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Some results on coupled coincidence points in vector quasi cone metric spaces over Banach algebras with satisfactory cones

Sahar Mohamed Ali Abou Bakr*, H. K. Hussein

Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt.

Abstract

Let $(X, C, \mathbb{D}_C, \chi)$ be a complete parametric vector quasi cone metric space over a Banach algebra $\mathbb{A}$, C be a cone in $\mathbb{A}$ that contains some semi-interior points, χ be a metric parameter in C with a spectral radius $\sigma(\chi) \geqslant 1$, and $T : X \times X \to X$ be a generalized contraction mapping where its parametric contractions are vectors in C, with these settings and without relying on the assumptions of normality and solidness of C, we prove the existence of coupled coincidence points of the mapping T and hence generalize many theorems concerned with the existence of coupled coincidence points of such types and we support these results with some illustrative examples.

Keywords: Banach algebra, normal cones, solid cones, cone metric spaces over Banach algebra, semi interior points, coupled coincidence points.

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1. Introduction and preliminaries

The interest in fixed, coupled fixed, and coupled coincidence point theorems and their generalizations, starting from the Banach contraction principle in 1922 [14] have gradually increased because their applications are widely used in solving various types of mathematical and computer science's problems including variational inequalities, optimization, differential equations, equilibrium problems, denotational semantics of programming languages, and many others [17, 33, 47].

The generalizations of these theorems have rich historical achievements by varying their frame of works, quietly moved from metric [1, 2, 21] to b-metric then to partial metric real-valued functions frame by Matthews [38], Zabrejko [49] and Moshokoa [39]. From metric cone valued functions by Huang and Zhang [28] to b-cone metric valued functions by Huang, Xu, and Shi [27, 46] then to partial cone metric valued functions frame by Jiang and Li [30]. From θ-cone metric valued functions to $\theta-$b-cone metric valued functions by Sahar [3–5]. From using the normality condition of the underlying cone [25, 29, 31, 41]

*Corresponding author
Email addresses: saharm_ali@sci.asu.edu.eg; saharm_ali@yahoo.com (Sahar Mohamed Ali Abou Bakr), h_karam@sci.asu.edu.eg (H. K. Hussein)

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to removing the normality condition of the underlying cone [3–5, 35, 44, 48]. Currently, attempts are to remove the solidness assumption of the underlying cone.

Meanwhile, in 2010 and 2012, equivalence characterizations between the topology generated by cone metric valued function and some topology generated by corresponding metric function have been given in the case of normal and solid cones [12, 13, 18–20, 22, 32, 34].

In 2013, Liu and Xu [36] generalized the concept of cone metric space to the concept of cone metric space over Banach algebra, then in 2016, Huang and Radenovic [26] generalized the concept to cone b-metric space over a Banach algebra, and then in 2019, Huang et al [24] gave some of its properties for normal and non-normal cones, while cones are assumed to be solid.

In 2017, Basile et al [15] started to define semi-interior points, this points permitted solving many equilibrium problems needless of the existence of interior points of the underlying non-solid cone, accordingly the exitance of semi-interior points reduced the limitation of some applications. Consequently, studies in the case of non-solid cones are optimistic tasks.

In 2019, Aleksi'c et al [9] gave some examples of normal cones with normal constant 1 and these cones are not solid. Since the structure of any topology generated by the cone metric valued function is mainly depending on the interior points of the underlying cone, any cone metric valued function with a non-solid cone can not generate a corresponding topology with prescribed characterizations. These confirmed Basile et al observations.

In 2020-2021, Sahar Mohamed Ali Abou Bakr [4, 5] used the concept of b-cone, theta cone, and b-theta cone metric spaces to generalize the previously mentioned results of coupled fixed point theorems in these frames.

In 2022, Sahar Mohamed Ali Abou Bakr [6] investigated some properties of satisfactory cones; particularly cones with some semi-interior points in normed algebras, introduced the notions of S-set and S-number for the semi-interior point, calculated these quantities for some examples, used the topology generated by ϖ-cone metric space over Banach algebra when cones are satisfactory to forward more generalized fixed point theorems.

Back in 1987, Gue and Lakshmikantham [23] initiated the concept of coupled fixed point on partially ordered complete metric space, and then in 2006, Bhaskar and Lakshmikantham [16] introduced the concept of mixed monotone property and showed that those mappings have coupled fixed points. Some other results have been given in 2009 by authors in [43].

In 2011 and then in 2013, Alotaibi et al [11] and Agarwal et al [8] separately proved the existence of coupled coincidence points for monotone operators and operators lacking the mixed monotone property in partially ordered complete metric spaces.

In 2013, Luong, Thuan, and Rao [37] proved the existence of coupled coincidence and coupled fixed point of two compatible mappings $F : X \times X \to X$ and $g : X \to X$, F is continuous, F has the mixed g-monotone property, F and g satisfying some generalized contraction conditions, X is sequentially partially ordered complete cone metric space.

In 2010, and 2012, Shatanawi [45] and Nashine et al [40] gave more results of coupled fixed points in partially ordered cone metric spaces settings.

In this paper, needless of the normality and solidness assumptions of the underlying cone and in the setting of parametric vector quasi cone metric space over Banach algebra with the cone having semi-interior points, the main theorem of this paper generalized the previously mentioned results concerning coupled coincidence and coupled fixed points, where the quasi cone metric parameter is replaced by a vector in the underlying cone and the contraction constants of the mappings are replaced by vectors in such a cone.

2. Basic Definitions and Notations

In this paper, the normed space $(\mathbb{A}, \|.\|)$ with extra third multiplicative operation $\diamond$ is given to be an unital Banach algebra, it will be simply written as $\mathbb{A}$, the elements θ and e denote the unit elements concerning addition and multiplication operations, respectively and x^{-1} denote the multiplicative inverse of the element $x \in \mathbb{A}$, if exists.

The spectral radius of any element ϑ in any unital Banach algebra $\mathbb{A}$ will be denoted by $\sigma(\vartheta)$ and some of its characteristics are given in [42].

Two types of cones are considered, cones in normed spaces and cones in unital normed algebras, any of these cones will be denoted by C.

The set of interior points of any cone will be denoted by $\mathrm{Int}(C)$. The set $\mathbb{U} := \{v : v \in \mathbb{A}, \|v\| < 1\}$ denote the open unit ball of the normed algebra $\mathbb{A}$, the positive part of $\mathbb{U}$ is defined as the intersection of $\mathbb{U}$ with C and will be denoted by $\mathbb{U}_+$; $\mathbb{U}_+ := \mathbb{U} \bigcap C$.

A point $v \in C$ is defined to be the semi-interior point of C iff there exists a positive real number $\delta > 0$ such that $v - \delta\mathbb{U}_+ \subseteq C$.

The set of semi-interior points of any cone will be denoted by $\mathrm{SInt}(C)$, a brief relationship between $\mathrm{Int}(C)$ and $\mathrm{SInt}(C)$ is given by examples in [6, 15]. We stress the commitment to the fact that the possibility of the existence of a semi-interior point is higher than the possibility of the existence of an interior point.

Any cone C of a normed space $\mathbb{A}$ defines four partial ordered relations; for $u, v \in \mathbb{A}$ as follows:

$$
\begin{aligned}
&u \preceq v \text{ is equivalent to } v - u \in C,\\
&u \prec v \text{ is equivalent to } v - u \in C \text{ and } u \neq v,\\
&u \ll v \text{ is equivalent to } v - u \in \mathrm{Int}(C),\\
&u \sqsubseteq v \text{ is equivalent to } v - u \in \mathrm{SInt}(C).
\end{aligned}
$$

We have $\theta \sqsubseteq u$ if and only if $u \in \mathrm{SInt}(C)$. We gave the following defined term: A cone C is a satisfactory cone if and only if it is either normal and solid, or not normal and solid, or normal and not solid but $\mathrm{SInt}(C) \neq \emptyset$, or not normal and not solid but $\mathrm{SInt}(C) \neq \emptyset$.

Since the concept of semi-interior points gives a wider range of study we focus mainly on the relation induced by cones with some semi-interior points, $\mathrm{SInt}(C) \neq \emptyset$, cones themselves may be non-solid $\mathrm{Int}(C) = \emptyset$. This implies more extensions and generalizations on fixed points theory. We have the following observations:

Lemma 2.1 ([6])**.** *If C is a cone in a normed space $\mathbb{A}$, then we have*

1. $\theta \sqsubseteq \lambda u$ *whenever* $\theta \sqsubseteq u$ *and* $\lambda \geqslant 0$.
2. $\theta \sqsubseteq \lambda u + \mu v$ *whenever* $\theta \sqsubseteq u, v$ *and* $\lambda, \mu > 0$.
3. $u \sqsubseteq \omega$ *whenever* $u \preceq v$ *and* $v \sqsubseteq \omega$.
4. $u + \omega \sqsubseteq \vartheta + \upsilon$ *whenever* $u \sqsubseteq \vartheta$ *and* $\omega \sqsubseteq \upsilon$.
5. *If* $\theta \sqsubseteq u$ *and* $\theta \sqsubseteq v$, *then there is* $\theta \sqsubseteq \omega$ *such that* $\omega \sqsubseteq u$ *and* $\omega \sqsubseteq v$.
6. *If* $u \preceq v$ *for all* v *with* $\theta \sqsubseteq v$, *then* $u = \theta$. *In particular; if* $u \sqsubseteq v$ *for every* $v \in \mathrm{SInt}(C)$, *then* $u = \theta$.

We also will require the following:

Lemma 2.2 ([6])**.** *If C is a cone in a unital normed algebra $\mathbb{A}$ and $u, v, \omega, \varpi \in C$ be any vectors, $\sigma \in C$ be an invertible element, its inverse $\sigma^{-1} \in C$, then*

1. *If* $\theta \preceq u \preceq v$ *and* $\theta \preceq \omega \preceq \varpi$, *then* $\theta \preceq u \diamond \omega \preceq v \diamond \varpi$.
2. *If* $\theta \sqsubseteq u$, *then* $\theta \sqsubseteq \sigma \diamond u$ *and* $\theta \sqsubseteq u \diamond \sigma$. *There are examples of Banach algebras with cones having non-invertible elements* $v \in C$ *for which* $v \diamond u = u \diamond v \notin \mathrm{SInt}(C)$ *for every* $u \in C$.
3. *If* $u \sqsubseteq v$, *then* $\sigma \diamond u \sqsubseteq \sigma \diamond v$ *and* $u \diamond \sigma \sqsubseteq v \diamond \sigma$.
4. *If* $\theta \sqsubseteq v$, $\varepsilon > 0$, *and* $n \in \mathbb{N}$. *Set* $\frac{\varepsilon}{n} u \diamond v := u \diamond v_n$, *then* $\theta \sqsubseteq v_n$ *and there are* $n_0 \in \mathbb{N}$ *such that* $u \diamond v_n \sqsubseteq v$ *and* $\|v_n\| = \frac{\varepsilon}{n}\|v\|$ *for every* $n \geqslant n_0$.

Definition 2.3 ([6])**.** Let C be a cone in the normed space $\mathbb{A}$, and $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{A}$. Then

1. $\{v_n\}_{n\in\mathbb{N}}$ is called θ-sequence (equivalently; convergent to θ in the norm topology of $\mathbb{A}$) if and only if
$$\text{for every } \epsilon > 0 \text{ there exists } n_0 \in \mathbb{N} \text{ such that } v_n \in \epsilon\mathbb{U} \text{ for all } n \geqslant n_0.$$
This type of convergence is denoted by $\|.\| - \lim_{n\to+\infty} v_n = \theta$. Equivalently;
$$\|.\| - \lim_{n\to+\infty} v_n = \theta \text{ if and only if } \lim_{n\to+\infty} \|v_n\| = 0.$$
2. $\{v_n\}_{n\in\mathbb{N}}$ is convergent to v if and only if $\{v_n - v\}_{n\in\mathbb{N}}$ is θ-sequence, $\|.\| - \lim_{n\to+\infty} v_n = v$.
3. If $\mathrm{SInt}(C) \neq \emptyset$, then $\{v_n\}_{n\in\mathbb{N}}$ is called $s - v$-sequence if and only if
$$\text{for every } \theta \sqsubseteq v \text{ there exists } n_0 \in \mathbb{N} \text{ such that } v_n \sqsubseteq v \text{ for all } n \geqslant n_0.$$
4. If $\mathrm{Int}(C) \neq \emptyset$, then $\{v_n\}_{n\in\mathbb{N}}$ is called v-sequence if and only if
$$\text{for every } \theta \ll v \text{ there exists } n_0 \in \mathbb{N} \text{ such that } v_n \ll v \text{ for all } n \geqslant n_0.$$
This definition is not usable and has no meaning if the cone C is not solid.

Lemma 2.4 ([6]). *Every θ-sequence is $s - v$-sequence, the converse is generally not true unless the underlying cone C is a normal cone.*

Examples of non-normal cones containing $s - v$-sequences which are not θ-sequences are given in [7]. The following lemma can be easily proved:

Lemma 2.5. *Let $\mathbb{A}$ be a normed algebra, C be a cone in $\mathbb{A}$, $\mathrm{SInt}(C) \neq \emptyset$, $u, \varpi, \omega \in C$ be arbitrarily vectors, $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be two sequences in $\mathbb{A}$. Then*

1. *If the spectral radius of u is strictly less than one, then $\{u^n\}_{n\in\mathbb{N}}$ is θ-sequence and consequently; it is $s - v$-sequence.*
2. *If $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are two $s - v$-sequences in $\mathbb{A}$, then $\{\varpi \diamond u_n + \omega \diamond v_n\}_{n\in\mathbb{N}}$ is $s - v$-sequence in $\mathbb{A}$.*
3. *If $\{v_n\}_{n\in\mathbb{N}}$ is $s - v$-sequence and there is $n_0 \in \mathbb{N}$ such that $u_n \preceq v_n$ for every $n \geqslant n_0$, then $\{u_n\}_{n\in\mathbb{N}}$ is $s - v$-sequence.*

We introduce the following generalized definition:

Definition 2.6. Let X be a nonempty set, $\mathbb{A}$ be a normed algebra, C be a cone in $\mathbb{A}$, λ be a given scalar $\lambda \geqslant 1$, χ be a given vector, $\chi \in C$ with $\sigma(\chi) \geqslant 1$, and $\mathbb{D}_c : X \times X \to C$ be a C valued function: Consider the following:

(1) $\mathbb{D}_c(x, y) = \theta$ iff $x = y$.
(2) $\mathbb{D}_c(x, y) = \mathbb{D}_c(y, x)$ for every $x, y \in X$.
(3) $\mathbb{D}_c(x, y) \preceq \mathbb{D}_c(x, z) + \mathbb{D}_c(z, y)$ for every $x, y, z \in X$.
(4) $\mathbb{D}_c(x, y) \preceq \lambda[\mathbb{D}_c(x, z) + \mathbb{D}_c(z, y)]$ for every $x, y, z \in X$.
(5) $\mathbb{D}_c(x, y) \preceq \chi \diamond [\mathbb{D}_c(x, z) + \mathbb{D}_c(z, y)]$ for every $x, y, z \in X$.

If $\mathbb{D}_c$ satisfies conditions (1), (2) and (3), then $\mathbb{D}_c$ is said to be a cone metric on X and the triple $(X, C, \mathbb{D}_c)$ is called cone metric space.

If $\mathbb{D}_c$ satisfies conditions (1), (2) and (4), then $\mathbb{D}_c$ is said to be a b_c-cone metric on X and $(X, C, \mathbb{D}_c, \lambda)$ is called b-cone metric space or quasi con metric space.

If $\mathbb{D}_c$ satisfies conditions (1), (2) and (5), then $\mathbb{D}_c$ is said to be parametric vector quasi cone metric on X or χ-cone metric on X and $(X, C, \mathbb{D}_C, \chi)$ is called χ-cone metric space.

Remark 2.7. The class of all χ-cone metric spaces is larger than the class of b_c-cone metric spaces and the later spaces are generalizations of cone metric spaces.

Let $(X, C, \mathbb{D}_C, \chi)$ be a parametric vector quasi cone metric space over Banach algebra $\mathbb{A}$, $x \in X$, $\nu \in \mathrm{SInt}(C)$. Denote

$$B(x,\nu) := \{y : y \in X, \mathbb{D}_c(x,y) \sqsubseteq \nu\}, \ \mathbb{B} := \{B(x,\nu) : x \in X, \nu \in \mathrm{SInt}(C)\}.$$

Lemma 2.8. *Let $(X, C, \mathbb{D}_C, \chi)$ be a χ-cone metric space over Banach algebra $\mathbb{A}$. Then the class of subsets*

$$\tau_s := \{O : O \subset X, \text{for every } x \in O \text{ there exists } \nu \in \mathrm{SInt}(C) \text{ such that } B(x,\nu) \subset O\}$$

is a topology on X and the class $\mathbb{B}$ is a base of neighborhoods.

Definition 2.9. Let $(X, C, \mathbb{D}_C, \chi)$ be a χ-cone metric space over Banach algebra $\mathbb{A}$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then

1. $\{x_n\}_{n\in\mathbb{N}}$ is said to be τ_s convergent to x iff the sequence $\{\mathbb{D}_c(x_n, x)\}_{n\in\mathbb{N}}$ is $s-\nu$-sequence in C. This means
$$\text{for every } \theta \sqsubseteq \nu \text{ there exists } n_0 \in \mathbb{N} \text{ such that } \mathbb{D}_c(x_n, x) \sqsubseteq \nu \text{ for all } n \geqslant n_0.$$
The convergence of this type is denoted by $\tau_s - \lim_{n\to+\infty} x_n = x$.
2. $\{x_n\}_{n\in\mathbb{N}}$ is said to be $s-\nu$-Cauchy sequence iff
$$\text{for every } \theta \sqsubseteq \nu \text{ there exists } n_0 \in \mathbb{N} \text{ such that } \mathbb{D}_c(x_n, x_m) \sqsubseteq \nu \text{ for all } n, m \geqslant n_0.$$
3. $(X, C, \mathbb{D}_C, \chi)$ is said to be τ_s-complete iff every $s-\nu$-Cauchy sequence in X is τ_s-convergent to some element $x \in X$.

Definition 2.10 ([8, 10, 16, 23, 45])**.** Let $(X, \leqslant)$ be a partially ordered set and T be a mapping $T : X \times X \to X$, G be a mapping $G : X \to X$, and $(x, y) \in X \times X$ be a given element. Then

1. (x, y) is a coupled fixed point of T iff $T(x, y) = x$ and $T(y, x) = y$.
2. (x, y) is a coupled coincidence point of T and G iff $T(x, y) = G(x)$ and $T(y, x) = G(y)$.
3. (x, y) is a coupled fixed point of T and G iff $T(x, y) = G(x) = x$ and $T(y, x) = G(y) = y$.
4. T is having a mixed monotone property iff T is both monotone non-decreasing in x and monotone non-increasing in y. That is;
$$T(x_1, y) \leqslant T(x_2, y) \text{ for every } y \in X \text{ whenever } x_1 \leqslant x_2 \text{ and}$$
$$T(x, y_2) \leqslant T(x, y_1) \text{ for every } x \in X \text{ whenever } y_1 \leqslant y_2.$$
5. T is having a mixed G-monotone property iff $T(x, y)$ is both monotone G-non-decreasing in x and monotone G-non-increasing in y. That is;
$$T(x_1, y) \leqslant T(x_2, y) \text{ for every } y \in X \text{ whenever } G(x_1) \leqslant G(x_2), \text{and}$$
$$T(x, y_2) \leqslant T(x, y_1) \text{ for every } x \in X \text{ whenever } G(y_1) \leqslant G(y_2).$$
6. The mappings T and G are commutative iff $G(T(x, y)) = T(G(x), G(y))$ for any $x, y \in X$.

We give the following:

Definition 2.11. A partially ordered vector cone metric space is a vector cone metric space endowed with some partial ordered relation $\leqslant$, $(X, C, \mathbb{D}_C, \chi, \leqslant)$.

Let $(X, C, \mathbb{D}_C, \chi, \leqslant)$ be a partially ordered vector cone metric space. Then X is sequentially ordered vector cone metric space iff it possesses the following two properties:

1. If $\{x_n\}_{n\in\mathrm{N}}$ is non-decreasing sequence with $\tau_s - \lim_{n\to+\infty} x_n = x$, then $x_n \leqslant x$ for all $n \in \mathrm{N}$.
2. If $\{y_n\}_{n\in\mathrm{N}}$ is non-increasing sequence with $\tau_s - \lim_{n\to+\infty} y_n = y$, then $y_n \geqslant y$ for all $n \in \mathrm{N}$.

3. Main Results

Our following constructed examples illustrate and confirm that the class of all cone metric spaces over Banach algebras is a proper subset of the class of vector quasi cone metric spaces $(X, C, \mathbb{D}_C, \chi)$.

The cone in the first example is normal with extra semi-interior points, the cone in the second example is not-normal but solid, and the cone in the third example is not-normal not-solid but with many semi-interior points.

Example 3.1. Let $\mathbb{A}$ be the unital Banach algebra of all 3×3 matrices of real numbers with the usual addition, scalar product and product operations $\diamond$ of matrices, $\mathbb{A} := \left\{[u_{ij}]_{1 \leqslant i,j \leqslant 3} : u_{ij} \in \mathbb{R}\right\}$, $\|[u_{ij}]_{1 \leqslant i,j \leqslant 3}\| := \sum_{1 \leqslant i,j \leqslant 3} |u_{ij}|$. The zero element θ is the zero matrix and the unit element e is the identity matrix, the cone of $\mathbb{A}$ is $C(\mathbb{A}) := \{[u_{ij}]_{1 \leqslant i,j \leqslant 3} : u_{ij} \in \mathbb{R}^+\}$, every $u = [u_{ij}]_{1 \leqslant i,j \leqslant 3} \in C$ with $\delta := \min\{u_{ij} : 1 \leqslant i, j \leqslant 3\} > 0$ is semi-interior point of $C(\mathbb{A})$ because $u - \delta\ \mathbb{U}_+(\mathbb{A}) \subset C(\mathbb{A})$. For $\chi = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have $\sigma(\chi) = max\{2, 3\} = 3 > 1$. Let $X = \{a, b, d\}$ and define $\mathbb{D}_c : X \times X \to C(\mathbb{A})$ as: $\mathbb{D}_c(a, a) = \mathbb{D}_c(b, b) = \mathbb{D}_c(d, d) = \theta$,

$$\mathbb{D}_c(a, b) = \mathbb{D}_c(b, a) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathbb{D}_c(b, d) = \mathbb{D}_c(d, b) = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and}$$

$$\mathbb{D}_c(a, d) = \mathbb{D}_c(d, a) = \begin{pmatrix} 3 & 7 & 11 \\ 10 & 16 & 17 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $(X, C, \mathbb{D}_c)$ is not cone metric space because

$$\mathbb{D}_c(d, a) = \begin{pmatrix} 3 & 7 & 11 \\ 10 & 16 & 17 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \mathbb{D}_c(a, b) + \mathbb{D}_c(b, d) = \begin{pmatrix} 2 & 6 & 10 \\ 6 & 10 & 12 \\ 0 & 0 & 0 \end{pmatrix},$$

this means, $\mathbb{D}_c(a, b) + \mathbb{D}_c(b, d) \prec \mathbb{D}_c(d, a)$. While $(X, C, \mathbb{D}_c, \chi)$ is χ-cone metric space over the unital Banach algebra $\mathbb{A}$, because $\mathbb{D}_c(a, b) \preceq \chi \diamond [\mathbb{D}_c(a, d) + \mathbb{D}_c(d, b)]$, $\mathbb{D}_c(a, d) \preceq \chi \diamond [\mathbb{D}_c(a, b) + \mathbb{D}_c(b, d)]$, and $\mathbb{D}_c(b, d) \preceq \chi \diamond [\mathbb{D}_c(b, a) + \mathbb{D}_c(a, d)]$.

Example 3.2. Let $\mathbb{A}$ be the commutative unital Banach algebra of all differentiable real-valued functions on the closed interval $[0, 1]$, $\mathbb{A} := C^1([0, 1])$ with usual addition, scalar multiplication and point-wise multiplication operations of mappings $(u \diamond v)(t) := u(t)v(t)$,

$$\mathbb{A} := \left\{u;\ u : [0, 1] \to \mathbb{R}; u \text{ differentiable}\right\}, \ \ \|u\| := \|u\|_\infty + \|\frac{du}{dt}\|_\infty.$$

The zero element θ is the zero mapping and the identity element e with respect to multiplication is $e(t) = 1$ for every $t \in [0, 1]$, and u is invertible if and only if $u(t) \neq 0$ for every $t \in [0, 1]$, $u^{-1}(t) = \frac{1}{u(t)}$, the spectrum of any element u is given by $Spec(u) = Range(u)$, and $\sigma(u) = \sup\{\ |\lambda| : \lambda \in Range(u)\}$. The cone $C := \{u : u \in \mathbb{A}, u(t) \geqslant 0 \text{ for all } t \in [0, 1]\}$ is non-normal and solid cone. Let $\chi \in C$ be the exponential function $\chi(t) = \exp^t$ for $t \in [0, 1]$, we have $\chi \in C$ and $\sigma(\chi) \simeq 2.7 > 1$. Let $X = \{a, b, d\}$ and define $\mathbb{D}_c : X \times X \to C$ by

$$\mathbb{D}_c(a, a) = \mathbb{D}_c(b, b) = \mathbb{D}_c(d, d) = \theta,$$
$$\mathbb{D}_c(a, b) = \mathbb{D}_c(b, a) = 2\exp^t, \ \ \mathbb{D}_c(b, d) = \mathbb{D}_c(d, b) = 4\exp^{2t},$$

and

$$\mathbb{D}_c(a, d) = \mathbb{D}_c(d, a) = \exp^{3t}.$$

Then $(X, C, \mathbb{D}_c)$ is not cone metric on X because $\Big(\mathbb{D}_c(b,a)+\mathbb{D}_c(a,d)\Big)(0) = 2+1 = 3$ and $\mathbb{D}_c(b,d)(0) = 4$, this means; $\Big(\mathbb{D}_c(b,a)+\mathbb{D}_c(a,d)\Big) - \mathbb{D}_c(b,d) \notin C$. While $(X, C, \mathbb{D}_c, \chi)$ is χ-cone metric on X over the unital Banach algebra $\mathbb{A}$.

Example 3.3. Let $\mathbb{A}$ be the unital Banach algebra of all $\mathbb{R}^2$ valued sequences $\{(\vartheta_1^n, \vartheta_2^n)\}_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \|(\vartheta_1^n, \vartheta_2^n)\|_n < +\infty$, where $\|(\vartheta_1^n, \vartheta_2^n)\|_n$ is given in either of the following:

$$\|(\vartheta_1^n, \vartheta_2^n)\|_n = \begin{cases} |\vartheta_1^n| + |\vartheta_2^n|, & \text{if } \vartheta_1^n \times \vartheta_2^n \geqslant 0; \\ \max\{|\vartheta_1^n|, |\vartheta_2^n|\} - (\frac{n-1}{n}) \min\{|\vartheta_1^n|, |\vartheta_2^n|\}, & \text{if } \vartheta_1^n \times \vartheta_2^n < 0. \end{cases} \tag{3.1}$$

or

$$\|(\vartheta_1^n, \vartheta_2^n)\|_n = \begin{cases} \sqrt{|\vartheta_1^n|^2 + |\vartheta_2^n|^2}, & \text{if } \vartheta_1^n \times \vartheta_2^n \geqslant 0; \\ \max\{|\vartheta_1^n|, |\vartheta_2^n|\} - (\frac{n-1}{n}) \min\{|\vartheta_1^n|, |\vartheta_2^n|\}, & \text{if } \vartheta_1^n \times \vartheta_2^n < 0. \end{cases} \tag{3.2}$$

Endow $\mathbb{A}$ with the usual addition, usual scalar multiplication and the multiplication $\diamond$ given by

$$u \diamond v := \{(\vartheta_1^n\, v_1^n, \vartheta_1^n v_2^n + \vartheta_2^n\, v_1^n)\}_{n\in\mathbb{N}} \ \text{ for every } u = \{(\vartheta_1^n, \vartheta_2^n)\}_{n\in\mathbb{N}},\ v = \{(v_1^n, v_2^n)\}_{n\in\mathbb{N}} \in \mathbb{A}.$$

The element $\theta = \{(0,0),\ (0,0),\ (0,0),\ \dots\}$ is the additive identity and $e = \{(1,0),\ (1,0),\ (1,0),\ \dots\}$ is the multiplicative identity, and $u = \{(\vartheta_1^n, \vartheta_2^n)\}_{n\in\mathbb{N}}$ is invertible if and only if $\vartheta_1^n \neq 0$ for every $n \in \mathbb{N}$, $u^{-1} = \{(\frac{1}{\vartheta_1^n}, -\frac{\vartheta_2^n}{(\vartheta_1^n)^2})\}_{n\in\mathbb{N}}$, the spectrum is given by $\mathrm{Spec}(u) = \{\vartheta_1^n\}_{n\in\mathbb{N}}$, and $\sigma(u) = \sup\{\ |\vartheta_1^n| :\ n \in \mathbb{N}\}$. The cone $C := \{u :\ u \in \mathbb{A}, \vartheta_i^n \geqslant 0 \text{ for all } n \in \mathbb{N},\ i = 1,2\}$ is normal and non-solid cone. Every element $u \in C$ with the two conditions $\lim_{n\to+\infty} \max\{\vartheta_1^n, \vartheta_2^n\} = \lim_{n\to+\infty} \min\{\vartheta_1^n, \vartheta_2^n\}$ and

$$\delta := \min\{\inf_{n\in\mathbb{N}} \vartheta_1^n,\ \inf_{n\in\mathbb{N}} \vartheta_2^n\} > 0$$

is semi-interior not interior point of C. Let $\chi = \{(2,1),\ (2,1),\ (2,1),\ \dots\} \in C$, we have $\sigma(\chi) = 2 > 1$. Now; let $X = \{a, b, d\}$ and define $\mathbb{D}_c : X \times X \to C$ by

$$\mathbb{D}_c(a,a) = \mathbb{D}_c(b,b) = \mathbb{D}_c(d,d) = \theta,$$

$$\mathbb{D}_c(a,b) = \mathbb{D}_c(b,a) = e,\ \ \mathbb{D}_c(b,d) = \mathbb{D}_c(d,b) = \{(3,2),\ (3,2),\ (3,2),\ \dots\},$$

and

$$\mathbb{D}_c(a,d) = \mathbb{D}_c(d,a) = u = \{(\vartheta_1^n, \vartheta_2^n)\}_{n\in\mathbb{N}},$$

where either ($\frac{1}{2} \leqslant \vartheta_1^n \leqslant 8$ and $2 < \vartheta_2^n \leqslant 8$) or ($4 < \vartheta_1^n \leqslant 8$ and $\frac{1}{2} \leqslant \vartheta_2^n \leqslant 8$). Then $(X, C, \mathbb{D}_c)$ is not cone metric on X while $(X, C, \mathbb{D}_c, \chi)$ is χ-cone metric on X over the unital Banach algebra $\mathbb{A}$.

We have the following main coupled coincidence point theorem:

Theorem 3.4. *Let $(X, C, \mathbb{D}_c, \leqslant)$ be a sequentially ordered χ-cone metric space, C be a satisfactory cone, T and G be two mappings, $T : X \times X \to X$ and $G : X \to X$ respectively. Assume*

1. $F(X \times X) \subseteq G(X)$;
2. $G(X)$ *is τ_s-complete subspace of* X;
3. F *has the mixed* G*-monotone property;*
4. *there exist $\{\vartheta_i\}_{i=1}^6 \subset C$ commuting with χ, $\sigma(\vartheta_4 + \vartheta_6 \diamond \chi) < 1$ and $\sigma([e - (\vartheta_4 + \vartheta_6 \diamond \chi)]^{-1} \diamond [\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_6 \diamond \chi]) < \frac{1}{\sigma(\chi)}$ such that for all $x, y, z, w \in X$ satisfying $G(x) \leqslant G(z)$ and $G(w) \leqslant G(y)$ the following holds*

$$\begin{aligned} \mathbb{D}_c(F(x,y), F(z,w)) \preceq\ & \vartheta_1 \diamond \mathbb{D}_c(G(x), G(z)) + \vartheta_2 \diamond \mathbb{D}_c(F(x,y), G(x)) \\ & + \vartheta_3 \diamond \mathbb{D}_c(G(y), G(w)) + \vartheta_4 \diamond \mathbb{D}_c(F(z,w), G(z)) \\ & + \vartheta_5 \diamond \mathbb{D}_c(F(x,y), G(z)) + \vartheta_6 \diamond \mathbb{D}_c(F(z,w), G(x)); \end{aligned} \tag{3.3}$$

5. *there exists* $(x_0, y_0) \in X \times X$ *such that*

$$G(x_0) \leqslant T(x_0, y_0) \ \textit{and} \ T(y_0, x_0) \leqslant G(y_0). \tag{3.4}$$

Then F *and* G *have a coupled coincidence point.*

Proof. Let $(x_0, y_0) \in X \times X$ be a point satisfy the inequalities (3.4) of the fifth condition and use the first condition with the third condition, we construct sequences $\{G(x_n)\}_{n\in\mathbb{N}}$ and $\{G(y_n)\}_{n\in\mathbb{N}}$ in X such that

$$G(x_n) = F(x_{n-1}, y_{n-1}) \leqslant G(x_{n+1}) = F(x_n, y_n) \text{ for all } n \in \mathbb{N}$$

and

$$G(y_{n+1}) = F(y_n, x_n) \leqslant G(y_n) = F(y_{n-1}, x_{n-1}) \text{ for all } n \in \mathbb{N}$$

Let $n \in \mathbb{N}$. Then by inequality (3.3), we have

$$\begin{aligned}
\mathbb{D}_c(G(x_n), G(x_{n+1})) &= \mathbb{D}_c(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\
&\preceq \vartheta_1 \diamond \mathbb{D}_c(G(x_{n-1}), G(x_n)) + \vartheta_2 \diamond \mathbb{D}_c(F(x_{n-1}, y_{n-1}), G(x_{n-1})) \\
&\quad + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y_n)) + \vartheta_4 \diamond \mathbb{D}_c(F(x_n, y_n), G(x_n)) \\
&\quad + \vartheta_5 \diamond \mathbb{D}_c(F(x_{n-1}, y_{n-1}), G(x_n)) + \vartheta_6 \diamond \mathbb{D}_c(F(x_n, y_n), G(x_{n-1})) \\
&\preceq \vartheta_1 \diamond \mathbb{D}_c(G(x_{n-1}), G(x_n)) + \vartheta_2 \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) \\
&\quad + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y_n)) + \vartheta_4 \diamond \mathbb{D}_c(G(x_{n+1}), G(x_n)) \\
&\quad + \vartheta_5 \diamond \mathbb{D}_c(G(x_n), G(x_n)) + \vartheta_6 \diamond \mathbb{D}_c(G(x_{n+1}), G(x_{n-1})) \\
&\preceq (\vartheta_1 + \vartheta_2) \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) \\
&\quad + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y_n)) + \vartheta_4 \diamond \mathbb{D}_c(G(x_{n+1}), G(x_n)) \\
&\quad + \vartheta_6 \diamond [\chi \diamond \{\mathbb{D}_c(G(x_{n+1}), G(x_n)) + \mathbb{D}_c(G(x_n), G(x_{n-1}))\}] \\
&\preceq (\vartheta_1 + \vartheta_2 + \vartheta_6 \diamond \chi) \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) \\
&\quad + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y_n)) + (\vartheta_4 + \vartheta_6 \diamond \chi) \diamond \mathbb{D}_c(G(x_{n+1}), G(x_n)).
\end{aligned}$$

This implies

$$\begin{aligned}
[e - (\vartheta_4 + \vartheta_6 \diamond \chi)] \diamond \mathbb{D}_c(G(x_n), G(x_{n+1})) \preceq & (\vartheta_1 + \vartheta_2 + \vartheta_6 \diamond \chi) \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) \\
& + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y_n)).
\end{aligned} \tag{3.5}$$

Similarly; we have

$$\begin{aligned}
[e - (\vartheta_4 + \vartheta_6 \diamond \chi)] \diamond \mathbb{D}_c(G(y_n), G(y_{n+1})) \preceq & (\vartheta_1 + \vartheta_2 + \vartheta_6 \diamond \chi) \diamond \mathbb{D}_c(G(y_n), G(y_{n-1})) \\
& + \vartheta_3 \diamond \mathbb{D}_c(G(x_{n-1}), G(x_n)).
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) gives

$$\begin{aligned}
[e - (\vartheta_4 + \vartheta_6 \diamond \chi)] \diamond [\mathbb{D}_c(G(x_{n+1}), G(x_n)) + \mathbb{D}_c(G(y_{n+1}), G(y_n))] \preceq & (\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_6 \diamond \chi) \\
& \diamond [\mathbb{D}_c(G(x_n), G(x_{n-1})) \\
& + \mathbb{D}_c(G(y_n), G(y_{n-1}))].
\end{aligned} \tag{3.7}$$

Since $(\vartheta_4 + \vartheta_6 \diamond \chi) \in C$ and $\sigma(\vartheta_4 + \vartheta_6 \diamond \chi) < 1$, then $[e - (\vartheta_4 + \vartheta_6 \diamond \chi)]$ is invertible element and its inverse is given by

$$[e - (\vartheta_4 + \vartheta_6 \diamond \chi)]^{-1} = \lim_{j \to +\infty} \sum_{i=0}^{j} (\vartheta_4 + \vartheta_6 \diamond \chi)^i = \sum_{i=0}^{+\infty} (\vartheta_4 + \vartheta_6 \diamond \chi)^i \in C.$$

Denote $D_{n-1} := \mathbb{D}_c(G(x_n), G(x_{n-1})) + \mathbb{D}_c(G(y_n), G(y_{n-1}))$ for every $n \in \mathbb{N}$ and multiply the two sides of (3.7) by $[e - (\vartheta_4 + \vartheta_6 \diamond \chi)]^{-1}$, we see that

$$D_n \preceq \Big[[e - (\vartheta_4 + \vartheta_6 \diamond \chi)]^{-1} \diamond (\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_6 \diamond \chi)\Big] \diamond D_{n-1}. \tag{3.8}$$

Denote $u := [e - (\vartheta_4 + \vartheta_6 \diamond \chi)]^{-1} \diamond [\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_6 \diamond \chi]$ and substitute in (3.8). Then we get

$$D_n \preceq u \diamond D_{n-1} \text{ for all } n \in \mathbb{N}. \tag{3.9}$$

Continuing the last processes gives

$$D_n \preceq u \diamond D_{n-1} \preceq u^2 \diamond D_{n-2} \preceq \cdots \preceq u^n \diamond D_0 \text{ for all } n \in \mathbb{N}. \tag{3.10}$$

Use the given condition $\sigma(u) = \sigma([e - (\vartheta_4 + \vartheta_6 \diamond \chi)]^{-1} \diamond [\vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_6 \diamond \chi]) < \frac{1}{\sigma(\chi)} \leqslant 1$, then (1) of lemma 2.5 insures that $\{u^n\}_{n\in\mathbb{N}}$ is $s-\nu$-sequence and (2) of lemma 2.5 proves that $\{u^n \diamond D_0\}_{n\in\mathbb{N}}$ is $s-\nu$-sequence, using (3) of lemma 2.5 with (3.9) yields that $\{D_n\}_{n\in\mathbb{N}}$ is $s-\nu$-sequence, since $\mathbb{D}_c(G(x_n), G(x_{n-1})) \preceq D_{n-1}$ and $\mathbb{D}_c(G(y_n), G(y_{n-1})) \preceq D_{n-1}$, then (3.10) gives

$$\begin{aligned}
&\mathbb{D}_c(G(x_n), G(x_{n-1})) \preceq u^{n-1} \diamond D_0 \text{ for all } n \in \mathbb{N} \text{ and}\\
&\mathbb{D}_c(G(y_n), G(y_{n-1})) \preceq u^{n-1} \diamond D_0 \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

Using 3 of lemma 2.5 insures that the sequences $\{\mathbb{D}_c(G(x_n), G(x_{n-1}))\}_{n\in\mathbb{N}}$ and $\{\mathbb{D}_c(G(y_n), G(y_{n-1}))\}_{n\in\mathbb{N}}$ are two $s-\nu$-sequences. Now, we show that $\{G(x_n)\}_{n\in\mathbb{N}}$ and $\{G(y_n)\}_{n\in\mathbb{N}}$ are two $s-\nu$-Cauchy sequences in $G(X)$, for this purpose, let $n, m \in \mathbb{N}$ such that $n \leqslant m$. Then

$$\begin{aligned}
\mathbb{D}_c(G(x_n), G(x_m)) \preceq& \chi \diamond [\mathbb{D}_c(G(x_n), G(x_{n+1})) + \mathbb{D}_c(G(x_{n+1}), G(x_m))]\\
\preceq& \chi \diamond \mathbb{D}_c(G(x_n), G(x_{n+1})) + \chi^2 \diamond [\mathbb{D}_c(G(x_{n+1}), G(x_{n+2}))\\
&+ \mathbb{D}_c(G(x_{n+2}), G(x_m))]\\
\preceq& \chi \diamond \mathbb{D}_c(G(x_n), G(x_{n+1})) + \chi^2 \diamond \mathbb{D}_c(G(x_{n+1}), G(x_{n+2}))\\
&+ \chi^3 \diamond [\mathbb{D}_c(G(x_{n+2}), G(x_{n+3})) + \mathbb{D}_c(G(x_{n+3}), G(x_m))]\\
&\vdots\\
\preceq& \chi \diamond \mathbb{D}_c(G(x_n), G(x_{n+1})) + \chi^2 \diamond \mathbb{D}_c(G(x_{n+1}), G(x_{n+2})) +\\
&\cdots + \chi^{(m-n)} \diamond \mathbb{D}_c(G(x_{n+(m-n-1)}), G(x_m))\\
\preceq& \sum_{j=1}^{(m-n)} \chi^j \diamond \mathbb{D}_c(G(x_{n+j-1}), G(x_{n+j})).
\end{aligned}$$

Hence; using the commutativity condition gives

$$\begin{aligned}
\mathbb{D}_c(G(x_n), G(x_m)) \preceq& \sum_{j=1}^{(m-n)} \chi^j \diamond u^{n+j-1} = \chi \diamond [\sum_{j=0}^{(m-n-1)} \chi^j \diamond u^j] \diamond u^n\\
=& \chi \diamond [\sum_{j=0}^{(m-n-1)} (\chi \diamond u)^j] \diamond u^n
\end{aligned} \tag{3.11}$$

Since $\sigma(\chi \diamond u) \leqslant \sigma(\chi)\sigma(u) < 1$, then $e - (\chi \diamond u)$ is invertible element and $[e - (\chi \diamond u)]^{-1} = \sum_{j=0}^{+\infty} (\chi \diamond u)^j = \lim_{m\to+\infty} \sum_{j=0}^{m} (\chi \diamond u)^j \in C$. Accordingly; the closed-ness of C together with inequalities (3.11) imply the following:

$$\mathbb{D}_c(G(x_n), G(x_m)) \preceq \Big[\chi \diamond [e - (\chi \diamond u)]^{-1}\Big] \diamond u^n. \tag{3.12}$$

Using (2) of lemma 2.5 proves that $\{\left[\chi \diamond [e-(\chi \diamond u)]^{-1}\right] \diamond u^n\}_{n\in\mathbb{N}}$ is $s-\nu$-sequence in C. Hence; let $\theta \sqsubseteq \nu$. Then there is $n_0 \in \mathbb{N}$ such that $\left[\chi \diamond [e-(\chi \diamond u)]^{-1}\right] \diamond u^n \sqsubseteq \nu$ for every $n \geqslant n_0$. Using (3) of lemma 2.1 gives

$$\mathbb{D}_c(G(x_n), G(x_m)) \sqsubseteq \nu \text{ for all } n \geqslant n_0. \tag{3.13}$$

This completes the proof that $\{G(x_n)\}_{n\in\mathbb{N}}$ is $s-\nu$-Cauchy sequence in $G(X)$. Similarly; $\{G(y_n)\}_{n\in\mathbb{N}}$ is $s-\nu$-Cauchy sequence in $G(X)$. Since $G(X)$ is τ_s-complete, then there are two points z and w in $G(X)$ such that

$$\tau_s - \lim_{n\to+\infty} G(x_n) = z \text{ and } \tau_s - \lim_{n\to+\infty} G(y_n) = w.$$

Let x and y in X be such that $G(x) = z$ and $G(y) = w$. Then

$$\tau_s - \lim_{n\to+\infty} G(x_n) = G(x) \text{ and } \tau_s - \lim_{n\to+\infty} G(y_n) = G(y).$$

Equivalently; the sequences $\{\mathbb{D}_c(G(x_n), G(x))\}_{n\in\mathbb{N}}$ and $\{\mathbb{D}_c(G(y_n), G(y))\}_{n\in\mathbb{N}}$ are $s-\nu$ sequences in C. Let $\theta \sqsubseteq \nu$. Then there is $n_0 \in \mathbb{N}$ such that

$$\mathbb{D}_c(G(x_n), G(x)) \sqsubseteq \nu \text{ and } \mathbb{D}_c(G(y_n), G(y)) \sqsubseteq \nu \text{ for all } n \geqslant n_0.$$

On the other hand, using the sequentially ordered property of X gives

$$G(x_n) \leqslant G(x) \text{ and }, G(y) \leqslant G(y_n) \text{ for all } n \in \mathbb{N}.$$

Accordingly; we have

$$\begin{aligned}
\mathbb{D}_c(G(x_n), F(x,y)) &= \mathbb{D}_c(F(x_{n-1}, y_{n-1}), F(x,y)) \\
&\preceq \vartheta_1 \diamond \mathbb{D}_c(G(x_{n-1}), G(x)) + \vartheta_2 \diamond \mathbb{D}_c(F(x_{n-1}, y_{n-1}), G(x_{n-1})) \\
&\quad + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y)) + \vartheta_4 \diamond \mathbb{D}_c(F(x,y), G(x)) \\
&\quad + \vartheta_5 \diamond \mathbb{D}_c(F(x_{n-1}, y_{n-1}), G(x)) + \vartheta_6 \diamond \mathbb{D}_c(F(x,y), G(x_{n-1})) \\
&\preceq \vartheta_1 \diamond \mathbb{D}_c(G(x_{n-1}), G(x)) + \vartheta_2 \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) \\
&\quad + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y)) + \vartheta_4 \diamond \mathbb{D}_c(F(x,y), G(x)) \\
&\quad + \vartheta_5 \diamond \mathbb{D}_c(G(x_n), G(x)) + \vartheta_6 \diamond \mathbb{D}_c(F(x,y), G(x_{n-1})) \\
&\preceq \vartheta_1 \diamond \mathbb{D}_c(G(x_{n-1}), G(x)) + \vartheta_2 \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y)) \\
&\quad + \vartheta_4 \diamond \left[\chi \diamond \left(\mathbb{D}_c(F(x,y), G(x_n)) + \mathbb{D}_c(G(x_n), G(x))\right)\right] \\
&\quad + \vartheta_5 \diamond \mathbb{D}_c(G(x_n), G(x)) + \vartheta_6 \diamond \left[\chi \diamond \left(\mathbb{D}_c(F(x,y), G(x_n)) + \mathbb{D}_c(G(x_n), G(x_{n-1}))\right)\right] \\
&\preceq \vartheta_1 \diamond \mathbb{D}_c(G(x_{n-1}), G(x)) + [\vartheta_2 + \vartheta_6 \diamond \chi] \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) \\
&\quad + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y)) + [(\vartheta_4 + \vartheta_6) \diamond \chi] \diamond \mathbb{D}_c(F(x,y), G(x_n)) \\
&\quad + [\vartheta_4 \diamond \chi + \vartheta_5] \diamond \mathbb{D}_c(G(x_n), G(x)).
\end{aligned}$$

This means,

$$\begin{aligned}
[e - (\vartheta_4 + \vartheta_6) \diamond \chi] \diamond \mathbb{D}_c(F(x,y), G(x_n)) \preceq\ & \vartheta_1 \diamond \mathbb{D}_c(G(x_{n-1}), G(x)) \\
& + [\vartheta_2 + \vartheta_6 \diamond \chi] \diamond \mathbb{D}_c(G(x_n), G(x_{n-1})) \\
& + \vartheta_3 \diamond \mathbb{D}_c(G(y_{n-1}), G(y)) \\
& + [\vartheta_4 \diamond \chi + \vartheta_5] \diamond \mathbb{D}_c(G(x_n), G(x)).
\end{aligned} \tag{3.14}$$

Since all of the sequences $\{\mathbb{D}_c(G(x_n), G(x))\}_{n\in\mathbb{N}}$, $\{\mathbb{D}_c(G(x_n), G(x_{n-1}))\}_{n\in\mathbb{N}}$, and $\{\mathbb{D}_c(G(y_{n-1}), G(y))\}_{n\in\mathbb{N}}$ are $s-\nu$-sequences in C, then the combination of the left hand side of (3.14) according to (2) of lemma 2.5

is $s-\nu$-sequence in C. Using (3) of lemma 2.5 shows that $\{\mathbb{D}_c(F(x,y),G(x_n)))\}_{n\in\mathbb{N}}$ is also $s-\nu$-sequence in C. Let $\theta \sqsubseteq \nu$ and $m \in \mathbb{N}$ be any natural number. Then $\theta \sqsubseteq \frac{\nu}{2m}$, hence there is $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ we get

$$\mathbb{D}_c(F(x,y),G(x)) \preceq \mathbb{D}_c(F(x,y),G(x_n)) + \mathbb{D}_c(G(x_n),G(x)) \sqsubseteq \frac{\nu}{2m} + \frac{\nu}{2m} = \frac{\nu}{m}.$$

This in particular means that $\frac{\nu}{m} - \mathbb{D}_c(F(x,y),G(x)) \in C$. Taking the limit as $m \to +\infty$ implies $-\mathbb{D}_c(F(x,y),G(x)) \in C$ and therefore $\mathbb{D}_c(F(x,y),G(x)) = \theta$, hence $F(x,y) = G(x)$. Similarly, we prove that $F(y,x) = G(y)$ and show that (x,y) is a coupled coincidence point of F and G. □

We have the following corollaries:

Corollary 3.5. *Let X be a sequentially ordered complete 1-cone metric space $(X, C = \mathbb{R}^+, \mathbb{D}_c(x,y) = \|x-y\|)$. Then use Theorem 3.4 with the numbers $\{\frac{k}{2}, \frac{k}{2}, 0, 0, 0, 0\}$, $0 < k < 1$ instead of the vectors $\{\vartheta_i\}_{i=1}^6$ and with the identity operator on X instead of G we get the main coupled fixed point theorem of Baskar-Lackshmikantham [16].*

Corollary 3.6. *Let $(X, C, \mathbb{D}_c, \leqslant)$ be a sequentially ordered complete $\chi = 1$-cone metric space, C be solid cone. Then use Theorem 3.4 with scalars $\{a_1, 0, a_3, 0, 0, 0\}$, $0 < a_1 + a_3 < 1$ instead of the vectors $\{\vartheta_i\}_{i=1}^6$ we get the Shatanawi coupled fixed point theorem (3.1) [45].*

Removing the conditions $G(x) \leqslant G(z)$ and $G(w) \leqslant G(y)$ in the main inequalities which give weaker results, we have the following:

Corollary 3.7. *Let $(X, C, \mathbb{D}_c, \leqslant)$ be a sequentially ordered complete $\chi = 1$-cone metric space, C be solid cone. Then use Theorem 3.4 with positive scalars $\{a_i\}_{i=1}^6$, $\sum_{i=1}^6 a_i < 1$ instead of the vectors $\{\vartheta_i\}_{i=1}^6$ we get the Agarwal et al coupled fixed point theorem [8].*

3.1. Conclusion

In this paper, needless of the normality and solidness assumptions of the underlying cone and in the setting of vector quasi (χ-) cone metric space over Banach algebra with the cone having semi-interior points, we generalized some previously proved theorems on the existence of coupled coincidence points and given in the cited reference of this paper. The metric parameter is replaced by a vector χ and the contraction constants are replaced by vectors in such a cone. Theorem 3.4 generalized many previous results, specifically; its corollaries, Corollary 3.5 is the Baskar-Lackshmikantham coupled fixed point [16], Corollary 3.6 is the Shatanawi coupled fixed point theorem [45], Corollary 3.7 is the Agarwal et al coupled fixed point theorem [8], their underlying cone C is assumed to be solid cone for the generated topology on X on the other hand removing the condition $G(x) \leqslant G(z)$ and $G(w) \leqslant G(y)$ gives weaker result. Moreover, the results of Sabetghadam et al [43], Sahar [10], and Alotaibi et al [11] can be similarly proved as corollaries of Theorem 3.4.

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