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On determinants, inverses, norms, and spread of skew circulant matrices involving the product of Pell and Pell-Lucas numbers

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Abstract

In this paper, we discuss skew circulant matrices involving the product of Pell and Pell-Lucas numbers. The invertibility of the skew circulant matrices is investigated, while the fundamental theorems on the determinants and inverses of such matrices are derived by simple construction matrices. Specifically, the determinant and inverse of $n \times n$ skew circulant matrices can be expressed by the $(n-1)$th, nth, $(n+1)$th, $(n+2)$th product of Pell and Pell-Lucas numbers. Some norms and bounds for spread of these matrices are given, respectively. In addition, we generalized these results to skew left circulant matrix involving the product of Pell and Pell-Lucas numbers. Finally, several numerical examples are illustrated to show the effectiveness of our theoretical results.

Keywords: Determinant, inverse, norm, spread, Pell number, skew circulant matrix.

2020 MSC: 15A09, 15A15, 15B05, 11B39, 65F40.

1. Introduction

It is worth mentioning that circulant and skew circulant matrices play a crucial role in chemistry [15, 18] and PDEs [5, 22, 26, 27, 33]. Circulant matrix not only was originally designed for fractional Brownian motion but also the sampling algorithm for the random fluctuating force [19]. Circulant matrix also had an application on the quantum optics effects in quasi-one-dimensional and two-dimensional carbon materials [35]. Besides, Houteghem et al. [7] put forward circulant matrix as a model for the Hessian of ring molecular structures and visualized of the physical breathing modes when projecting atomic velocities on eigenvectors of the circulant matrix. Furthermore, Hadamard circulant matrices have taken up in Hadamard transform spectroscopy, the construction of optimal chemical designs as well as block designs [1]. Especially, Hadamard transform spectroscopy is derived from Hadamard matrices, which employs spectroscopic multiplexing techniques and is a good analysis of complex spectra.

On the other hand, the importance of determinants, inverses, norms and spread in special matrix analysis, several authors [4, 6, 10, 11, 13, 17, 20, 23, 28–32] have done some research on these special

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matrices. Recently, Jiang and Hong [12] studied the explicit form of determinants and inverses of Tribonacci r-circulant type matrices, while Zheng and Shon [36] gave the exact determinants and inverses of generalized Lucas skew circulant type matrices. Besides, Shen et al. [24] did some work on the explicit determinants and inverses of circulant matrices with Fibonacci and Lucas numbers. What's more, AhmetÍpek [8] investigated an improved estimation for spectral norms of circulant matrices with classical Fibonacci and Lucas number entries. Bose et al. [3] discussed the convergence in probability and in distribution of the spectral norm of scaled Toeplitz, circulant, skew circulant, symmetric circulant, and a class of k-circulant matrices.

The well-known Pell and Pell-Lucas numbers form a unifying thread intertwining analysis, geometry, trigonometry, number theory, graph theory, linear algebra, combinatorics and physics [2, 16]. For example, as shown in [25], number theoretical transforms have contributed to reducing the number of multiplications for discrete Fourier transforms computations. Number theoretical transforms related to the famous numbers, which are often used to deal with problems of digital filtering and convolution of discrete signals in [21].

Motivated by the universal existence and extremely importance of Pell and Pell-Lucas numbers and skew circulant matrices, we present, therefore, some results on them. More specifically, we study the determinants, inverses, multiple norms, lower and upper bounds for the spread of such matrices, which are going to have potential to be useful for realistic applications. More work continuing the present paper is forthcoming.

Now we introduce the main objects we study in the paper. Pell numbers P_n and Pell-Lucas numbers Q_n are often defined recursively [16]:

$$\begin{aligned} P_{n+1} &= 2P_n + P_{n-1}, \quad \text{where} \quad P_1 = 1, P_2 = 2, \\ Q_{n+1} &= 2Q_n + Q_{n-1}, \quad \text{where} \quad Q_1 = 1, Q_2 = 3. \end{aligned}$$

Accordingly, the product $\mathbb{P}_n$ of Pell and Pell-Lucas numbers satisfies the following recurrence relations:

$$\mathbb{P}_{n+1} = 6\mathbb{P}_n - \mathbb{P}_{n-1}, \quad \text{where} \quad \mathbb{P}_1 = 1, \mathbb{P}_2 = 6. \tag{1.1}$$

The Binet formula of the sequence $\{\mathbb{P}_n\}$ is given by

$$\mathbb{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are the roots of the characteristic equation $x^2 - 6x + 1 = 0$. The first few values of the sequences are given in the following table:

n	1	2	3	4	5	6	7
$\mathbb{P}_n$	1	6	35	204	1189	6930	40391

We define a skew circulant matrix involving the product of Pell and Pell-Lucas numbers as follows:

$$\mathrm{SCirc}(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n) = \begin{pmatrix} \mathbb{P}_1 & \mathbb{P}_2 & \ldots & \mathbb{P}_n \\ -\mathbb{P}_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{P}_2 \\ -\mathbb{P}_2 & \ldots & -\mathbb{P}_n & \mathbb{P}_1 \end{pmatrix}_{n\times n}.$$

We also define a skew left circulant matrix involving the product of Pell and Pell-Lucas numbers as follows:

$$\mathrm{SLCirc}(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n) = \begin{pmatrix} \mathbb{P}_1 & \ldots & \mathbb{P}_{n-1} & \mathbb{P}_n \\ \vdots & ⋰ & ⋰ & -\mathbb{P}_1 \\ \mathbb{P}_{n-1} & ⋰ & ⋰ & \vdots \\ \mathbb{P}_n & -\mathbb{P}_1 & \ldots & -\mathbb{P}_{n-1} \end{pmatrix}_{n\times n}.$$

2. Preliminaries

In this section, we list some lemmas which will be used in the latter proofs of main theorems.

Lemma 2.1. *Let $\{\mathbb{P}_n\}$ be the product of Pell and Pell-Lucas numbers. Then*

$$\sum_{i=1}^{n}\mathbb{P}_i=\frac{(\mathbb{P}_{n+1}-\mathbb{P}_n-1)}{4}, \tag{2.1}$$

$$\sum_{i=1}^{n}i\mathbb{P}_i=\frac{n\mathbb{P}_{n+1}-(n+1)\mathbb{P}_n}{4}, \tag{2.2}$$

$$\sum_{i=1}^{n}\mathbb{P}_i^2=\frac{\mathbb{P}_{n+1}^2-\mathbb{P}_n^2-2n-1}{32}, \tag{2.3}$$

$$\sum_{i=1}^{n}\mathbb{P}_i a^i=\frac{\mathbb{P}_n a^{n+2}-\mathbb{P}_{n+1}a^{n+1}+a}{a^2-6a+1},\quad (a\neq 3\pm 2\sqrt{2}). \tag{2.4}$$

Proof. From (1.1), we obtain that

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{P}_i =&\mathbb{P}_1+\mathbb{P}_2+\cdots+\mathbb{P}_n\\ =&\mathbb{P}_1+(6\mathbb{P}_1-\mathbb{P}_0)+\cdots+(6\mathbb{P}_{n-1}-\mathbb{P}_{n-2})=5\sum_{i=1}^{n}\mathbb{P}_i-5\mathbb{P}_n+\mathbb{P}_{n-1}+\mathbb{P}_1-\mathbb{P}_0,\\ \sum_{i=1}^{n}i\mathbb{P}_i =&\mathbb{P}_1+2\mathbb{P}_2+\cdots+n\mathbb{P}_n\\ =&\mathbb{P}_1+2(6\mathbb{P}_1-\mathbb{P}_0)+\cdots+n(6\mathbb{P}_{n-1}-\mathbb{P}_{n-2})\\ =&\sum_{i=1}^{n}(5i+4)\mathbb{P}_i+(n+1)\mathbb{P}_{n-1}-(5n+4)\mathbb{P}_n+\mathbb{P}_1-2\mathbb{P}_0\\ =&5\sum_{i=1}^{n}i\mathbb{P}_i+4\sum_{i=1}^{n}\mathbb{P}_i+(n+1)\mathbb{P}_{n-1}-(5n+4)\mathbb{P}_n+\mathbb{P}_1-2\mathbb{P}_0.\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}\mathbb{P}_i=\frac{1}{4}(\mathbb{P}_{n+1}-\mathbb{P}_n-1),\qquad \sum_{i=1}^{n}i\mathbb{P}_i=\frac{1}{4}[n\mathbb{P}_{n+1}-(n+1)\mathbb{P}_n].$$

Let $X_i=\begin{pmatrix}\mathbb{P}_{i-1} & \mathbb{P}_i\\ \mathbb{P}_i & \mathbb{P}_{i+1}\end{pmatrix}$. By Lemma 1 in [34], for $i\geqslant 1$, we get

$$|X_i|=\mathbb{P}_{i-1}\mathbb{P}_{i+1}-\mathbb{P}_i^2=-1.$$

In view of

$$\mathbb{P}_{i+1}=6\mathbb{P}_i-\mathbb{P}_{i-1},$$

we obtain

$$\mathbb{P}_i=\frac{\mathbb{P}_{i+1}+\mathbb{P}_{i-1}}{6},$$

so,

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{P}_i^2=\sum_{i=1}^{n}\left(\frac{\mathbb{P}_{i+1}+\mathbb{P}_{i-1}}{6}\right)^2&=\frac{\sum_{i=1}^{n}\mathbb{P}_{i+1}^2+\sum_{i=1}^{n}\mathbb{P}_{i-1}^2+2\sum_{i=1}^{n}\mathbb{P}_{i+1}\mathbb{P}_{i-1}}{36}\\&=\frac{\sum_{i=1}^{n}\mathbb{P}_{i+1}^2+\sum_{i=1}^{n}\mathbb{P}_{i-1}^2+2\sum_{i=1}^{n}\mathbb{P}_{i}^2-2n}{36}\\&=\frac{4\sum_{i=1}^{n}\mathbb{P}_{i}^2+\mathbb{P}_{n+1}^2-\mathbb{P}_n^2-\mathbb{P}_1^2+\mathbb{P}_0^2-2n}{36}.\end{aligned}$$

Hence,

$$\sum_{i=1}^{n}\mathbb{P}_i^2=\frac{\mathbb{P}_{n+1}^2-\mathbb{P}_n^2-2n-1}{32}.$$

Let

$$S_n=\sum_{i=1}^{n}\mathbb{P}_i a^i=\mathbb{P}_1 a+\mathbb{P}_2 a^2+\cdots+\mathbb{P}_n a^n. \tag{2.5}$$

From (2.5) and the above recurrence relations, we get

$$(a^2-6a+1)S_n=\mathbb{P}_n a^{n+2}-\mathbb{P}_{n+1}a^{n+1}+\mathbb{P}_1 a,$$

thus,

$$S_n=\frac{\mathbb{P}_n a^{n+2}-\mathbb{P}_{n+1}a^{n+1}+a}{a^2-6a+1},\quad (a\neq 3\pm 2\sqrt{2}).$$

□

Lemma 2.2. *Let the matrix* $\mathcal{H}=[h_{ij}]_{i,j=1}^{n-2}$ *be of the form*

$$h_{ij}=\begin{cases}\mathbb{P}_1+\mathbb{P}_{n+1}, & i=j,\\ -\mathbb{P}_n, & i=j+1,\\ 0, & \text{otherwise}.\end{cases}$$

Then $\mathcal{H}^{-1}=[h'_{i,j}]_{i,j=1}^{n-2}$ *is given by*

$$h'_{ij}=\begin{cases}\frac{\mathbb{P}_n^{i-j}}{(\mathbb{P}_1+\mathbb{P}_{n+1})^{i-j+1}}, & i\geqslant j,\\ 0, & i<j.\end{cases}$$

Proof. Let $e_{ij}=\sum\limits_{k=1}^{n-2}h_{ik}h'_{kj}$. Apparently, $e_{ij}=0$ for $i<j$. For $i=j$, we obtain

$$e_{ii}=h_{ii}h'_{ii}=(\mathbb{P}_1+\mathbb{P}_{n+1})\cdot\frac{1}{\mathbb{P}_1+\mathbb{P}_{n+1}}=1.$$

For $i\geqslant j+1$, we get

$$e_{ij}=\sum_{k=1}^{n-2}h_{ik}h'_{kj}=h_{i,i-1}h'_{i-1,j}+h_{ii}h'_{ij}=-\frac{\mathbb{P}_n\cdot\mathbb{P}_n^{i-j-1}}{(\mathbb{P}_1+\mathbb{P}_{n+1})^{i-j}}+\frac{(\mathbb{P}_1+\mathbb{P}_{n+1})\cdot\mathbb{P}_n^{i-j}}{(\mathbb{P}_1+\mathbb{P}_{n+1})^{i-j+1}}=0.$$

Hence, we get $\mathcal{H}\mathcal{H}^{-1}=I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathcal{H}^{-1}\mathcal{H}=I_{n-2}$. Thus, the proof is completed. □

Lemma 2.3 ([14]). *Let* A=SLCirc$(a_1, a_2, \dots, a_n)$ *be a skew left circulant matrix and n be odd. Then,*

$$\lambda_j = \pm \left| \sum_{k=1}^{n} a_k \omega^{(j-\frac{1}{2})(k-1)} \right|, \quad (j = 1, 2, \dots, \frac{n-1}{2}), \qquad \lambda_{\frac{n+1}{2}} = \sum_{k=1}^{n} \left| a_k (-1)^{k-1} \right|,$$

where $\lambda_j (j = 1, 2, \dots, \frac{n-1}{2}, \frac{n+1}{2})$ *are the eigenvalues of* A.

Lemma 2.4 ([6]). *Let the orthogonal skew left circulant matrix* $\Delta = \mathrm{SLCirc}(1, 0, 0, \dots, 0)$. *Then*

$$\mathrm{SCirc}(a_1, a_2, \dots, a_n) = \Delta \mathrm{SLCirc}(a_1, a_2, \dots, a_n).$$

3. Main results

In this section, we obtain an explicit formula for the determinant of skew circulant matrix involving the product of Pell and Pell-Lucas numbers. We prove that the matrix is invertible for every positive interger n. Then we compute its inverse, maximum column sum matrix norm, maximum row sum matrix norm, spectral norm, Euclidean (or Frobenius) norm as well as lower and upper bounds for spread. Based on the relationship between skew circulant and skew left circulant matrices, we generalize these results to skew left circulant matrix involving the product of Pell and Pell-Lucas numbers.

3.1. Determinant and inverse of skew circulant matrix involving the product of Pell and Pell-Lucas numbers

Suppose that $A_n = \mathrm{SCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ is a skew circulant matrix. On the one hand, we obtain an explicit formula for the determinant of A_n. On the other hand, we prove that A_n is invertible for every positive integer n, and then we compute the inverse of A_n.

Theorem 3.1. *Let* $A_n = \mathrm{SCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ *be a skew circulant matrix for a positive integer* n. *Then*

$$\det A_n = \frac{\mathbb{P}_n^n + (1 + \mathbb{P}_{n+1})^n}{2 - \mathbb{P}_{n-1} + \mathbb{P}_{n+1}}, \tag{3.1}$$

where $\mathbb{P}_n$ *is the* n^{th} *product of Pell and Pell-Lucas numbers.*

Proof. For $n \leqslant 3$, it is easy to check that $\det A_1 = 1$, $\det A_2 = 37$, and $\det A_3 = 43290$. Therefore, (3.1) is satisfied. Now we consider the case $n > 3$, let

$$\Sigma = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 6 & \vdots & & & & \iddots & 1 \\ -1 & \vdots & & & \iddots & \iddots & -6 \\ 0 & \vdots & & \iddots & \iddots & \iddots & 1 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & 0 \\ \vdots & 0 & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & -6 & 1 & 0 & \cdots & 0 \end{pmatrix}_{n \times n}$$

and

$$\Omega_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & (\frac{\mathbb{P}_n}{\mathbb{P}_1 + \mathbb{P}_{n+1}})^{n-2} & \vdots & & \iddots & 1 \\ \vdots & \vdots & \vdots & \iddots & \iddots & 0 \\ \vdots & (\frac{\mathbb{P}_n}{\mathbb{P}_1 + \mathbb{P}_{n+1}})^{2} & 0 & \iddots & \iddots & \vdots \\ \vdots & \frac{\mathbb{P}_n}{\mathbb{P}_1 + \mathbb{P}_{n+1}} & 1 & \iddots & & \vdots \\ 0 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}_{n \times n}.$$

Multiplying A_n by Σ from the left and Ω_1 from the right, we obtain

$$\Sigma A_n \Omega_1 = \begin{pmatrix} \mathbb{P}_1 & l'_n & \mathbb{P}_{n-1} & \mathbb{P}_{n-2} & \cdots & \cdots & \mathbb{P}_2 \\ 0 & l_n & \mathbb{P}_{n-2} & \mathbb{P}_{n-3} & \cdots & \cdots & \mathbb{P}_1 \\ \vdots & 0 & \mathbb{P}_1+\mathbb{P}_{n+1} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & -\mathbb{P}_n & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\mathbb{P}_n & \mathbb{P}_1+\mathbb{P}_{n+1} \end{pmatrix}_{n\times n},$$

where

$$l'_n = \sum_{k=1}^{n-1} \mathbb{P}_{k+1}\left(\frac{\mathbb{P}_n}{\mathbb{P}_1+\mathbb{P}_{n+1}}\right)^{n-k-1}, \qquad l_n = \mathbb{P}_1 + 6\mathbb{P}_n + \sum_{k=1}^{n-2} \mathbb{P}_k\left(\frac{\mathbb{P}_n}{\mathbb{P}_1+\mathbb{P}_{n+1}}\right)^{n-k-1}.$$

By (2.4), we can further simplify the above results as follows:

$$l'_n = \frac{\mathbb{P}_n(\mathbb{P}_1+\mathbb{P}_{n+1})^{n-2} + (1-\mathbb{P}_{n-1})\mathbb{P}_n^{n-1}}{(2-\mathbb{P}_{n-1}+\mathbb{P}_{n+1})(\mathbb{P}_1+\mathbb{P}_{n+1})^{n-2}}, \quad l_n = \frac{\mathbb{P}_n^n + (\mathbb{P}_1+\mathbb{P}_{n+1})^n}{(2-\mathbb{P}_{n-1}+\mathbb{P}_{n+1})(\mathbb{P}_1+\mathbb{P}_{n+1})^{n-2}}. \tag{3.2}$$

We conclude that

$$\det(\Sigma A_n \Omega_1) = \mathbb{P}_1 l_n (\mathbb{P}_1+\mathbb{P}_{n+1})^{n-2} = \frac{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}{2-\mathbb{P}_{n-1}+\mathbb{P}_{n+1}}.$$

From the definition of Σ and Ω_1, we get

$$\det \Sigma = \det \Omega_1 = (-1)^{\frac{(n-1)(n-2)}{2}}.$$

Therefore, we have

$$\det A_n = \frac{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}{2-\mathbb{P}_{n-1}+\mathbb{P}_{n+1}}.$$

□

Theorem 3.2. *Let* $A_n = \mathrm{SCirc}(\mathbb{P}_1, \ldots, \mathbb{P}_n)$ *be a skew circulant matrix for every positive integer* n. *Then* A_n *is invertible.*

Proof. Taking n=1 in Theorem 3.1, $\det A_1 = 1 \neq 0$. For this reason, A_1 is invertible. In the case $n > 1$, according to [13], we have the eigenvalues of A_n,

$$f(\omega^k \eta) = \sum_{j=1}^{n} \mathbb{P}_j (\omega^k \eta)^{j-1}, \quad (k = 1, 2, \ldots, n-1),$$

where $\omega = \exp(\frac{2\pi i}{n}), \eta = \exp(\frac{\pi i}{n})$. Since $\mathbb{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha\beta = 1, \alpha + \beta = -6$, we have

$$f(\omega^k \eta) = \sum_{j=1}^{n} \frac{\alpha^j - \beta^j}{\alpha - \beta} (\omega^k \eta)^{j-1}.$$

Since $|\omega^k\eta| = 1$, and $|\alpha| \neq 1$, $|\beta| \neq 1$, we have $\omega^k\eta \neq \alpha$ and $\omega^k\eta \neq \beta$, and hence

$$\begin{aligned} f(\omega^k\eta) = \sum_{j=1}^{n} \frac{\alpha^j-\beta^j}{\alpha-\beta}(\omega^k\eta)^{j-1} &= \frac{\alpha}{\alpha-\beta}\sum_{j=1}^{n}(\alpha\omega^k\eta)^{j-1} - \frac{\beta}{\alpha-\beta}\sum_{j=1}^{n}(\beta\omega^k\eta)^{j-1} \\ &= \frac{\alpha-\beta+\alpha^{n+1}-\beta^{n+1}}{(\alpha-\beta)(1-\alpha\omega^k\eta)(1-\beta\omega^k\eta)} - \frac{\alpha\beta\omega^k\eta(\alpha^n-\beta^n)}{(\alpha-\beta)(1-\alpha\omega^k\eta)(1-\beta\omega^k\eta)} \\ &= \frac{1+\mathbb{P}_{n+1}-\mathbb{P}_n\omega^k\eta}{1-6\omega^k\eta+\omega^{2k}\eta^2}, \quad (k=1,2,\ldots,n-1). \end{aligned}$$

Assume that there exists $\omega^l\eta (l = 1, 2, \ldots, n-1)$ such that $f(\omega^l\eta) = 0$. We obtain $1 + \mathbb{P}_{n+1} - \mathbb{P}_n\omega^l\eta = 0$. Hence, $\omega^l\eta = \frac{1+\mathbb{P}_{n+1}}{\mathbb{P}_n}$ is a real number. That is to say, the imaginary part of $\omega^l\eta$ is $\sin\frac{(2l+1)\pi}{n} = 0$. Therefore, $\omega^l\eta = \pm 1$, and $1 + \mathbb{P}_{n+1} - \mathbb{P}_n\omega^l\eta \neq 0$ $(n > 0)$. Consequently, we obtain $f(\omega^k\eta) \neq 0$ for any $\omega^k\eta$ $(k = 1, 2, \ldots, n-1)$. □

Theorem 3.3. *Let $A_n = \mathrm{SCirc}(\mathbb{P}_1, \ldots, \mathbb{P}_n)$ be a skew circulant matrix for a positive integer $n > 1$. Then*

$$A_n^{-1} = \mathrm{SCirc}(y_1, y_2, \ldots, y_n), \tag{3.3}$$

where

$$y_1 = \frac{(\mathbb{P}_{n-1}-1)\mathbb{P}_n^{n-2} + (1+\mathbb{P}_{n+1})^{n-1}}{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}, \tag{3.4}$$

$$y_2 = -\frac{\mathbb{P}_n^{n-1} + (6+\mathbb{P}_{n+2})(1+\mathbb{P}_{n+1})^{n-2}}{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}, \tag{3.5}$$

$$y_k = \frac{(2-\mathbb{P}_{n-1}+\mathbb{P}_{n+1})\mathbb{P}_n^{k-3}(1+\mathbb{P}_{n+1})^{n-k}}{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}, \quad (k = 3, 4, \ldots, n). \tag{3.6}$$

Proof. Let

$$\Omega_2 = \begin{pmatrix} 1 & -\frac{l_n'}{\mathbb{P}_1} & \omega_3 & \omega_4 & \cdots & \omega_n \\ 0 & 1 & -\frac{\mathbb{P}_{n-2}}{l_n} & -\frac{\mathbb{P}_{n-3}}{l_n} & \cdots & -\frac{\mathbb{P}_1}{l_n} \\ \vdots & \ddots & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{n\times n},$$

where

$$\omega_j = \frac{1}{\mathbb{P}_1}\Big(\frac{l_n'}{l_n}\mathbb{P}_{n+1-j} - \mathbb{P}_{n+2-j}\Big), \quad (j = 3, 4, \ldots, n),$$

l_n' and l_n are given in (3.2). Multiplying $\Sigma A_n\Omega_1$ by Ω_2 in Theorem 3.1 from the left, we have

$$\Sigma A_n\Omega_1\Omega_2 = \begin{pmatrix} \mathbb{P}_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & l_n & \ddots & & & & \vdots \\ \vdots & 0 & \mathbb{P}_1+\mathbb{P}_{n+1} & \ddots & & & \vdots \\ \vdots & \vdots & -\mathbb{P}_n & \ddots & \ddots & & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\mathbb{P}_n & \mathbb{P}_1+\mathbb{P}_{n+1} \end{pmatrix}_{n\times n},$$

where Σ and Ω_1 are the same as Theorem 3.1. The matrix $\Sigma A_n \Omega_1 \Omega_2$ admits a block partition of the form

$$\Sigma A_n \Omega_1 \Omega_2 = \mathcal{D} \oplus \mathcal{H},$$

where $D = \mathrm{diag}(2, l_n)$ is a diagonal matrix, $\mathcal{H}$ is as defined in Lemma 2.2, and $\mathcal{D} \oplus \mathcal{H}$ is the direct sum of $\mathcal{D}$ and $\mathcal{H}$. If we write $\Omega = \Omega_1 \Omega_2$, then we obtain $A_n^{-1} = \Omega(\mathcal{D}^{-1} \oplus \mathcal{H}^{-1})\Sigma$.

Since the last row elements of the matrix Ω are $(0, 1, -\frac{\mathbb{P}_{n-2}}{l_n}, -\frac{\mathbb{P}_{n-3}}{l_n}, \ldots, -\frac{\mathbb{P}_2}{l_n}, -\frac{\mathbb{P}_1}{l_n})$, the last row elements of the matrix $\Omega(\mathcal{D}^{-1} \oplus \mathcal{H}^{-1})$ are $(0, \frac{1}{l_n}, T_3, T_4, \ldots, T_n)$, where

$$T_k = \sum_{i=1}^{n+1-k} -\frac{\mathbb{P}_{n+2-k-i}\mathbb{P}_n^{i-1}}{l_n(\mathbb{P}_1 + \mathbb{P}_{n+1})^i}, \quad (k = 3, 4, \ldots, n).$$

Using (2.4), we have

$$T_k = \frac{\left[\mathbb{P}_{n+2-k}\mathbb{P}_n - \mathbb{P}_{n+1-k}(\mathbb{P}_1 + \mathbb{P}_n)\right](\mathbb{P}_1 + \mathbb{P}_{n+1})^{n-2}}{\mathbb{P}_n^n + (\mathbb{P}_1 + \mathbb{P}_{n+1})^n} - \frac{\mathbb{P}_n^{n+2-k}(\mathbb{P}_1 + \mathbb{P}_{n+1})^{k-3}}{\mathbb{P}_n^n + (\mathbb{P}_1 + \mathbb{P}_{n+1})^n}, \quad (k = 3, 4, \ldots, n).$$

The last row of $A_n^{-1} = \mathrm{SCirc}(y_1, y_2, \ldots, y_n)$ is $(-y_2, -y_3, \ldots, -y_n, y_1)$, which is given by the following equations:

$$\begin{aligned}
-y_2 &= \frac{6}{l_n} - T_3, \\
-y_3 &= T_n, \\
-y_4 &= T_{n-1} - 6T_n, \\
-y_5 &= T_{n-2} - 6T_{n-1} + T_n, \\
&\vdots \\
-y_k &= T_{n-k+3} - 6T_{n-k+4} + T_{n-k+5}, \\
&\vdots \\
-y_n &= T_3 - 6T_4 + T_5, \\
y_1 &= \frac{1}{l_n} - 6T_3 + T_4.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
y_1 &= \frac{(\mathbb{P}_{n-1} - 1)\mathbb{P}_n^{n-2} + (1 + \mathbb{P}_{n+1})^{n-1}}{\mathbb{P}_n^n + (1 + \mathbb{P}_{n+1})^n}, \\
y_2 &= -\frac{\mathbb{P}_n^{n-1} + (6 + \mathbb{P}_{n+2})(1 + \mathbb{P}_{n+1})^{n-2}}{\mathbb{P}_n^n + (1 + \mathbb{P}_{n+1})^n}, \\
y_k &= \frac{(2 - \mathbb{P}_{n-1} + \mathbb{P}_{n+1})\mathbb{P}_n^{k-3}(1 + \mathbb{P}_{n+1})^{n-k}}{\mathbb{P}_n^n + (1 + \mathbb{P}_{n+1})^n}, \quad (k = 3, 4, \ldots, n).
\end{aligned}$$

□

3.2. Norms and spread of skew circulant matrix involving the product of Pell and Pell-Lucas numbers

Theorem 3.4. *Let $A_n = \mathrm{SCirc}(\mathbb{P}_1, \ldots, \mathbb{P}_n)$ be a skew circulant matrix. Then three kinds of norms of A_n are given by*

$$\|A_n\|_1 = \|A_n\|_\infty = \frac{1}{4}(\mathbb{P}_{n+1} - \mathbb{P}_n - 1), \tag{3.7}$$

$$\|A_n\|_F = \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{32}}. \tag{3.8}$$

Proof. By Definition 4 in [17], (2.1), and (2.2), we have

$$\|A_n\|_1 = \|A_n\|_\infty = \sum_{i=1}^{n} \mathbb{P}_i = \frac{1}{4}(\mathbb{P}_{n+1} - \mathbb{P}_n - 1), \quad \| A_n \|_F = \sqrt{n \sum_{i=1}^{n} \mathbb{P}_i^2} = \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{32}}.$$

□

Theorem 3.5. *Let* $A_n^{'} = \mathrm{SCirc}(\mathbb{P}_1, -\mathbb{P}_2, \dots, -\mathbb{P}_{n-1}, \mathbb{P}_n)$ *be an odd-order alternative skew circulant matrix. Then*

$$\|A_n^{'}\|_2 = \sum_{i=1}^{n} \mathbb{P}_i = \frac{1}{4}(\mathbb{P}_{n+1} - \mathbb{P}_n - 1).$$

Proof. By Lemma 1 in [9] , we get

$$\lambda_j(A_n^{'}) = \sum_{i=1}^{n} (-1)^{i-1} \mathbb{P}_i (\omega^j \eta)^{i-1},$$

therefore,

$$|\lambda_j(A_n^{'})| \leqslant \sum_{i=1}^{n} \left|(-1)^{i-1}\mathbb{P}_i\right| \cdot \left|(\omega^j \eta)^{i-1}\right| = \sum_{i=1}^{n} \mathbb{P}_i,$$

for $j = 0, 1, \dots, n-1$. Since n is odd, $\sum_{i=1}^{n} \mathbb{P}_i$ is an eigenvalue of $A_n^{'}$, that is

$$\begin{pmatrix} \mathbb{P}_1 & -\mathbb{P}_2 & \mathbb{P}_3 & \cdots & \mathbb{P}_n \\ -\mathbb{P}_n & \ddots & \ddots & \ddots & \vdots \\ \mathbb{P}_{n-1} & \ddots & \ddots & \ddots & \mathbb{P}_3 \\ \vdots & \ddots & \ddots & \ddots & -\mathbb{P}_2 \\ \mathbb{P}_2 & \cdots & \mathbb{P}_{n-1} & -\mathbb{P}_n & \mathbb{P}_1 \end{pmatrix}_{n\times n} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1} = \sum_{i=1}^{n} \mathbb{P}_i \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}.$$

To sum up, we have

$$\max_{0\leqslant j\leqslant n-1} \left|\lambda_j(A_n^{'})\right| = \sum_{i=1}^{n} \mathbb{P}_i. \tag{3.9}$$

Since all skew circulant matrices are normal, by Lemma 7 in [17], (2.1), and (3.9), we have

$$\|A_n^{'}\|_2 = \sum_{i=1}^{n} \mathbb{P}_i = \frac{1}{4}(\mathbb{P}_{n+1} - \mathbb{P}_n - 1),$$

which completes the proof. □

Theorem 3.6. *Let* $A_n = \mathrm{SCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ *be a skew circulant matrix. Then lower and upper bounds (denoted as* $s(A_n)$*) for spread of* A_n *are given as follows*

$$\frac{1}{4(n-1)}\left|n\mathbb{P}_n + (2-n)\mathbb{P}_{n+1} - 4n\right| \leqslant s(A_n) \leqslant \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{16} - 2n}. \tag{3.10}$$

Proof. Note that the trace of A_n is $\mathrm{tr}A_n = n\mathbb{P}_1 = n$. Denote the sum of the off-diagonal elements of A_n as $\chi(A_n)$,

$$\chi(A_n) = \sum_{k=2}^{n}(n-k+1)\mathbb{P}_k - \sum_{k=2}^{n}(k-1)\mathbb{P}_k = (n+2)\sum_{k=2}^{n}\mathbb{P}_k - 2\sum_{k=2}^{n}k\mathbb{P}_k.$$

By (2.1) and (2.3),

$$\chi(A_n) = \frac{n\mathbb{P}_n + (2-n)\mathbb{P}_{n+1} - 4n}{4}.$$

Since A_n is a normal and real matrix, by Definition 5 and Lemma 6 in [17], and (3.8), we get

$$\frac{1}{4(n-1)}\left|n\mathbb{P}_n + (2-n)\mathbb{P}_{n+1} - 4n\right| \leqslant s(A_n) \leqslant \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{16} - 2n}.$$

□

3.3. Determinant and inverse of skew left circulant matrix involving the product of Pell and Pell-Lucas numbers

In this subsection, let $A''_n = \mathrm{SLCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ be a skew left circulant matrix. Using the obtained conclusions in Section 2, we get a determinant explicit formula for the matrix A''_n. And then, we prove that A''_n is an invertible matrix for any positive interger n. The inverse of the matrix A''_n is also presented.

Theorem 3.7. *Let $A''_n = \mathrm{SLCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ be a skew left circulant matrix for a positive integer n. Then*

$$\det A''_n = (-1)^{\frac{n(n-1)}{2}} \frac{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}{2 - \mathbb{P}_{n-1} + \mathbb{P}_{n+1}}, \tag{3.11}$$

where $\mathbb{P}_n$ is the n^{th} product of Pell and Pell-Lucas numbers.

Theorem 3.8. *Let $A''_n = \mathrm{SLCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ be a skew left circulant matrix for every positive interger n. Then A''_n is invertible.*

Theorem 3.9. *Let $A''_n = \mathrm{SLCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ be a skew left circulant matrix for positive integer $n > 1$. Then*

$$(A''_n)^{-1} = \mathrm{SLCirc}(y''_1, y''_2, \dots, y''_n), \tag{3.12}$$

where

$$y''_1 = \frac{(\mathbb{P}_{n-1} - 1)\mathbb{P}_n^{n-2} + (1+\mathbb{P}_{n+1})^{n-1}}{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}, \tag{3.13}$$

$$y''_k = -\frac{(2 - \mathbb{P}_{n-1} + \mathbb{P}_{n+1})\mathbb{P}_n^{n-1-k}(1+\mathbb{P}_{n+1})^{k-2}}{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}, \quad (k = 2, 3, \dots, n-1), \tag{3.14}$$

$$y''_n = \frac{\mathbb{P}_n^{n-1} + (6+\mathbb{P}_{n+2})(1+\mathbb{P}_{n+1})^{n-2}}{\mathbb{P}_n^n + (1+\mathbb{P}_{n+1})^n}. \tag{3.15}$$

3.4. Norms and spread of skew left circulant matrix involving the product of Pell and Pell-Lucas numbers

Theorem 3.10. *Let $A''_n = \mathrm{SLCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ be a skew left circulant matrix. Then three kinds of norms of A''_n are given by*

$$\|A''_n\|_1 = \|A''_n\|_\infty = \frac{1}{4}(\mathbb{P}_{n+1} - \mathbb{P}_n - 1), \tag{3.16}$$

$$\|A''_n\|_F = \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{32}}. \tag{3.17}$$

Proof. Using the similar method as in Theorem 3.4, the conclusion is obtained. □

Theorem 3.11. *Let $A_n''' = \mathrm{SLCirc}(\mathbb{P}_1, -\mathbb{P}_2, \dots, -\mathbb{P}_{n-1}, \mathbb{P}_n)$ be an odd-order alternative skew left circulant matrix. Then*

$$\|A_n'''\|_2 = \sum_{i=1}^{n} \mathbb{P}_i = \frac{1}{4}(\mathbb{P}_{n+1} - \mathbb{P}_n - 1).$$

Proof. According to Lemma 2.3,

$$\lambda_j(A_n''') = \pm \left| \sum_{i=1}^{n} (-1)^{i-1} \mathbb{P}_i \omega^{(j-\frac{1}{2})(k-1)} \right|, \quad j = (1, 2, \dots, \frac{n-1}{2}), \quad \lambda_{\frac{n+1}{2}}(A_n''') = \sum_{i=1}^{n} \mathbb{P}_i. \tag{3.18}$$

So

$$|\lambda_j(A_n''')| \leqslant \sum_{i=1}^{n} |(-1)^{i-1} \mathbb{P}_i (-1)^{i-1}| = \sum_{i=1}^{n} \mathbb{P}_i \quad (j = 1, 2, \dots, \frac{n-1}{2}). \tag{3.19}$$

By (3.18) and (3.19), we have

$$\max_{0 \leqslant i \leqslant n+1} \left| \lambda_i(A_n''') \right| = \sum_{i=1}^{n} \mathbb{P}_i. \tag{3.20}$$

Since all skew left circulant matrices are symmetrical, by Lemma 7 in [17], (2.1), and (3.20), we obtain

$$\|A_n'''\|_2 = \frac{1}{4}(\mathbb{P}_{n+1} - \mathbb{P}_n - 1).$$

□

Theorem 3.12. *Let $A_n'' = \mathrm{SLCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ be a skew left circulant matrix. Then the bounds for the spread of A_n'' are,*

$$2\mathbb{P}_n \leqslant s(A_n'') \leqslant \begin{cases} \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{16} - \frac{(\mathbb{P}_{n+1} + \mathbb{P}_n - 1)^2}{32n}}, & \text{if } n \text{ is odd}, \\ \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{16}}, & \text{if } n \text{ is even}. \end{cases}$$

Proof. Since A_n'' is a symmetric matrix, by Definition 5 and Lemma 6 in [17],

$$2 \max |A_i''| \leqslant s(A_n'') \leqslant \sqrt{2\|A_n''\|_F^2 - \frac{2}{n} |\mathrm{tr} A_n''|^2}.$$

For skew left circulant matrix A_n'',

$$2 \max |A_i''| = 2\mathbb{P}_n.$$

According to Theorem 3.10,

$$\|A_n''\|_F = \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{32}}.$$

If n is odd, consider $A = \sum_{i=0}^{\frac{n-1}{2}} \mathbb{P}_{2i+1}$ and $B = \sum_{i=1}^{\frac{n-1}{2}} \mathbb{P}_{2i}$, we have

$$\mathrm{tr}(A_n'') = \mathbb{P}_1 - \mathbb{P}_2 + \mathbb{P}_3 - \dots + \mathbb{P}_n = \sum_{i=0}^{\frac{n-1}{2}} \mathbb{P}_{2i+1} - \sum_{i=1}^{\frac{n-1}{2}} \mathbb{P}_{2i} = A - B.$$

According to

$$\mathbb{P}_{n+1} = 6\mathbb{P}_n - \mathbb{P}_{n-1}, \text{ where } \mathbb{P}_1 = 1, \mathbb{P}_2 = 6,$$

we have

$$\begin{cases} A - B = 5B - A + \mathbb{P}_1 + \mathbb{P}_n, \\ B = 6A - 6\mathbb{P}_1 - 6\mathbb{P}_n - B + \mathbb{P}_2 + \mathbb{P}_{n-1}. \end{cases}$$

For this reason, we have

$$\operatorname{tr}(A_n'') = A - B = \frac{1}{8}(\mathbb{P}_{n+1} + \mathbb{P}_n - 1).$$

If n is even, then

$$\operatorname{tr}(A_n'') = \mathbb{P}_1 - \mathbb{P}_1 + \mathbb{P}_3 - \mathbb{P}_3 + \cdots - \mathbb{P}_{n-1} = 0.$$

So the result is as follows:

$$2\mathbb{P}_n \leqslant s(A_n'') \leqslant \begin{cases} \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{16} - \frac{(\mathbb{P}_{n+1} + \mathbb{P}_n - 1)^2}{32n}}, & \text{if } n \text{ is odd}, \\ \sqrt{\frac{n(\mathbb{P}_{n+1}^2 - \mathbb{P}_n^2 - 2n - 1)}{16}}, & \text{if } n \text{ is even}. \end{cases}$$

□

4. Algorithms and numerical computations

In this section, we give two algorithms for computing inverses of skew circulant and skew left circulant matrices involving the product of Pell and Pell-Lucas numbers. Meanwhile, we list several examples to calculate their determinants, inverses, three kinds of norms and lower and upper bounds for spread.

Firstly, based on Theorem 3.3, the inverse of skew circulant matrix involving the product of Pell and Pell-Lucas numbers $A_n = \operatorname{SCirc}(\mathbb{P}_1, \ldots, \mathbb{P}_n)$ is calculated by the following algorithm.

Algorithm 4.1.

Step 1: Input n and generate the product of Pell and Pell-Lucas numbers by (1.1).

Step 2: Compute y_i $(i = 1, 2, \ldots, n)$ via (3.4)-(3.6).

Step 2: Output $A_n^{-1} = \operatorname{SCirc}(y_1, y_2, \ldots, y_n)$ by (3.3).

Example 4.2. Consider a 4×4 skew circulant matrix involving the product of Pell and Pell-Lucas numbers:

$$A_4 = \begin{bmatrix} 1 & 6 & 35 & 204 \\ -204 & 1 & 6 & 35 \\ -35 & -204 & 1 & 6 \\ -6 & -35 & -204 & 1 \end{bmatrix}.$$

From (3.1), we get $\det A_4 = 1736220676$. In particular, by (3.7) and (3.8), the three kinds of norms of A_4 are given by

$$\|A_4\|_1 = \|A_4\|_\infty = 246, \quad \|A_4\|_F = 2\sqrt{42878}.$$

By (3.10), the bounds for the spread of A_4 are

$$-\frac{263}{2} \leqslant s(A_4) \leqslant 2\sqrt{85754}.$$

As for A_4^{-1}, according to Algorithm 4.1, we have

$$y_1 = \frac{42911}{51065314}, \quad y_2 = -\frac{125058}{25532657}, \quad y_3 = \frac{35}{51065314}, \quad y_4 = \frac{3}{25532657}.$$

From (3.4), we obtain

$$A_4^{-1} = \begin{bmatrix} \frac{42911}{51065314} & -\frac{125058}{25532657} & \frac{35}{51065314} & \frac{3}{25532657} \\ -\frac{3}{25532657} & \frac{42911}{51065314} & -\frac{125058}{25532657} & \frac{35}{51065314} \\ -\frac{35}{51065314} & -\frac{3}{25532657} & \frac{42911}{51065314} & -\frac{125058}{25532657} \\ \frac{125058}{25532657} & -\frac{35}{51065314} & -\frac{3}{25532657} & \frac{42911}{51065314} \end{bmatrix}.$$

Secondly, based on Theorem 3.9, the inverse of skew left circulant matrix involving the product of Pell and Pell-Lucas numbers $A''_n = \mathrm{SLCirc}(\mathbb{P}_1, \dots, \mathbb{P}_n)$ is calculated by the following algorithm.

Algorithm 4.3.

Step 1: Input n and generate the product of Pell and Pell-Lucas numbers by (1.1).

Step 2: Compute y''_i $(i = 1, 2, \dots, n)$ via (3.13)-(3.15).

Step 2: Output $A''^{-1}_n = \mathrm{SLCirc}(y''_1, y''_2, \dots, y''_n)$ by (3.12).

Example 4.4. Consider a 4×4 skew left circulant matrix involving the product of Pell and Pell-Lucas numbers A''_4:

$$A''_4 = \begin{bmatrix} 1 & 6 & 35 & 204 \\ 6 & 35 & 204 & -1 \\ 35 & 204 & -1 & -6 \\ 204 & -1 & -6 & -35 \end{bmatrix}.$$

From (3.11), we get $\det A''_4 = 1736220676$. In particular, by (3.16) and (3.17), the three kinds of norms of A''_4 are given by

$$\|A''_4\|_1 = \|A''_4\|_\infty = 246, \quad \|A''_4\|_F = 2\sqrt{42878}.$$

By Theorem 3.12, the bounds for the spread of A''_4 are

$$408 \leqslant s(A''_4) \leqslant 4\sqrt{21439}.$$

According to Algorithm 4.3, we have

$$y''_1 = \frac{42911}{51065314}, \quad y''_2 = -\frac{3}{25532657}, \quad y''_3 = -\frac{35}{51065314}, \quad y''_4 = \frac{125058}{25532657}.$$

From (3.12), we obtain

$$A''^{-1}_4 = \begin{bmatrix} \frac{42911}{51065314} & -\frac{3}{25532657} & -\frac{35}{51065314} & \frac{125058}{25532657} \\ -\frac{3}{25532657} & -\frac{35}{51065314} & \frac{125058}{25532657} & -\frac{42911}{102130628} \\ -\frac{35}{51065314} & \frac{125058}{25532657} & -\frac{42911}{102130628} & \frac{3}{25532657} \\ \frac{125058}{25532657} & -\frac{42911}{102130628} & \frac{3}{25532657} & \frac{35}{51065314} \end{bmatrix}.$$

5. Conclusions

In this paper, we discuss the invertibility of the skew circulant and skew left circulant matrices involving the product of Pell and Pell-Lucas numbers and compute determinant and inverse by constructing the transformation matrices. The four kinds of norms, lower and upper bounds for spread of these matrices are given, respectively. Besides, we design two algorithms (Algorithms 4.1 and 4.3) and two examples to verify our algorithm's effectiveness.

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