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Existence and uniqueness of solutions for fuzzy initial value problems under granular differentiability

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Abstract

In this paper, we introduce the notion of second and higher-order granular differentiability for fuzzy number-valued functions. A weighted granular metric is defined for continuously granular differentiable mappings and proves that it is a complete metric space. Fuzzy initial value problems are investigated for second and higher-order fuzzy differential equations under granular differentiability. Sufficient conditions are established for the existence and uniqueness of solutions for the fuzzy initial value problems. An algorithm is developed to determine the solution to the fuzzy initial value problem under granular differentiability. Moreover, examples are presented to verify our theoretical results and algorithm.

Keywords: Fuzzy number, granular differentiability, contraction mapping, fuzzy differential equations, horizontal membership function.

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1. Introduction

Fuzzy set theory is one of the most effective tools used in handling vagueness and uncertainty in the data using mathematical models for many real-life applications. While modeling dynamical systems with a lack of sufficient information about their behavior, fuzzy differential equations (FDEs) provide a powerful tool. When the parameters or the states of the differential equations are uncertain, they can be modeled with FDEs. But, most practical situations demand that FDEs with initial conditions, which are called fuzzy initial value problems (FIVPs). Zadeh introduced the fuzzy set theory in 1965 [24] . In 1972, Chang and Zadeh [10], presented the idea of a derivative of a fuzzy function (FF). Different versions of derivatives for FFs have been presented by many researchers. Dubois and Prade in 1982 [12] presented a definition using the extension principle, Puri-Ralescu [20] presented the Hukuhara derivative (H-derivative) in 1983, and Seikkala et al. [21] presented the Seikkala derivative in 1987. Applying these

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definitions to solve FDEs results that, a solution exists only for the non-decreasing length of support, which limits these concepts unable to address many of the real-world problems.

The generalized Hukuhara difference (gH-difference) of two interval numbers is the generalization of H-difference, the gH-derivative for interval-valued functions using gH-difference was introduced by Stefanini and Bede [23] and also applied for solving interval differential equations. The concept of strongly generalized differentiability was introduced by Bede and Gal [6, 7] for fuzzy number valued functions and applied them to solve the FDEs. Bede and Stefanini [8], introduced and studied properties of generalized Hukuhara derivative for fuzzy valued functions using gH-difference. With this new definition based on gH-difference, we could address some of the problems associated with prior definitions of uncertainty support closure length, such as their non-decreasing nature. However, this approach has some flaws. It is necessary for monotonic uncertainty to have a monotonous support closure length for the solution. Additionally, there is a wide array of solutions generated from this method. It is possible to have two solutions for a first-order FDE: one for the first kind of differentiability and another for the second kind. The doubling property is used to solve each first-order FDE. Even though, we cannot guarantee that the gH-difference of two fuzzy numbers (FN) exists. The two approaches (i) gH-differentiability and (ii) SGH-differentiability have suffered from unnatural behavior in modeling. Recently, the granular derivative (gr-derivative) concept was introduced by Mazandarani et al. [16] using the horizontal membership function (HMF) [19]. It was successfully applied to solve first-order FDEs and resolved all the above-mentioned issues.

The initial value problem is permitted as a FIVP if the variables, parameters, and initial conditions are fuzzy sets. Research into FIVPs is relatively scarce, yet it has many applications in the formation of civil engineering, physics, control theory, economics, population models. Buckley and Feuring [9] presented solutions of first-order FIVPs under various derivatives. Georgiou et al. [13] established the existence and uniqueness of solutions of Cauchy problems for FDEs of second order and FIVPs of higher order in [14] under H-differentiability. Further, Murty and Suresh Kumar [17] obtained the existence and uniqueness criteria for fuzzy initial and boundary value problems. Furthermore, Allahviranloo et al. [3] studied the existence and uniqueness of solutions of second-order FDEs under SGH-differentiability. Jmeel and Altaie [15] presented finite difference method for numerical solution of fuzzy boundary value problems. Son et al. [22] presented conditions for the existence and uniqueness of solutions of fuzzy delay differential equations under granular differentiability.

The purpose of this work is to (i) investigate FIVPs associated with second and higher-order FDEs under granular differentiability; (ii) establish existence and uniqueness criteria; (iii) develop an algorithm to determine their solutions. The rest of this paper is as follows. Section 2, presents basic definitions and results related to HMFs, granular metric, gr-differentiability, and gr-integration. The existence and uniqueness theorems for second and higher-order non-linear FIVPs under gr-differentiability are established in Section 3. Section 4 presents an algorithm to solve FIVPs under gr-differentiability and highlighted the proposed results and algorithm with suitable examples. Concluding remarks and future work are discussed in Section 5.

2. Preliminaries

This section present some useful definitions, notations and results to establish the main results.

Definition 2.1 ([18])**.** A non-empty fuzzy subset of $\mathbf{R}$, with membership function $p : \mathbf{R} \to [0, 1]$, is said to be a fuzzy number, if it satisfies the following conditions.

(i) $p(y_0) = 1$ for at least one $y_0 \in \mathbf{R}$.
(ii) $p(\beta y + (1 - \beta)z) \geqslant \min\{p(y), p(z)\}, \forall \beta \in [0, 1],\ y, z \in \mathbf{R}$.
(iii) p is upper semi continuous on $\mathbf{R}$.
(iv) $cl\{y \in \mathbf{R} : p(y) > 0\}$ is compact.

Here, $p(y)$ is the membership degree of y, for every $y \in \mathbf{R}$. Let $\mathbf{R}_F$ denotes the space of FNs in $\mathbf{R}$. The β-level sets of p are defined by $[p]^\beta = \{y \in \mathbf{R} : p(y) \geqslant \beta\} = [p_l^\beta, p_r^\beta]$, for $0 < \beta \leqslant 1$ and $[p]^0 = cl\{y \in \mathbf{R} : p(y) > 0\}$.

Definition 2.2 ([16]). Suppose that $q : [a, b] \to [0, 1]$, where $[a, b] \subseteq \mathbf{R}$ be a fuzzy number. Then the HMF $q_{gr} : [0, 1] \times [0, 1] \to [a, b]$, is a representation of $q(y)$ as $q_{gr}(\beta, \alpha_q) = y$ in which "gr" stands for granule of information include in $y \in [a, b]$, $\beta \in [0, 1]$ is the membership degree of y in $q(y)$, where $\alpha_q \in [0, 1]$ is said to be RDM variable, and $q_{gr}(\beta, \alpha_q) = q_l^\beta + (q_r^\beta - q_l^\beta)\alpha_q$.

Proposition 2.1 ([16]). *The HMF of* $q(y) \in \mathbf{R}_F$ *is also denoted by* $H(q) \triangleq q_{gr}(\beta, \alpha_q)$. *Moreover, the* β*-level sets of the VMF of* $q(y)$ *can be obtained from*

$$H^{-1}(q_{gr}(\beta, \alpha_q)) = [q]^\beta = \left[\inf_{\beta \leqslant \alpha \leqslant 1} \min_{\alpha_q} q_{gr}(\alpha, \alpha_q), \sup_{\beta \leqslant \alpha \leqslant 1} \max_{\alpha_q} q_{gr}(\alpha, \alpha_q) \right],$$

which is the span of the information granule.

Note 2.1 ([19]). The HMF of triangular and trapezoidal FNs, $p = (p_1, p_2, p_3)$ and $q = (p_1, p_2, p_3, p_4)$ are as $H(p) = [p_1 + (p_2 - p_1)\beta] + [(1 - \beta)(p_3 - p_1)]\alpha_p$ and $H(q) = [p_1 + (p_2 - p_1)\beta] + [(p_4 - p_1) + (p_4 - p_3 + p_2 - p_1)\beta]\alpha_q$, for all $\beta, \alpha_p, \alpha_q \in [0, 1]$, respectively.

Definition 2.3 ([16]). Let p and q be two FNs. Then $H(p) = H(q)$ for all $\alpha_p = \alpha_q \in [0, 1]$ if and only if p and q are said to be equal.

Definition 2.4 ([22]). For two fuzzy numbers $p, q \in \mathbf{R}_F$, whose HMFs are $H(p)$ and $H(q)$, we define $H(p \odot q) \triangleq H(p) * H(q)$, where $\odot$, $*$ used to present arithmetic operations in $\mathbf{R}_F$ and $\mathbf{R}$ such as addition, multiplication, subtraction or division, respectively. Moreover, $0 \notin q_{gr}(\beta, \alpha_q)$ if $\odot$ denotes the division operator.

Proposition 2.2 ([16]). *Let* $r = p \odot q$. *Then* $[r]^\beta = H^{-1}(p_{gr}(\alpha, \alpha_p) * q_{gr}(\alpha, \alpha_q))$ *always present* β*-level sets of r.*

Note 2.2. In the Definition 2.4 the addition and difference are called granular addition (gr-addition) and granular difference (gr- difference), respectively, and are denoted by "$\oplus_{gr}$", "$\ominus_{gr}$" and $r = p \ominus_{gr} q \iff p = r \oplus_{gr} q$

Example 2.1. Let $p = (1, 2, 3)$, $q = (2, 3, 4, 5)$ be two FNs, then the HMFs of p and q are $p_{gr}(\beta, \alpha_p) = 1 + \beta + 2(1 - \beta)\alpha_p$, and $q_{gr}(\beta, \alpha_q) = 2 + \beta + (3 - 2\beta)\alpha_q$, for β, α_p, $\alpha_q \in [0, 1]$, respectively. Let $r = p \ominus_{gr} q$, then $H(r) = H(p \ominus_{gr} q) = p_{gr}(\beta, \alpha_p) - q_{gr}(\beta, \alpha_q) = -1 + 2(1 - \beta)\alpha_p - (3 - 2\beta)\alpha_q = r_{gr}(\beta, \alpha_r)$, where $\alpha_r = (\alpha_p, \alpha_q)$ and

$$H^{-1}(r_{gr}(\beta, \alpha_r)) = [r]^\beta = \left[\inf_{\beta \leqslant \alpha \leqslant 1} \min_{\alpha_r} r_{gr}(\alpha, \alpha_r), \sup_{\beta \leqslant \alpha \leqslant 1} \max_{\alpha_r} r_{gr}(\alpha, \alpha_r) \right],$$

$$\begin{aligned}
[p \ominus_{gr} q]^\beta &= \left[\inf_{\beta \leqslant \alpha \leqslant 1} \min_{\alpha_p, \alpha_q} r_{gr}(\alpha, \alpha_p, \alpha_q), \sup_{\beta \leqslant \alpha \leqslant 1} \max_{\alpha_p, \alpha_q} r_{gr}(\alpha, \alpha_p, \alpha_q) \right] \\
&= \left[\inf_{\beta \leqslant \alpha \leqslant 1} \min_{\alpha_p, \alpha_q} (-1 + 2(1 - \beta)\alpha_p - (3 - 2\beta)\alpha_q), \sup_{\beta \leqslant \alpha \leqslant 1} \max_{\alpha_p, \alpha_q} (-1 + 2(1 - \beta)\alpha_p - (3 - 2\beta)\alpha_q) \right] \\
&= [-4 + 2\beta, 1 - 2\beta].
\end{aligned}$$

Definition 2.5 ([16]). If $g : [a, b] \subseteq \mathbf{R} \to \mathbf{R}_F$ include $n \in N$ distinct FNs $p_1, p_2, \ldots, p_n$, then the HMF of g is indicated by $H(g(x)) \triangleq g_{gr}(x, \beta, \alpha_g)$, and interpreted as $g_{gr} : [a, b] \times [0, 1] \times \underbrace{[0, 1] \times [0, 1] \times \cdots \times [0, 1]}_{n \text{ times}} \to [c, d] \subseteq \mathbf{R}$, in which $\alpha_g \triangleq (\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_n})$, where $\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_n}$, are the RDM variables related to p_1, p_2, ..., p_n, respectively.

Definition 2.6 ([16]). Let $p, q \in \mathbf{R}_F$. The function $\mathcal{D}_{gr} : \mathbf{R}_F \times \mathbf{R}_F \to \mathbf{R}^+ \cup \{0\}$, defined by

$$\mathcal{D}_{gr}(p, q) = \sup_{\beta} \max_{\alpha_p, \alpha_q} |p_{gr}(\beta, \alpha_p) - q_{gr}(\beta, \alpha_q)|,$$

is the granular distance (gr-distance) between two FNs p and q.

Proposition 2.3 ([16]). *$(\mathbf{R}_F, \mathcal{D}_{gr})$ is a metric space. The metric $\mathcal{D}_{gr}$ is called granular metric on $\mathbf{R}_F$.*

Theorem 2.1 ([16]). *$(\mathbf{R}_F, \mathcal{D}_{gr})$ is a complete metric space (CMS).*

Lemma 2.1 ([16]). *Suppose that $p, q, r \in \mathbf{R}_F$ and $\mu \in \mathbf{R}$, then the below results hold:*

(i) $\mathcal{D}_{gr}(p \oplus_{gr} r, q \oplus_{gr} r) = \mathcal{D}_{gr}(p, q)$;
(ii) $\mathcal{D}_{gr}(\mu \odot p, \mu \odot q) = |\mu| \mathcal{D}_{gr}(p, q)$.

Lemma 2.2. *Suppose that $p, q, r, s \in \mathbf{R}_F$, then*

$$\mathcal{D}_{gr}(p \oplus_{gr} q, r \oplus_{gr} s) \leqslant \mathcal{D}_{gr}(p, r) + \mathcal{D}_{gr}(q, s).$$

Proof. From Definition 2.6, we have

$$\begin{aligned}\mathcal{D}_{gr}(p \oplus_{gr} q, r \oplus_{gr} s) &= \sup_{\beta} \max_{\alpha_p, \alpha_q, \alpha_r, \alpha_s} |(p_{gr}(\beta, \alpha_p) + q_{gr}(\beta, \alpha_q)) - (r_{gr}(\beta, \alpha_r) + s_{gr}(\beta, \alpha_s))| \\ &= \sup_{\beta} \max_{\alpha_p, \alpha_q, \alpha_r, \alpha_s} |(p_{gr}(\beta, \alpha_p) - r_{gr}(\beta, \alpha_r)) + (q_{gr}(\beta, \alpha_q) - s_{gr}(\beta, \alpha_s))| \\ &\leqslant \sup_{\beta} \max_{\alpha_p, \alpha_r} |p_{gr}(\beta, \alpha_p) - r_{gr}(\beta, \alpha_r)| + \sup_{\beta} \max_{\alpha_q, \alpha_s} |(q_{gr}(\beta, \alpha_q) - s_{gr}(\beta, \alpha_s))| \\ &= \mathcal{D}_{gr}(p, r) + \mathcal{D}_{gr}(q, s).\end{aligned}$$

□

Definition 2.7 ([16]). Let $g : [b, c] \to \mathbf{R}_F$ be a FF. The function $g(t)$ is said to be continuous at $t = t_0$ if $g(t_0) \in \mathbf{R}_F$, which is subject to have following conditions.

(i) If $t_0 \in (b, c)$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $|t - t_0| < \delta_1 \implies \mathcal{D}_{gr}(g(t), g(t_0)) < \epsilon_1$, and write it as $\lim_{t \to t_0} g(t) = g(t_0)$.
(ii) If $t_0 = b$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $0 < t - b < \delta_1 \implies \mathcal{D}_{gr}(g(t), g(b)) < \epsilon_1$, and write it as $\lim_{t \to b^+} g(t) = g(b)$.
(iii) If $t_0 = c$, for all $\epsilon_1 > 0$, there exits $\delta_1 > 0$ such that $0 < c - t < \delta_1 \implies \mathcal{D}_{gr}(g(t), g(c)) < \epsilon_1$, and write it as $\lim_{t \to c^-} g(t) = g(c)$.

Definition 2.8 ([16]). Let $\mathbf{g} : [b, c] \to \mathbf{R}_F$, where $[b, c] \subset \mathbf{R}$ be the FF. If there exists $\frac{d_{gr}\mathbf{g}(t_0)}{dt} \in \mathbf{R}_F$, such that

$$\lim_{h \to 0} \frac{\mathbf{g}(t_0 + h) \ominus_{gr} \mathbf{g}(t_0)}{h} = \frac{d_{gr}\mathbf{g}(t_0)}{dt} = \mathbf{g}'_{gr}(t_0),$$

this limit is taken in the metric space $(\mathbf{R}_F, \mathcal{D}_{gr})$. Then g is said to be granular differentiable (gr-differentiable) at a point $t_0 \in [b, c]$. In particular, $\frac{d_{gr}\mathbf{g}(t_0)}{dt}$ is said to be the gr-derivative of g at the point t_0. If the gr-derivative exists for every point $t_0 \in [b, c]$, then $\mathbf{g}$ is gr-differentiable on $[b, c]$.

Theorem 2.2 ([16]). *Let $\mathbf{g} : [b, c] \to \mathbf{R}_F$. Then $\mathbf{g}$ is gr-differentiable if and only if its HMF is differentiable with respect to $t \in [b, c]$. Moreover,*

$$H(\frac{d_{gr}\mathbf{g}(t)}{dt}) = \frac{\partial \mathbf{g}_{gr}(t, \beta, \alpha_g)}{\partial t}.$$

Now, we define second and higher order gr-derivatives for FF.

Definition 2.9. Let $\mathbf{g} : [b, c] \to \mathbf{R}_F$, be the FF. If there exists $\frac{d^2_{gr}\mathbf{g}(t_0)}{dt^2} \in \mathbf{R}_F$, such that

$$\lim_{h\to 0} \frac{\mathbf{g}'(t_0 + h) \ominus_{gr} \mathbf{g}'(t_0)}{h} = \frac{d^2_{gr}\mathbf{g}(t_0)}{dt^2} = \mathbf{g}''_{gr}(t_0),$$

then $\mathbf{g}$ is said to be second order gr-differentiable at a point $t_0 \in [b, c]$.

Theorem 2.3. *Let $\mathbf{g} : [b, c] \to \mathbf{R}_F$. Then $\mathbf{g}$ is twice gr-differentiable if and only if its HMF is twice differentiable with respect to $t \in [b, c]$. Moreover,*

$$H\left(\frac{d^2_{gr}\mathbf{g}(t)}{dt^2}\right) = \frac{\partial^2 \mathbf{g}_{gr}(t, \beta, \alpha_g)}{\partial t^2}.$$

Proof. From the Definitions 2.9, and Theorem 2.2, we have

$$\begin{aligned}
H\left(\frac{d^2_{gr}\mathbf{g}(t)}{dt^2}\right) &= H\left(\lim_{h\to 0} \frac{\mathbf{g}'(t + h) \ominus_{gr} \mathbf{g}'(t)}{h}\right) \\
&= \lim_{h\to 0} H\left(\frac{\mathbf{g}'(t + h) \ominus_{gr} \mathbf{g}'(t)}{h}\right) \\
&= \lim_{h\to 0} \frac{H\left(\mathbf{g}'(t + h)\right) - H\left(\mathbf{g}'(t)\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(\frac{\partial \mathbf{g}_{gr}(t+h,\beta,\alpha_g)}{\partial t}\right) - \left(\frac{\partial \mathbf{g}_{gr}(t,\beta,\alpha_g)}{\partial t}\right)}{h} = \frac{\partial^2 \mathbf{g}_{gr}(t, \beta, \alpha_g)}{\partial t^2}.
\end{aligned}$$

□

Example 2.2. Suppose $\mathbf{g}(t) = e^t \odot \tilde{3} \oplus_{gr} t \odot \tilde{2}$, in which $\tilde{3} = (2, 3, 4)$, $\tilde{2} = (1, 2, 3)$. Then the HMF of $\mathbf{g}(t)$ is

$$\mathbf{g}_{gr}(t, \beta, \alpha_g) = [2 + \beta + 2(1 - \beta)\alpha_3]e^t + [1 + \beta + 2(1 - \beta)\alpha_2]t, \text{ where } \alpha_g = (\alpha_2, \alpha_3) \in [0, 1],$$

and

$$\frac{\partial \mathbf{g}_{gr}(t, \beta, \alpha_g)}{\partial t} = [2 + \beta + 2(1 - \beta)\alpha_3]e^t + [1 + \beta + 2(1 - \beta)\alpha_2].$$

Again

$$\frac{\partial^2 \mathbf{g}_{gr}(t, \beta, \alpha_g)}{\partial t^2} = [2 + \beta + 2(1 - \beta)\alpha_3]e^t.$$

Therefore, the granular function $\mathbf{g}_{gr}(t, \beta, \alpha_g)$ is twice differentiable with respect to t. Thus, from Theorem 2.3 the function $\mathbf{g}(t)$ is also twice differentiable. Taking the inverse HMF, we get

$$\left[\frac{\partial^2 \mathbf{g}_{gr}(t, \beta, \alpha_g)}{\partial t^2}\right]^\beta = [2 + \beta, 4 - \beta]e^t.$$

Using the β-level sets representation, we have

$$\frac{d^2_{gr}\mathbf{g}(t)}{dt^2} = \bigcup_\beta \left\{[2 + \beta, 4 - \beta]e^t\right\} = e^t \odot (2, 3, 4) = e^t \odot \tilde{3}.$$

Definition 2.10. Let $\mathbf{g} : [b, c] \to \mathbf{R}_F$, be the FF. If there exists $\frac{d^m_{gr}\mathbf{g}(t_0)}{dt^m} \in \mathbf{R}_F$, such that

$$\lim_{h\to 0} \frac{\mathbf{g}^{(m-1)}(t_0 + h) \ominus_{gr} \mathbf{g}^{(m-1)}(t_0)}{h} = \frac{d^m_{gr}\mathbf{g}(t_0)}{dt^m} = \mathbf{g}^{(m)}_{gr}(t_0),$$

then $\mathbf{g}$ is said to be m^{th} order gr-differentiable at a point $t_0 \in [b, c]$.

Theorem 2.4. *Let* $\mathbf{g} : [b, c] \to \mathbf{R}_F$. *Then* $\mathbf{g}$ *is m times gr-differentiable if and only if its HMF is m times differentiable with respect to* $t \in [b, c]$. *Moreover,*

$$H\left(\frac{d^m_{gr}\mathbf{g}(t)}{dt^m}\right) = \frac{\partial^m \mathbf{g}_{gr}(t, \beta, \alpha_g)}{\partial t^m}.$$

Proof. The proof is similar to the proof of Theorem 2.3. □

Definition 2.11 ([16]). Suppose that $\mathbf{g} : [b, c] \to \mathbf{R}_F$ is continuous and the HMF $H(\mathbf{g}(t)) = \mathbf{g}_{gr}(t, \beta, \alpha_f)$ is integrable on $[b, c]$. If there exists an FF $\mathbf{n} = \int_b^c \mathbf{g}(t)dt$ such that $H(\mathbf{n}) = \int_b^c H(\mathbf{g}(t))dt$, then $\mathbf{g}(t)$ is called gr-integrable on $[b, c]$.

Proposition 2.4 ([16]). *Assume that* $\mathbf{F} : [b, c] \to \mathbf{R}_F$ *is gr-differentiable and* $\mathbf{f}(t) = \frac{d_{gr}\mathbf{F}(t)}{dt}$ *is continuous on* $[b, c]$. *Then,* $\int_b^c \mathbf{f}(t)dt = \mathbf{F}(c) \ominus_{gr} \mathbf{F}(b)$.

Example 2.3. Consider a function $\mathbf{F} : [-3, 3] \to \mathbf{R}_F$, defined by $\mathbf{F}(t) = e^t \odot p \oplus_{gr} \frac{t^3}{3} \odot q \oplus_{gr} r$, with $p = (1, 2, 3)$, $q = (-3, -2, -1)$ and $r = (-1, 0, 1) \in \mathbf{R}_F$. Then,

$$H(\mathbf{F}(t)) = [1 + \beta + 2(1 - \beta)\alpha_p]e^t + [-3 + \beta + 2(1 - \beta)\alpha_q]\frac{t^3}{3} + [-1 + \beta + 2(1 - \beta)\alpha_r],$$

and

$$\frac{\partial \mathbf{F}_{gr}}{\partial t} = [1 + \beta + 2(1 - \beta)\alpha_p]e^t + [-3 + \beta + 2(1 - \beta)\alpha_q]t^2.$$

Let us take

$$\frac{d\mathbf{F}(t)}{dt} = \mathbf{g}(t),$$

it is continuous on [-3, 3] and gr-integrable. From Proposition 2.4, we have

$$\begin{aligned}\int_{-3}^{3} \mathbf{g}(t)dt = \mathbf{F}(3) \ominus_{gr} \mathbf{F}(-3) &= (e^3 \odot p \oplus_{gr} 9 \odot q \oplus_{gr} r) \ominus_{gr} (e^{-3} \odot p \oplus_{gr} (-9) \odot q \oplus_{gr} r)\\ &= (e^3 - e^{-3}) \odot p \oplus_{gr} 18 \odot q.\end{aligned}$$

Theorem 2.5 ([22]). *If* $\mathbf{g}$, $\mathbf{h}$ *are mappings from* $[b, c] \to \mathbf{R}_F$ *and are two gr-integrable FFs and* $l, m \in \mathbf{R}$, *then*

(i) $\int_b^c [l \odot \mathbf{g}(t) \oplus_{gr} m \odot \mathbf{h}(t)]dt = l \odot \int_b^c \mathbf{g}(t)dt \oplus_{gr} m \odot \int_b^c \mathbf{h}(t)dt$;
(ii) $\mathcal{D}_{gr}(\mathbf{g}, \mathbf{h})$ *is integrable;*
(iii) $\mathcal{D}_{gr}\left(\int_b^c \mathbf{g}(t)dt, \int_b^c \mathbf{h}(t)dt\right) \leqslant \int_b^c \mathcal{D}_{gr}(\mathbf{g}(t), \mathbf{h}(t))dt$;
(iv) $\int_b^c \mathbf{g}(t)dt = \int_b^a \mathbf{g}(t)dt \oplus_{gr} \int_a^c \mathbf{g}(t)dt$, *for each* $a \in (b, c)$.

Note 2.3 ([22]). If $\mathbf{g}, \mathbf{h} : [b, c] \to \mathbf{R}_F$, then,

$$\mathcal{D}_{gr}(\mathbf{g}(x), \mathbf{h}(x)) = \sup_{\beta} \max_{\alpha_g, \alpha_h} |\mathbf{g}_{gr}(x, \beta, \alpha_g) - \mathbf{h}_{gr}(x, \beta, \alpha_h)|,$$

where $x \in [b, c]$ and $\beta, \alpha_g, \alpha_h \in [0, 1]$.

The set of all continuous mappings from $[x_0, b]$ to $\mathbf{R}_F$ is denoted by $\mathbb{C}([x_0, b], \mathbf{R}_F)$, which is a CMS with the distance,

$$\mathcal{D}(y, z) = \sup_{x \in [x_0, b]} \left\{\mathcal{D}_{gr}(y(x), z(x))e^{-\mu x}\right\}, \text{ where } [x_0, b] \subset \mathbf{R}, \mu \in \mathbf{R} \text{ is a fixed real number.}$$

Denote $\mathbb{C}'([x_0, b], \mathbf{R}_F)$ be the set of all continuously gr-differentiable mappings from $[x_0, b]$ to $\mathbf{R}_F$. For $y, z \in \mathbb{C}'([x_0, b], \mathbf{R}_F)$, we define the distance

$$\mathcal{D}_1(y, z) = \mathcal{D}(y, z) + \mathcal{D}(y'_{gr}, z'_{gr}).$$

Lemma 2.3. $(\mathbb{C}'([x_0,b],\mathbf{R}_F),\mathcal{D}_1)$ *is a CMS.*

Proof. Let $\{s_n\}_{n\geqslant 1}\in\mathbb{C}'\big([x_0,b],\mathbf{R}_F\big)$ be a Cauchy sequence in $\big(\mathbb{C}'\big([x_0,b],\mathbf{R}_F\big),\mathcal{D}_1\big)$, then for each $\epsilon>0$, there exists $n_\epsilon\in N$ such that

$$\mathcal{D}_1(s_m,s_n)<\epsilon,\ m,n\geqslant n_\epsilon,\ \text{and}\ \mathcal{D}(s_m,s_n)+\mathcal{D}(s'_m,s'_n)<\epsilon.$$

Since $\{s_n\}_{n\geqslant 1}$ and $\{s'_n\}_{n\geqslant 1}$ are Cauchy sequences in CMS $\big(\mathbb{C}\big([x_0,b],\mathbf{R}_F\big),\mathcal{D}\big)$, then there exist $s,t\in\mathbb{C}\big([x_0,b],\mathbf{R}_F\big)$ such that $\{s_n\}\to s$ and $\{s'_n\}\to t$ as $n\to\infty$. That is

$$\mathcal{D}(s_n,s)\to 0\ \text{and}\ \mathcal{D}(s'_n,t)\to 0\ \text{as}\ n\to\infty. \tag{2.1}$$

Now, we will prove that $s\in\big(\mathbb{C}'\big([x_0,b],\mathbf{R}_F\big)$ and $s'=t$. For that we will verify that $s(x)=s(x_0)\oplus_{gr}\int_{x_0}^{x}t(\tau)d\tau$. Let us take $s_n(x)=s_n(x_0)\oplus_{gr}\int_{x_0}^{x}s'_n(\tau)d\tau$. Suppose that $\phi(x)=s(x_0)\oplus_{gr}\int_{x_0}^{x}t(\tau)d\tau$, then

$$\begin{aligned}
\mathcal{D}(s,\phi)&=\sup_{x\in[x_0,b]}\{\mathcal{D}_{gr}(s(x),\phi(x))e^{-\mu x}\}\\
&\leqslant\sup_{x\in[x_0,b]}\left\{\mathcal{D}_{gr}(s(x),s_n(x))e^{-\mu x}+\mathcal{D}_{gr}(s_n(x),\phi(x))e^{-\mu x}\right\}\\
&=\sup_{x\in[x_0,b]}\left\{\mathcal{D}_{gr}(s(x),s_n(x))e^{-\mu x}+\mathcal{D}_{gr}\left(s_n(x_0)\oplus_{gr}\int_{x_0}^{x}s'_n(\tau)d\tau,s(x_0)\oplus_{gr}\int_{x_0}^{x}t(\tau)d\tau\right)e^{-\mu x}\right\}\\
&\leqslant\mathcal{D}(s,s_n)+\sup_{x\in[x_0,b]}\left\{\mathcal{D}_{gr}(s_n(x_0),s(x_0))+\int_{x_0}^{x}\mathcal{D}_{gr}(s'_n(\tau),t(\tau))d\tau e^{-\mu x}\right\}\\
&\leqslant\mathcal{D}(s,s_n)+\mathcal{D}(s_n,s)e^{\mu x_0}e^{-\mu x}+\sup_{x\in[x_0,b]}\left(\int_{x_0}^{x}\mathcal{D}(s'_n,t)e^{\mu\tau}d\tau\right)e^{-\mu x}\\
&=\mathcal{D}(s,s_n)+\mathcal{D}(s_n,s)e^{-\mu(x-x_0)}+\mathcal{D}(s'_n,t)\sup_{x\in[x_0,b]}\left(\frac{1-e^{-\mu(x-x_0)}}{\mu}\right).
\end{aligned}$$

From (2.1), $\mathcal{D}(s,\phi)\to 0$ as $n\to\infty$. Therefore,

$$s(x)=\phi(x)=s(x_0)\oplus_{gr}\int_{x_0}^{x}t(\tau)d\tau,\quad\forall x\in[x_0,b].$$

That is $s'=t$ and $s\in\big(\mathbb{C}'\big([x_0,b],\mathbf{R}_F\big)$. Hence $\big(\mathbb{C}'\big([x_0,b],\mathbf{R}_F\big),\mathcal{D}_1\big)$ is a CMS.

3. Second order FDEs under gr-differentiability

Consider a non-linear second order FIVP under gr-differentiability

$$z''_{gr}(x)=g(x,z(x),z'_{gr}(x)), \tag{3.1}$$
$$z(x_0)=a_1,\ z'_{gr}(x_0)=a_2, \tag{3.2}$$

where $g:[x_0,b]\times\mathbf{R}_F\times\mathbf{R}_F\to\mathbf{R}_F$ is continuous and $a_1,a_2\in\mathbf{R}_F$. Taking gr-integral on both sides of (3.1) from x_0 to x_1, we have

$$z'_{gr}(x_1)=a_2\oplus_{gr}\int_{x_0}^{x_1}g(\tau,z(\tau),z'_{gr}(\tau))d\tau. \tag{3.3}$$

Again taking gr-integral on both sides of (3.3) from x_0 to x, we get

$$z(x)=a_1\oplus_{gr}a_2\odot(x-x_0)\oplus_{gr}\int_{x_0}^{x}\int_{x_0}^{x_1}g(\tau,z(\tau),z'_{gr}(\tau))d\tau dx_1.$$

□

The fixed point theory and their applications were presented in [1]. Now, we establish the following existence and uniqueness theorem for the solutions of FIVP (3.1)-(3.2) using Banch contraction mapping theorem.

Theorem 3.1. *Let* $g \in \mathbb{C}([x_0, b] \times \mathbf{R}_F \times \mathbf{R}_F, \mathbf{R}_F)$ *and* $\mathcal{D}_{gr}(g(x, y, y'_{gr}), g(x, z, z'_{gr})) \leqslant L\mathcal{D}_{gr}(y, z) + M\mathcal{D}_{gr}(y'_{gr}, z'_{gr})$*, for* $(x, z, z'_{gr}), (x, y, y'_{gr}) \in [x_0, b] \times \mathbf{R}_F \times \mathbf{R}_F$*, where* $L, M \geqslant 0$*, then the FIVP (3.1)-(3.2) have a unique solution.*

Proof. Let $\mathcal{S} = \mathbb{C}'([x_0, b], \mathbf{R}_F)$ with the metric

$$\mathcal{D}_1(y, z) = \mathcal{D}(y, z) + \mathcal{D}(y'_{gr}, z'_{gr}).$$

Define the operator $\mathbf{F} : \mathcal{S} \to \mathcal{S}$ as

$$\mathbf{F}z(x) = a_1 \oplus_{gr} a_2 \odot (x - x_0) \oplus_{gr} \int_{x_0}^{x}\int_{x_0}^{x_1} g(\tau, z(\tau), z'_{gr}(\tau))d\tau dx_1.$$

Consider,

$$\begin{aligned}
\mathcal{D}(\mathbf{F}y, \mathbf{F}z) &= \sup_{x\in[x_0,b]} \Big\{\mathcal{D}_{gr}\Big(a_1 \oplus_{gr} a_2 \odot (x - x_0) \oplus_{gr} \int_{x_0}^{x}\int_{x_0}^{x_1} g(\tau, y(\tau), y'_{gr}(\tau))d\tau dx_1,\\
&\qquad a_1 \oplus_{gr} a_2 \odot (x - x_0) \oplus_{gr} \int_{x_0}^{x}\int_{x_0}^{x_1} g(\tau, z(\tau), z'_{gr}(\tau))d\tau dx_1\Big) e^{-\mu x}\Big\}\\
&= \sup_{x\in[x_0,b]} \Big\{\mathcal{D}_{gr}\Big(\int_{x_0}^{x}\int_{x_0}^{x_1} g(\tau, y(\tau), y'_{gr}(\tau))d\tau dx_1, \int_{x_0}^{x}\int_{x_0}^{x_1} g(\tau, z(\tau), z'_{gr}(\tau))d\tau dx_1\Big) e^{-\mu x}\Big\}\\
&\leqslant \sup_{x\in[x_0,b]} \Big\{\int_{x_0}^{x}\int_{x_0}^{x_1} \mathcal{D}_{gr}\left(g(\tau, y, y'_{gr}), g(\tau, z, z'_{gr})\right) d\tau dx_1 e^{-\mu x}\Big\}\\
&\leqslant \sup_{x\in[x_0,b]} \Big\{\int_{x_0}^{x}\int_{x_0}^{x_1} \left[L\mathcal{D}_{gr}(y, z) + M\mathcal{D}_{gr}(y'_{gr}, z'_{gr})\right] d\tau dx_1 e^{-\mu x}\Big\}\\
&\leqslant \left[L\mathcal{D}(y, z) + M\mathcal{D}(y'_{gr}, z'_{gr})\right] \sup_{x\in[x_0,b]} \Big\{\int_{x_0}^{x}\int_{x_0}^{x_1} e^{\mu\tau} d\tau dx_1 e^{-\mu x}\Big\}\\
&\leqslant \max\{L, M\}\Big\{\sup_{x\in[x_0,b]}\Big(\frac{1 - e^{-\mu(x-x_0)} - \mu(x - x_0)e^{-\mu(x-x_0)}}{\mu^2}\Big)\Big\}\mathcal{D}_1(y, z).
\end{aligned}$$

Therefore,

$$\mathcal{D}(\mathbf{F}y, \mathbf{F}z) \leqslant \max\{L, M\}\Big\{\frac{1}{\mu^2}(1 - e^{-\mu(b-x_0)}[1 + \mu(b - x_0)])\Big\}\mathcal{D}_1(y, z). \tag{3.4}$$

Now, consider

$$\begin{aligned}
\mathcal{D}\left((\mathbf{F}y)'_{gr}, (\mathbf{F}z)'_{gr}\right) &= \sup_{x\in[x_0,b]} \Big\{\mathcal{D}_{gr}\Big(a_2 \oplus_{gr} \int_{x_0}^{x} g(\tau, y(\tau), y'_{gr}(\tau))d\tau, a_2 \oplus_{gr} \int_{x_0}^{x} g(\tau, z(\tau), z'_{gr}(\tau))d\tau\Big) e^{-\mu x}\Big\}\\
&= \sup_{x\in[x_0,b]} \Big\{\mathcal{D}_{gr}\Big(\int_{x_0}^{x} g(\tau, y(\tau), y'_{gr}(\tau))d\tau, \int_{x_0}^{x} g(\tau, z(\tau), z'_{gr}(\tau))d\tau\Big) e^{-\mu x}\Big\}\\
&\leqslant \sup_{x\in[x_0,b]} \Big\{\Big(\int_{x_0}^{x} \mathcal{D}_{gr}\left(g(\tau, y(\tau), y'_{gr}(\tau)), g(\tau, z(\tau), z'_{gr}(\tau))d\tau\right)\Big) e^{-\mu x}\Big\}\\
&\leqslant \sup_{x\in[x_0,b]} \Big\{\Big(\int_{x_0}^{x} \mathcal{D}_{gr}\left(L\mathcal{D}_{gr}(y, z) + M\mathcal{D}_{gr}(y'_{gr}, z'_{gr})\right) d\tau\Big) e^{-\mu x}\Big\}
\end{aligned}$$

$$\leqslant \max\{L, M\}\left[\mathcal{D}(y,z)+\mathcal{D}(y'_{gr}, z'_{gr})\right] \sup_{x\in[x_0,b]} \left\{\int_{x_0}^{x} e^{\mu\tau}d\tau \quad e^{-\mu x}\right\}$$
$$= \max\{L, M\}\mathcal{D}_1(y,z) \sup_{x\in[x_0,b]} \left\{\frac{(1-e^{-\mu(x-x_0)})}{\mu}\right\}.$$

Therefore,

$$\mathcal{D}\left((\mathbf{F}y)'_{gr}, (\mathbf{F}z)'_{gr}\right) \leqslant \max\{L, M\}\mathcal{D}_1(y,z)\left\{\frac{1}{\mu}\left(1-e^{-\mu(b-x_0)}\right)\right\}. \tag{3.5}$$

From (3.4) and (3.5), we get

$$\mathcal{D}_1(\mathbf{F}y, \mathbf{F}z) \leqslant k\mathcal{D}_1(y,z),$$

where $k = \max\{L, M\}\left\{\frac{1}{\mu^2}(1-e^{-\mu(b-x_0)}[1+\mu(b-x_0)]) + \frac{1}{\mu}\left(1-e^{-\mu(b-x_0)}\right)\right\}$. Clearly, $k \to 0$ as $\mu \to \infty$. Therefore, $k < 1$ for a suitable choose of $\mu > 0$. Thus, $\mathbf{F}$ is a contraction mapping. Hence, from the contraction mapping theorem, there exists a unique fixed point $z \in \mathbb{S}$ such that $Fz = z$ and it is the unique solution of FIVP (3.1)-(3.2). □

Example 3.1. Consider the linear second order FIVP under gr-differentiability,

$$z''_{gr}(x) = p \odot z'_{gr}(x) \oplus_{gr} q \odot z(x) \oplus_{gr} r(x), \tag{3.6}$$
$$z(x_0) = \mu_1, z'_{gr}(x_0) = \mu_2, \tag{3.7}$$

where $x \in [x_0, b]$ and $p, q \in \mathbf{R}$. Clearly, $g(x, z, z'_{gr}) = p \odot z'_{gr}(x) \oplus_{gr} q \odot z(x) \oplus_{gr} r(x)$ is fuzzy continuous on $[x_0, b]$. Now consider,

$$\begin{aligned}
\mathcal{D}_{gr}(g(x,y,y'_{gr}), g(x,z,z'_{gr})) &= \mathcal{D}_{gr}(p \odot y'_{gr}(x) \oplus_{gr} q \odot y(x) \oplus_{gr} r(x), p \odot z'_{gr}(x) \oplus_{gr} q \odot z(x) \oplus_{gr} r(x)) \\
&= \mathcal{D}_{gr}(p \odot y'_{gr}(x) \oplus_{gr} q \odot y(x), p \odot z'_{gr}(x) \oplus_{gr} q \odot z(x)) \\
&\leqslant \mathcal{D}_{gr}(p \odot y'_{gr}(x), p \odot z'_{gr}(x)) + \mathcal{D}_{gr}(q \odot y(x), q \odot z(x)) \\
&= |p|\mathcal{D}_{gr}(y'_{gr}(x), z'_{gr}(x)) + |q|\mathcal{D}_{gr}(y(x), z(x)),
\end{aligned}$$

for all $x \in [x_0, b], y, y'_{gr}, z, z'_{gr} \in \mathbf{R}_F$. Therefore, from Theorem 3.1, the FIVP (3.6)-(3.7) has one and only one solution.

4. Higher order FDEs under granular differentiability

In this section, we establish existence and uniqueness of solutions for the FIVP associated with m^{th} order FDE of the form

$$z^{(m)}_{gr}(x) = g(x, z(x), z'_{gr}(x), \ldots, z^{(m-1)}_{gr}(x)), x \in [x_0, b], \tag{4.1}$$
$$z(x_0) = a_1, z'_{gr}(x_0) = a_2, \ldots, z^{(m-1)}_{gr}(x_0) = a_m, \tag{4.2}$$

where $g : [x_0, b] \times \underbrace{\mathbf{R}_F \times \cdots \times \mathbf{R}_F}_{m \text{ times}} \to \mathbf{R}_F$ and $a_1, a_2, \ldots, a_m \in \mathbf{R}_F$.

Lemma 4.1. *Suppose that the space* $\mathbb{C}^m([x_0, b], \mathbf{R}_F) = \{z \in \mathbb{C}([x_0, b], \mathbf{R}_F) : z'_{gr}, z''_{gr}, \ldots, z^{(m)}_{gr} \in \mathbb{C}([x_0, b], \mathbf{R}_F)\}$, *with the distance*

$$\mathcal{D}_m(y,z) = \sum_{j=0}^{m} \mathcal{D}(y^{(j)}_{gr}, z^{(j)}_{gr}),$$

where $y^{(0)}_{gr} = y$, $z^{(0)}_{gr} = z$. *Then, for every* $m \in N$, $m \geqslant 0$, $\left(\mathbb{C}^m([x_0, b], \mathbf{R}_F), \mathcal{D}_m\right)$ *is a CMS.*

Proof. For $m \geqslant 2$ and letting $\{s_n\}_{n\geqslant 1}$ is a Cauchy sequence in $(\mathbb{C}^m([x_0, b], \mathbf{R}_F), \mathcal{D}_m)$, then $\{s_n\}_{n\geqslant 1}$, $\{s'_n\}_{n\geqslant 1}, \ldots, \{s_n^{(m)}\}_{n\geqslant 1}$ are Cauchy sequences in the $(\mathbb{C}([x_0, b], \mathbf{R}_F), \mathcal{D})$. So that, there exists $s, t_1, t_2, \ldots, t_m \in (\mathbb{C}([x_0, b], \mathbf{R}_F)$ such that $\{s_n\} \to s, \{s'_n\} \to t_1, \ldots, \{s_n^{(m)}\} \to t_m$ as $n \to \infty$. In a similar argument as in Lemma 2.3, we get $t'_{m-1} = t_m, t'_{m-2} = t_{m-1}, \ldots, t'_1 = t_2, s' = t_1$. It implies that $s, t_1, t_2, \ldots, t_m \in (\mathbb{C}([x_0, b], \mathbf{R}_F)$ and $s' = t_1, s'' = t_2, \ldots, s^{(m)} = t_m$. Therefore,

$$\mathcal{D}_m(s_n, s) = \mathcal{D}(s_n, s) + \mathcal{D}(s'_n, t_1) + \cdots + \mathcal{D}(s_n^{(m)}, t_m) \to 0 \text{ as } n \to \infty.$$

Thus, $(\mathbb{C}^m([x_0, b], \mathbf{R}_F), \mathcal{D}_m)$ is a CMS. □

Theorem 4.1. *The fuzzy function $z \in (\mathbb{C}^{(m-1)}([x_0, b], \mathbf{R}_F))$, is a solution of FIVP (4.1)-(4.2) if and only if z satisfies the integral equation*

$$\begin{aligned} z(x) = {} & a_1 \oplus_{gr} a_2 \odot (x - x_0) \oplus_{gr} a_3 \odot \int_{x_0}^{x} (x_1 - x_0) dx_1 \oplus_{gr} a_4 \odot \int_{x_0}^{x} \int_{x_0}^{x_1} (x_2 - x_0) dx_2 dx_1 \oplus_{gr} \cdots \\ & \oplus_{gr} a_m \odot \int_{x_0}^{x} \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_{m-3}} (x_{m-2} - x_0) dx_{m-2} dx_{m-3} \ldots dx_2 dx_1 \\ & \oplus_{gr} \int_{x_0}^{x} \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_{m-1}} g(\tau, z(\tau), z'_{gr}(\tau), \ldots, z_{gr}^{(m-1)}(\tau)) d\tau dx_{m-1} \ldots dx_1. \end{aligned}$$

Proof. The proof follows as similar lines as discussed in Section 3. Hence, we omit the proof. □

Theorem 4.2. *Let $g : [x_0, b] \times \underbrace{\mathbf{R}_F \times \cdots \times \mathbf{R}_F}_{m \text{ times}} \to \mathbf{R}_F$ and $a_1, a_2, \ldots, a_m \in \mathbf{R}_F$ be continuous and suppose that there exist $N_j \geqslant 0$, $j = 0, 1, 2, \ldots, m-1$ such that*

$$\mathcal{D}_{gr}(g(x, y, y_1, \ldots, y_{m-1}), g(x, z, z_1, z_2, \ldots, z_{m-1})) \leqslant \sum_{j=0}^{m-1} N_j \mathcal{D}_{gr}(y_j, z_j),$$

for all $x \in [x_0, b]$, where $y = y_0$, $y_1 = y'_{gr}$, $\ldots$, $y_{m-1} = y_{gr}^{(m-1)}$, $z = z_0$, $z_1 = z'_{gr}$, $\ldots$, $z_{m-1} = z_{gr}^{(m-1)} \in \mathbf{R}_F$. Then, the FIVP (4.1)-(4.2) has a unique solution on $[x_0, b]$.

Proof. Let $S_1 = (\mathbb{C}^{m-1}([x_0, b], \mathbf{R}_F), \mathcal{D}_{m-1})$ be a CMS with the metric

$$\mathcal{D}_{m-1}(y, z) = \sum_{j=0}^{m-1} \mathcal{D}(y_{gr}^{(j)}, z_{gr}^{(j)}),$$

where $y_{gr}^{(0)} = y, z_{gr}^{(0)} = z$. Define the operator $\mathbf{F} : S_1 \to S_1$ as

$$\begin{aligned} (\mathbf{F}z)(x) = {} & a_1 \oplus_{gr} a_2 \odot (x - x_0) \oplus_{gr} a_3 \odot \int_{x_0}^{x} (x_1 - x_0) dx_1 \oplus_{gr} a_4 \odot \int_{x_0}^{x} \int_{x_0}^{x_1} (x_2 - x_0) dx_2 dx_1 \oplus_{gr} \cdots \\ & \oplus_{gr} a_m \odot \int_{x_0}^{x} \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_{m-3}} (x_{m-2} - x_0) dx_{m-2} dx_{m-3} \ldots dx_1 \\ & \oplus_{gr} \int_{x_0}^{x} \int_{x_0}^{x_1} \cdots \int_{x_0}^{x_{m-1}} g(\tau, z(\tau), z'_{gr}(\tau) \ldots, z_{gr}^{(m-1)}(\tau)) d\tau dx_{m-1} \ldots dx_1. \end{aligned}$$

Consider,

$$\mathcal{D}(\mathbf{F}y, \mathbf{F}z) = \sup_{x \in [x_0, b]} \Bigg\{ \mathcal{D}_{gr} \Bigg(a_1 \oplus_{gr} a_2 \odot (x - x_0) \oplus_{gr} a_3 \odot \int_{x_0}^{x} (x_1 - x_0) dx_1$$

$$
\begin{aligned}
&\oplus_{gr}\ a_4 \odot \int_{x_0}^{x}\int_{x_0}^{x_1}(x_2-x_0)dx_2dx_1 \oplus_{gr} \dots \\
&\oplus_{gr}\ a_m \odot \int_{x_0}^{x}\int_{x_0}^{x_1}\dots\int_{x_0}^{x_{m-3}}(x_{m-2}-x_0)dx_{m-2}dx_{m-3}\dots dx_2dx_1 \\
&\oplus_{gr} \int_{x_0}^{x}\int_{x_0}^{x_1}\dots\int_{x_0}^{x_{m-1}} g(\tau, y(\tau),\dots,y_{gr}^{(m-1)}(\tau))d\tau dx_{m-1}\dots dx_1, \\
&a_1 \oplus_{gr} a_2 \odot (x-x_0) \oplus_{gr} a_3 \odot \int_{x_0}^{x}(x_1-x_0)dx_1 \\
&\oplus_{gr}\ a_4 \odot \int_{x_0}^{x}\int_{x_0}^{x_1}(x_2-x_0)dx_2dx_1 \oplus_{gr} \dots \\
&\oplus_{gr}\ a_m \odot \int_{x_0}^{x}\int_{x_0}^{x_1}\dots\int_{x_0}^{x_{m-3}}(x_{m-2}-x_0)dx_{m-2}dx_{m-3}\dots dx_1 \\
&\left.\left.\oplus_{gr} \int_{x_0}^{x}\int_{x_0}^{x_1}\dots\int_{x_0}^{x_{m-1}} g(\tau, z(\tau),\dots,z_{gr}^{(m-1)}(\tau))d\tau dx_{m-1}\dots dx_1\right) e^{-\mu x}\right\} \\
= &\sup_{x\in[x_0,b]}\left\{\mathcal{D}_{gr}\left(\int_{x_0}^{x}\dots\int_{x_0}^{x_{m-1}} g(\tau, y(\tau),\cdots,y_{gr}^{(m-1)}(\tau))d\tau dx_{m-1}\dots dx_1,\right.\right. \\
&\left.\left.\int_{x_0}^{x}\dots\int_{x_0}^{x_{m-1}} g(\tau, z(\tau),\cdots,z_{gr}^{(m-1)}(\tau))d\tau dx_{m-1}\dots dx_2dx_1\right) e^{-\mu x}\right\} \\
\leqslant &\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{m-1}}\mathcal{D}_{gr}(g(\tau, y(\tau),\dots,y_{gr}^{(m-1)}(\tau)),\right. \\
&\left. g(\tau, z(\tau),\dots,z_{gr}^{(m-1)}(\tau)))d\tau dx_{m-1}\dots dx_2dx_1\right\} \\
= &\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{m-1}}\sum_{j=1}^{m}N_j\mathcal{D}_{gr}(y_{gr}^{(j-1)}, z_{gr}^{(j-1)})d\tau dx_{m-1}\dots dx_2dx_1\right\} \\
\leqslant &\sum_{j=0}^{m-1}N_j\mathcal{D}(y_{gr}^{(j)}, z_{gr}^{(j)})\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{m-1}}e^{\mu\tau}d\tau dx_{m-1}\dots dx_1\right\}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\mathcal{D}((\mathbf{F}y)'_{gr}, (\mathbf{F}z)'_{gr}) &\leqslant \sum_{j=0}^{m-1}N_j\mathcal{D}(y_{gr}^{(j)}, z_{gr}^{(j)})\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\cdots\int_{x_0}^{x_{m-2}}e^{\mu\tau}d\tau dx_{m-2}\dots dx_1\right\}, \\
&\ \vdots \\
\mathcal{D}((\mathbf{F}y)_{gr}^{(m-1)}, (\mathbf{F}z)_{gr}^{(m-1)}) &\leqslant \sum_{j=0}^{m-1}N_j\mathcal{D}(y_{gr}^{(j)}, z_{gr}^{(j)})\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}e^{\mu\tau}d\tau\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathcal{D}_{m-1}(\mathbf{F}y, \mathbf{F}z) = &\sum_{j=0}^{m-1}\mathcal{D}((\mathbf{F}y)_{gr}^{(j)}, (\mathbf{F}z)_{gr}^{(j)}) \\
\leqslant &\sum_{j=0}^{m-1}N_j\mathcal{D}(y_{gr}^{(j)}, z_{gr}^{(j)})\left[\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\cdots\int_{x_0}^{x_{m-1}}e^{\mu\tau}d\tau dx_{m-1}\dots dx_1\right\}\right. \\
&\left.+\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{m-2}}e^{\mu\tau}d\tau dx_{m-2}\dots dx_1\right\}+\cdots+\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}e^{\mu\tau}d\tau\right\}\right]
\end{aligned}
$$

$$\leqslant \max\{N_0, N_1, \ldots, N_{m-1}\}\mathcal{D}_{m-1}(y,z)\sum_{j=1}^{m}\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\cdots\int_{x_0}^{x_{m-j}} e^{\mu\tau}d\tau dx_{m-j}\ldots dx_1\right\}.$$

Since

$$\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x} e^{\mu\tau}d\tau\right\} = \frac{1}{\mu}(1-e^{-\mu(x-x_0)}) \leqslant \frac{1}{\mu}(1-e^{-\mu(b-x_0)}),$$

$$\begin{aligned}\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\int_{x_0}^{x_1} e^{\mu\tau}d\tau dx_1\right\} &= \sup_{x\in[x_0,b]}\left\{\frac{1}{\mu^2}(1-e^{-\mu(x-x_0)}[1+\mu(x-x_0)])\right\}\\ &\leqslant \frac{1}{\mu^2}(1-e^{-\mu(b-x_0)}[1+\mu(b-x_0)]),\\ &\ \ \vdots\end{aligned}$$

$$\begin{aligned}&\sup_{x\in[x_0,b]}\left\{e^{-\mu x}\int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{m-1}} e^{\mu\tau}d\tau dx_{m-1}\cdots dx_1\right\}\\ &= \sup_{x\in[x_0,b]}\left\{\frac{1}{\mu^m}\left(1-e^{-\mu(x-x_0)}\left[1+\frac{\mu(x-x_0)}{1!}+\ldots+\frac{\mu^{m-1}(x-x_0)^{m-1}}{(m-1)!}\right]\right)\right\}\\ &\leqslant \frac{1}{\mu^m}\left(1-e^{-\mu(b-x_0)}\left[1+\frac{\mu(b-x_0)}{1!}+\ldots+\frac{\mu^{m-1}(b-x_0)^{m-1}}{(m-1)!}\right]\right)\\ &= \frac{1}{\mu^m}\left(1-e^{-\mu(b-x_0)}\sum_{i=0}^{m-1}\frac{\mu^i}{i!}(b-x_0)^i\right),\end{aligned}$$

then

$$\mathcal{D}_{m-1}(\mathbf{F}y,\mathbf{F}z) \leqslant k\mathcal{D}_{m-1}(y,z),$$

where $k = \max\{N_0, N_1, \ldots, N_{m-1}\}\sum_{j=1}^{m}\frac{1}{\mu^j}\left[1-e^{-\mu(b-x_0)}\sum_{i=0}^{j-1}\frac{\mu^i}{i!}(b-x_0)^i\right]$. Clearly, $k \to 0$ as $\mu \to \infty$. Therefore, $k < 1$ for suitable choose of $\mu > 0$. Thus, $\mathbf{F}$ is a contraction mapping. Hence, from the contraction mapping theorem there exists a unique fixed point in $\mathbb{C}^{m-1}([x_0,b],\mathbf{R}_F)$ such that $(\mathbf{F}z)(x) = z(x)$, which is a unique solution of FIVP (4.1)-(4.2). □

5. A working method for solving FIVPs under gr-differentiability

Consider the following m^{th} order FIVP under gr-differentiability

$$z_{gr}^{(m)}(x) = g(x, z(x), \ldots, z_{gr}^{(m-1)}(x)), \tag{5.1}$$

$$z(x_0) = a_1, \ldots, z_{gr}^{(m-1)}(x_0) = a_m, \tag{5.2}$$

where $a_1, a_2, \ldots a_m \in \mathbf{R}_F$ and g is a fuzzy continuous function. The following algorithm describes the procedure to compute β-cut solution of FIVP (5.1)-(5.2), provided solution exists.

Step 1: Applying HMF on both sides of (5.1) and (5.2), we get

$$\frac{\partial^m z_{gr}(x,\beta,\alpha_z)}{\partial x^m} = g_{gr}(x,\beta,\alpha_g), \tag{5.3}$$

$$z_{gr}(x_0) = a_{1gr}(\beta,\alpha_1), \ldots, \frac{\partial^{m-1}z(x_0)}{\partial x^{m-1}} = a_{mgr}(\beta,\alpha_m), \tag{5.4}$$

where $\alpha_z = (\alpha_g, \alpha_1, \alpha_2, \ldots, \alpha_m)$ and β, α_g, α_1, α_2, …, $\alpha_m \in [0,1]$. Here, (5.3) is an m^{th} order partial differential equation with single independent variable x. Therefore, (5.3) and (5.4) can be taken as an initial value problem with ordinary derivatives.

Step 2: Solving (5.3) and (5.4), we get the solution granule

$$H(z(x)) = z_{gr}(x, \beta, \alpha_z). \tag{5.5}$$

Step 3: Applying inverse HMF on both sides of (5.5), we get

$$[z(x)]^{\beta} = [\inf_{\beta\leqslant\alpha\leqslant 1} \min_{\alpha_z} z_{gr}(x, \alpha, \alpha_z), \sup_{\beta\leqslant\alpha\leqslant 1} \max_{\alpha_z} z_{gr}(x, \alpha, \alpha_z)],$$

which is the required β-cut solution of FIVP (5.1)-(5.2).

Example 5.1. Consider the simple FIVP as in Example 4.1 of [2]

$$\tilde{1}z''(x) = \tilde{2},\ 0 \leqslant x \leqslant 2, \tag{5.6}$$
$$z(0) = \tilde{0}, z'(0) = \tilde{1}, \tag{5.7}$$

where β-cut set of initial values are $[\tilde{0}]^{\beta} = [\beta-1, 1-\beta]$, $[\tilde{1}]^{\beta} = [\beta, 2-\beta]$ and β-cut set of coefficients are $[\tilde{1}]^{\beta} = [1, 2-\beta]$, $[\tilde{2}]^{\beta} = [1+\beta, 3-\beta]$. This problem doesn't have a fuzzy solution using the algorithm described in [2].

Now, we apply our proposed method to solve this problem using granular differentiability. The FIVP (5.6)-(5.7) is taken as

$$\tilde{1}z''_{gr}(x) = \tilde{2}, \tag{5.8}$$
$$z(0) = \tilde{0}, z'_{gr}(0) = \tilde{1}. \tag{5.9}$$

Here, $g(x, z(x), z'_{gr}(x)) = \frac{\tilde{2}}{\tilde{1}}$ is a fuzzy constant. Clearly, g is continuous and satisfies Lipschitz condition with $L = M = 0$. From Theorem 3.1, the FIVP (5.6)-(5.7) has a unique solution. To find the solution we use the proposed algorithm.

Taking HMF on both sides of (5.8) and (5.9), we have

$$[\tilde{1}]_{gr} \frac{\partial^2 z_{gr}(x, \beta, \alpha_z)}{\partial x^2} - [\tilde{2}]_{gr} = 0, \text{ where } [\tilde{1}]_{gr} = [1 + (1-\beta)\alpha_2] \text{ and } [\tilde{2}]_{gr} = 1 + \beta + 2(1-\beta)\alpha_3, \tag{5.10}$$
$$z_{gr}(0) = \beta - 1 + 2(1-\beta)\alpha_0,\ z'_{gr}(0) = \beta + 2(1-\beta)\alpha_1, \tag{5.11}$$

where $\alpha_z = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\beta, \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in [0,1]$.

The solution of initial value problem (IVP) (5.10)-(5.11) is

$$z_{gr}(x, \beta, \alpha_0, \alpha_1, \alpha_2, \alpha_3) = \beta - 1 + 2(1-\beta)\alpha_0 + (\beta + 2(1-\beta)\alpha_1)x + \left[\frac{(\beta + 1 + 2(1-\beta)\alpha_3)}{1 + (1-\beta)\alpha_2}\right]\frac{x^2}{2}. \tag{5.12}$$

Applying inverse HMF on both sides of (5.12), we get

$$\begin{aligned}[z(x)]^{\beta} &= [\inf_{\beta\leqslant\alpha\leqslant 1} \min_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} z_{gr}(x, \alpha, \alpha_0, \alpha_1, \alpha_2, \alpha_3), \sup_{\beta\leqslant\alpha\leqslant 1} \max_{\alpha_0,\alpha_1,\alpha_2,\alpha_3} z_{gr}(x, \alpha, \alpha_0, \alpha_1, \alpha_2, \alpha_3)] \\ &= \left[\beta - 1 + \beta x + (1+\beta)\frac{x^2}{2}, 1 - \beta + (2-\beta)x + \left(\frac{3-\beta}{2-\beta}\right)\frac{x^2}{2}\right].\end{aligned}$$

The β-cut solution is computed using MATLAB and is depicted in Figure 1.

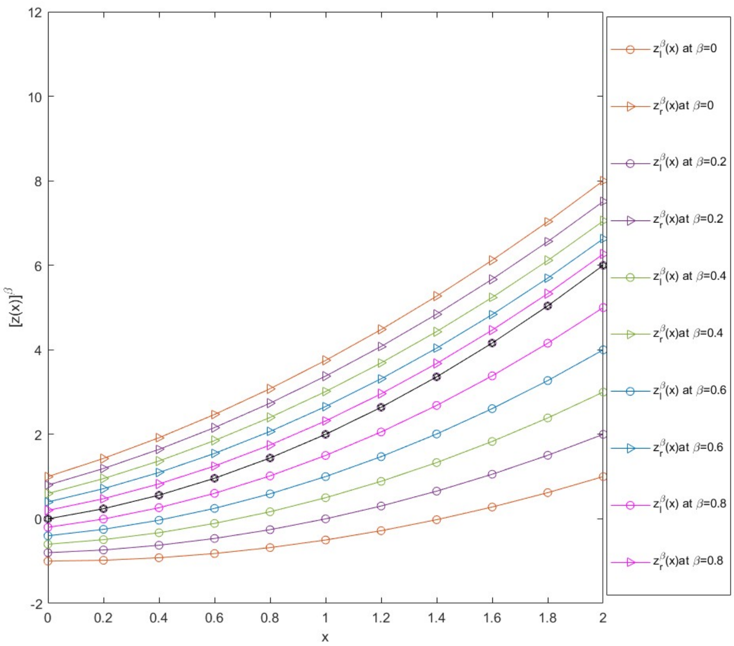

Figure 1: The black curve represents $z(x)$ at $\beta = 1$.

Example 5.2. Consider the non-linear FIVP as in Example 4 of [4],

$$z''(x) = -(z'(x))^2, \; 0 \leqslant x \leqslant 3, \quad z(0) = \tilde{1}, \; z'(0) = \tilde{2}. \tag{5.13}$$

The β-cut set of initial values are $[\tilde{1}]^\beta = [\beta, 2-\beta]$, $[\tilde{2}]^\beta = [1+\beta, 3-\beta]$. In [4], the authors obtained approximate β-cut solution using one-step hybrid block method. Now, we apply our proposed method to solve this problem using granular differentiability. The FIVP (5.13) can be taken as

$$z''_{gr}(x) = -(z'_{gr}(x))^2, \quad z(0) = \tilde{1}, \; z'_{gr}(0) = \tilde{2}. \tag{5.14}$$

Since $z(x)$ is fuzzy continuous, then $g(x, z(x), z'_{gr}(x)) = -(z'_{gr}(x))^2$ is also fuzzy continuous and satisfies Lipschitz condition with $L = 0$ and $M = \sup_{x \in J, \beta} \max_{\alpha_1, \alpha_2} |z'_{gr}(x, \beta, \alpha_1) + y'_{gr}(x, \beta, \alpha_2)| \geqslant 0$, where $J = [0, b]$ is a compact interval, b is a finite real number and $\alpha_1, \alpha_2 \in [0, 1]$. From Theorem 3.1, the FIVP (5.14) has a unique solution. Taking the HMF on both sides of (5.14), we have

$$\frac{\partial^2 z_{gr}(x, \beta, \alpha_z)}{\partial x^2} = -\left(\frac{\partial z_{gr}(x, \beta, \alpha_z)}{\partial x}\right)^2, \tag{5.15}$$

$$z_{gr}(0) = \beta + 2(1-\beta)\alpha_1, \; z'_{gr}(0) = 1 + \beta + 2(1-\beta)\alpha_2, \tag{5.16}$$

where $\alpha_z = (\alpha_1, \alpha_2)$ and $\beta, \alpha_1, \alpha_2 \in [0, 1]$. The solution of IVP (5.15)-(5.16) is

$$z_{gr}(x, \beta, \alpha_1, \alpha_2) = \ln\left[e^{[\beta + 2(1-\beta)\alpha_1]}(x[1 + \beta + 2(1-\beta)\alpha_2] + 1)\right]. \tag{5.17}$$

By taking the inverse HMF on both sides of (5.17), we get

$$\begin{aligned} [z(x)]^\beta &= \left[\inf_{\beta \leqslant \alpha \leqslant 1} \min_{\alpha_1, \alpha_2} z_{gr}(x, \alpha, \alpha_1, \alpha_2), \sup_{\beta \leqslant \alpha \leqslant 1} \max_{\alpha_1, \alpha_2} z_{gr}(x, \alpha, \alpha_1, \alpha_2)\right] \\ &= \left[\ln\left(e^\beta(x(1+\beta)+1)\right), \ln\left(e^{2-\beta}(x(3-\beta)+1)\right)\right], \end{aligned}$$

is the β-cut solution and depicted in Figure 2 using MATLAB.

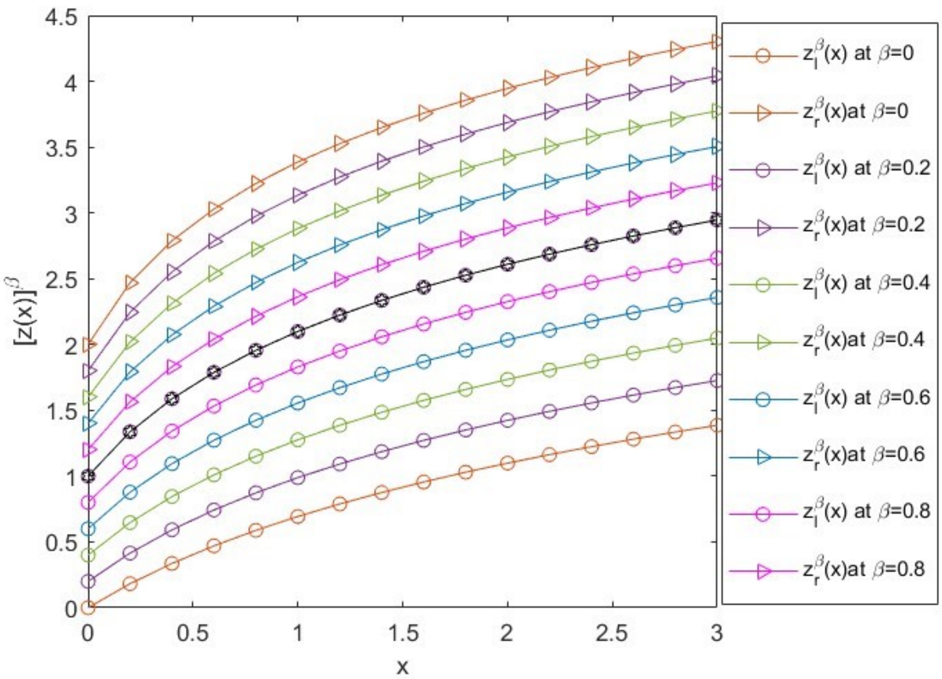

Figure 2: The black curve represents $z(x)$ at $\beta = 1$.

Example 5.3. Consider the fourth order linear FIVP as in Example 1 of [11],

$$z^{(4)}(x) = z(x),\ 0 \leqslant x \leqslant 1, \quad z(0) = z'(0) = z''(0) = z'''(0) = z^{(4)}(0) = \tilde{0}. \tag{5.18}$$

In [11], authors obtain multiple fuzzy solutions for FIVP (5.18) using Taylor series method under gH-differentiability. Now, we use the concept of granular differentiability. The granular FIVP of (5.18) is

$$z_{gr}^{(4)}(x) = z(x), \quad z(0) = z'_{gr}(0) = \cdots = z_{gr}^{(4)}(0) = \tilde{0}. \tag{5.19}$$

Clearly, $g(x, z(x), z_{gr}(x), z'_{gr}(x), z''_{gr}(x), z'''_{gr}(x)) = z(x)$ is fuzzy continuous and satisfies Lipschitz condition with $N_0 = 1$ and $N_i = 0$, $i = 1, 2, 3$. Therefore, from Theorem 4.2, the FIVP (5.19) has a unique solution. Taking HMF on both sides of (5.19), we have

$$\frac{\partial^4 z_{gr}(x, \beta, \alpha_z)}{\partial x^4} = z_{gr}(x, \beta, \alpha_z), \tag{5.20}$$

$$z_{gr}(0) = z'_{gr}(0) = z''_{gr}(0) = z'''_{gr}(0) = z_{gr}^{(4)}(0) = -1 + \beta + 2(1-\beta)\alpha_0, \tag{5.21}$$

where $\beta, \alpha_z = \alpha_0 \in [0, 1]$. The solution of IVP (5.20)-(5.21) is

$$z_{gr}(x, \beta, \alpha_0) = e^x\left[-1 + \beta + 2(1-\beta)\alpha_0\right]. \tag{5.22}$$

By applying the inverse HMF on both sides of (5.22), we get

$$[z(x)]^\beta = \left[\inf_{\beta \leqslant \alpha \leqslant 1} \min_{\alpha_0} z_{gr}(x, \alpha, \alpha_0),\ \sup_{\beta \leqslant \alpha \leqslant 1} \max_{\alpha_0} z_{gr}(x, \alpha, \alpha_0)\right] = \left[e^x(\beta - 1), e^x(1-\beta)\right],$$

is the β-cut solution and depicted in Figure 3 using MATLAB.

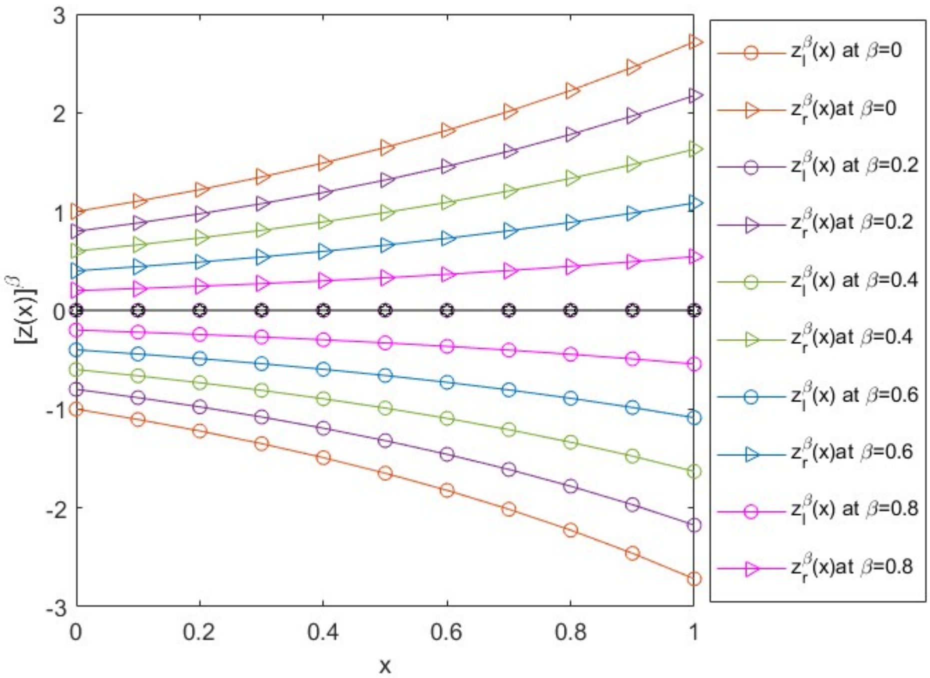

Figure 3: The black curve represents $z(x)$ at $\beta = 1$.

6. Conclusions

The proposed results of this paper are useful for testing and determine solutions for FIVPs. The granular differentiability for the fuzzy function is extended to second and higher-order derivatives. The second and higher-order FIVPs are investigated using the granular differentiability. Sufficient conditions are established for second and higher-order FIVPs. An algorithm is developed to solve the FIVPs. Some examples are given to illustrate the applicability and effectiveness of our method. In the future, we extend this work for fuzzy boundary value problems and fuzzy partial differential equations. Also, investigate the modeling of these problems in real-life applications.

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