{\in}(\gamma_A, \beta)$ of X are I-energetic subsets of X whenever they are nonempty.*

Proof. Assume that $X \backslash U^{\in}(\mu_A, \alpha)$ and $X \backslash U^{\in}(\gamma_A, \beta)$ are nonempty sets for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. Let x, y_1, y_2, a, b_1, b_2 such that $y_1 * (y_2 * x)) * x \in X \backslash U^{\in}(\mu_A, \alpha)$ and $b_1 * (b_2 * a)) * a \in X \backslash U^{\in}(\gamma_A, \beta)$, then $\alpha > \mu_A((y_1 * (y_2 * x)) * x) \geqslant \min\{\mu_A(y_1), \mu_A(y_2)\}$ and $\beta < \gamma_A((b_1 * (b_2 * a)) * a) \leqslant \max\{\gamma_A(b_1), \gamma_A(b_2)\}$. Then we have $\mu_A(y_1) < \alpha$ or $\mu_A(y_2) < \alpha$, that is, $y_1 \in X \backslash U^{\in}(\mu_A, \alpha)$ or $y_2 \in X \backslash U^{\in}(\mu_A, \alpha)$. Hence $\{y_1, y_2\} \cap (X \backslash U^{\in}(\mu_A, \alpha)) \neq \emptyset$. Then we get $\gamma_A(b_1) > \beta$ or $\gamma_A(b_2) > \beta$, that is, $b_1 \in X \backslash U^{\in}(\gamma_A, \beta)$ or $b_2 \in X \backslash U^{\in}(\gamma_A, \beta)$. Thus $\{b_1, b_2\} \cap (X \backslash U^{\in}(\gamma_A, \beta)) \neq \emptyset$. Therefore $X \backslash U^{\in}(\mu_A, \alpha)$ and $X \backslash U^{\in}(\gamma_A, \beta)$ are I-energetic subsets of X for all $(\alpha, \beta) \in [0, 1] \times [0, 1]$. □

Definition 3.19. Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. Then (α, β) is called an intutionstic fuzzy permeable I-value for $A = (\mu_A, \gamma_A)$ if the following assertion is valid:

$$(\forall x, y_1, y_2 \in X)\left(\begin{array}{l} (y_1 * (y_2 * x)) * x \in U^{\in}(\mu_A, \alpha) \Rightarrow \mu_A(y_1) \vee \mu_A(y_2) \geqslant \alpha \\ (y_1 * (y_2 * x)) * x \in U^{\in}(\gamma_A, \beta) \Rightarrow \gamma_A(y_1) \wedge \gamma_A(y_2) \leqslant \beta \end{array}\right). \tag{3.16}$$

Lemma 3.20. *If an intutionstic fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra X satisfies the condition of (3.7), then*

$$(\forall x \in X)\left(\begin{array}{l} \mu_A(1) \geqslant \mu_A(x) \\ \gamma_A(1) \leqslant \gamma_A(x) \end{array}\right). \tag{3.17}$$

Proof. Straightforward. □

Theorem 3.21. *Let $A = (\mu_A, \gamma_A)$ be an intutionstic fuzzy set in a Hilbert algebra X and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$. If (α, β) is an intutionstic fuzzy permeable* I*-value for $A = (\mu_A, \gamma_A)$, then the subsets $U^{\in}(\mu_A, \alpha)$ and $U^{\in}(\gamma_A, \beta)$ of X are* I*-energetic subsets of X.*

Proof. Let $x, y_1, y_2, u, v_1, v_2 \in X$ and $(\alpha, \beta) \in \Lambda_\alpha \times \Lambda_\beta$, where Λ_α and Λ_β are subsets of $[0, 1]$ such that $(y_1 * (y_2 * x)) * x \in U^{\in}(\mu_A, \alpha)$ and $(v_1 * (v_2 * u)) * u \in U^{\in}(\gamma_A, \beta)$. Because (α, β) is an intutionstic fuzzy permeable I-value for $A = (\mu_A, \gamma_A)$, it follows from (3.16) that $\mu_A(y_1) \vee \mu_A(y_2) \geqslant \alpha$ and $\gamma_A(v_1) \wedge \gamma_A(v_2) \leqslant \beta$. Hence $\mu_A(y_1) \geqslant \alpha$ or $\mu_A(y_2) \geqslant \alpha$, that is, $y_1 \in U^{\in}(\mu_A, \alpha)$ or $y_2 \in U^{\in}(\mu_A, \alpha)$ and $\gamma_A(v_1) \leqslant \beta$ or $\gamma_A(v_2) \leqslant \beta$, that is, $v_1 \in U^{\in}(\gamma_A, \beta)$ or $v_2 \in U^{\in}(\gamma_A, \beta)$. Hence $\{y_1, y_2\} \cap U^{\in}(\mu_A, \alpha) \neq \emptyset$ and $\{v_1, v_2\} \cap U^{\in}(\gamma_A, \beta) \neq \emptyset$, and $U^{\in}(\mu_A, \alpha)$ and $U^{\in}(\gamma_A, \beta)$ of X are I-energetic subsets of X. □

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