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Infinite rank solution for conformable degenerate abstract Cauchy problem in Hilbert spaces

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Abstract

In this paper, we find an infinite rank solution of a conformable abstract Cauchy problem. The involved derivative is the conformable one. The main idea of the proofs are based on the theory of tensor product of Banach spaces.

Keywords: Tensor product of Banach spaces, conformable derivative, abstract Cauchy problem.

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1. Introduction

Fractional calculus is a generalization to derivatives and integrals of integer order to non-integer orders. There are several kinds of fractional derivatives, the most common ones are the Riemann-Liouville and Caputo definitions. Such classical (global) fractional derivatives have been used in differential equations, but in the case of local fractional derivatives, this type of research is very limited.

Let X be a Banach space, $C(I)$ be the Banach space of all real valued continuous functions on $I = [0, 1]$, under the sup-norm, and $C(I, X)$ be the Banach space of all continuous functions defined on I with values on X. Consider the fractional abstract Cauchy problem

$$\begin{cases} Bu^{(\alpha)}(t) = Au(t) + f(t)z, & 0 < t \leqslant 1, \\ u(0) = x. \end{cases}$$

Here u is a continuously α-differentiable function from I to X and A, B are densely defined linear operators on the codomain of u. This problem is called degenerate, if the operator B is not invertible. Recently, these types of fractional abstract Cauchy problems have been studied using different methods, see [2, 3]. Moreover, similar results where the derivative is the classical one can be found in [22, 23].

We would like to solve the above fractional abstract Cauchy problem using tensor product technique. In fact, we are looking for an infinite rank solution, i.e., a solution of the form $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i$, where

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$\{\delta_1, \delta_2, \ldots\}$ is the natural basis of ℓ^2. The derivative in the above problem is the conformable derivative, precisely: for $f : [0; \infty) \to \mathbb{R}$ and $0 < \alpha \leqslant 1$, the conformable fractional derivative of f of order α is defined by

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

We denote $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$. We say f is α-differentiable if the conformable fractional derivative of f of order α exists.

For all $t > 0$, if f is α-differentiable on $(0, b)$, where $b > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then we define $f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

For $0 < \alpha \leqslant 1$ and f, g be α-differentiable at a point $t > 0$, we have the following properties:

(1) $D_\alpha(af + bg) = aD_\alpha(f) + bD_\alpha(g)$, for all $a, b \in \mathbb{R}$;
(2) $D_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$;
(3) $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$;
(4) $D_\alpha(\frac{f}{g}) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$;
(5) $D_\alpha(\lambda) = 0$, for all λ is constant function;
(6) if f is differentiable, then $D_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

The α-fractional integral of a function f starting from $a \geqslant 0$ is:

$$I_\alpha^a(f(t)) = I_1^a(t^{\alpha-1} f(t)) = \int_a^t \frac{f(s)}{s^{1-\alpha}} ds.$$

For more on conformable fractional derivative we refer to [1, 4–10, 12–14, 16, 17, 19–21].

2. Basics of tensor product in Banach spaces

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper. Firstly, we recall some basic concepts of the tensor product in Banach spaces. Let X and Y be Banach spaces, and let X^* denote the dual of X. For $x \in X$ and $y \in Y$ define the map $x \otimes y : X^* \to Y$ with $x \otimes y(x^*) = \langle x, x^* \rangle y$, for all $x^* \in X^*$. We know that $x \otimes y$ is a bounded linear operator and $\| x \otimes y \| = \| x \| \| y \|$, the element $x \otimes y$ is called an atom. The set $X \otimes Y = \text{span}\{x \otimes y : x \in X \text{ and } y \in Y\}$ is a subspace of $L(X^*, Y)$, the space of all bounded linear operators from X^* into Y.

Lemma 2.1 ([15]). *Let $x_1 \otimes y_1$ and $x_2 \otimes y_2$ be two nonzero atoms in $X \otimes Y$ such that*

$$x_1 \otimes y_1 = x_2 \otimes y_2.$$

Then $x_1 = x_2$ and $y_1 = y_2$.

Lemma 2.2 ([15]). *If x_1, x_2 and x_3 are in X and y_1, y_2, and y_3 are in Y such that*

$$x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3.$$

Then either $x_1 = x_2$ or $y_1 = y_2$.

Let us now define some important norms on $X \otimes Y$.

2.1. Injective norm

For $T = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ define

$$\|T\|_\vee = \sup_{\|x^*\|=1} |\sum_{i=1}^n \langle x^*, x_i \rangle y_i|.$$

So, $\| \cdot \|_\vee$ is just the operator norm on $L(X^*, Y)$. This is called the injective norm of T. The space

$(X \otimes Y, \|\cdot\|_{\vee})$ need not be complete. Let $X \overset{\vee}{\otimes} Y$ denote the completion of $(X \otimes Y, \|\cdot\|_{\vee})$ and it is called the completed injective tensor product of X with Y. One of the nice results associated with the injective norm is the following.

Theorem 2.3 ([18]). *For any compact Housdorff space* K, *and any Banach space* X, $C(K, X)$ *is isometrically isomorphic to* $C(K) \overset{\vee}{\otimes} X$. *In particular,* $C(K \times S) = C(K) \overset{\vee}{\otimes} C(S)$, *for any compact metric spaces* S *and* K.

2.2. The projective norm

For $T \in X \otimes Y$, the projective norm of T is defined by

$$\|T\|_{\wedge} = \{\inf \textstyle\sum_{i=1}^{n} \|x_i\| \|y_i\| : \text{where infimum taken over all representations of } T\}.$$

The space $(X \otimes Y, \|\cdot\|_{\wedge})$ need not be complete. Let $X \overset{\wedge}{\otimes} Y$ be the completion of $(X \otimes Y, \|\cdot\|_{\wedge})$ and it is called the completed projective tensor product of X with Y.

Theorem 2.4 ([18]). *Let* I *and* J *be two compact intervals, then*

$$L^1(I \times J) = L^1(I) \overset{\wedge}{\otimes} L^1(J),$$

where $L^1(I \times J)$ *is the Banach space of Lebesgue integrable.*

For more on tensor product of Banach spaces we refer to [11, 15, 18].

Let ℓ^2 be the Hilbert space of square summable sequences. Then we define

$$C^{(\alpha)}(I, \ell^2) = \{u : I \to \ell^2 : u \text{ is continuously } \alpha\text{-differentiable}\}.$$

Any function in $C^{(\alpha)}(I, \ell^2)$ can be written in the form $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i$, where $\{\delta_1, \delta_2, \ldots\}$ is the natural basis of ℓ^2. We don't guarantee that $\sum_{i=1}^{\infty} \|u_i\|_{\infty} < \infty$, so we introduce the following subspace of functions in $C(I, \ell^2)$:

$$\Sigma = \{u \in C^{(\alpha)}(I, \ell^2) \text{ such that } u = \textstyle\sum_{i=1}^{\infty} u_i \otimes \delta_i \text{ and } \|u\| = \sum_{i=1}^{\infty} \|u_i\|_{\infty} + \|u_i^{(\alpha)}\|_{\infty} < \infty\}.$$

Lemma 2.5. *Let* f_n *be* α*-differentiable on* (a, b) *such that* $a > 0$. *If* f_n *converges to* f *and* $f_n^{(\alpha)}$ *converges uniformly to* g, *then* f *is* α*-differentiable and* $g = f^{(\alpha)}$.

Proof. Firstly we observe that g is well defined and continuous since $f_n^{(\alpha)}$ converge uniformly to g. Let $x \in (a, b)$ and fix $x_0 \in (a, x)$, we have

$$f_n(x) - f_n(x_0) = \int_{x_0}^{x} \frac{f_n^{(\alpha)}(t)}{t^{1-\alpha}} dt.$$

Taking the limit of both sides as $n \to \infty$, we get

$$f(x) - f(x_0) = \int_{x_0}^{x} \frac{g(t)}{t^{1-\alpha}} dt,$$

where $\lim_{n\to\infty} f_n(x_0) = f(x_0)$, so f is α-differentiable and $f^{(\alpha)}(x) = g(x)$. ☐

Lemma 2.6. Σ *with the norm* $\|u\| = \sum_{i=1}^{\infty} \|u_i\|_{\infty} + \|u_i^{(\alpha)}\|_{\infty}$ *is a Banach space.*

Proof. Let $f_n = \sum_{i=1}^{\infty} f_{n_i} \otimes \delta_i$ be a sequence in Σ. Assume that $\sum_{n=1}^{\infty} \|f_n\| < \infty$. Then

$$\sum_{n=1}^{\infty} \|f_n\| = \sum_{n=1}^{\infty} (\sum_{i=1}^{\infty} \|f_{n_i}\|_\infty + \sum_{i=1}^{\infty} \|f_{n_i}^{(\alpha)}\|_\infty) < \infty = \sum_{i=1}^{\infty} (\sum_{n=1}^{\infty} \|f_{n_i}\|_\infty + \sum_{n=1}^{\infty} \|f_{n_i}^{(\alpha)}\|_\infty) < \infty.$$

So, $\sum_{n=1}^{\infty} \|f_{n_i}\|_\infty < \infty$ and $\sum_{n=1}^{\infty} \|f_{n_i}^{(\alpha)}\|_\infty < \infty$ in $C(I)$ and $C(I)$ is a Banach space. Thus, $\sum_{n=1}^{\infty} f_{n_i}$ converges to some $g_i \in C(I)$ and $\sum_{n=1}^{\infty} f_{n_i}^{(\alpha)}$ converges to some $h_i \in C(I)$. But, $\sum_{n=1}^{\infty} f_{n_i}^{(\alpha)}$ converges uniformly in $C(I)$, then $\sum_{n=1}^{\infty} f_{n_i}^{(\alpha)}$ converges to $g_i^{(\alpha)} \in C(I)$.

Now, let us define $f = \sum_{i=1}^{\infty} g_i \otimes \delta_i$. Claim $f \in \Sigma$.

$$\begin{aligned}\|f\| = \sum_{i=1}^{\infty} \|g_i\|_\infty + \|g_i^{(\alpha)}\|_\infty &= \sum_{i=1}^{\infty} \| \sum_{n=1}^{\infty} f_{n_i}\|_\infty + \| \sum_{n=1}^{\infty} f_{n_i}^{(\alpha)}\|_\infty \\ &\leqslant \sum_{i=1}^{\infty} (\sum_{n=1}^{\infty} \|f_{n_i}\|_\infty + \sum_{n=1}^{\infty} \|f_{n_i}^{(\alpha)}\|_\infty) \leqslant \sum_{n=1}^{\infty} (\sum_{i=1}^{\infty} \|f_{n_i}\|_\infty + \sum_{i=1}^{\infty} \|f_{n_i}^{(\alpha)}\|_\infty) < \infty.\end{aligned}$$

Now, we need to prove that $\sum_{n=1}^{\infty} f_n$ in Σ. Let $S_n = \sum_{k=1}^{n} f_k$. We claim that S_n converges to f in Σ.

$$\begin{aligned}\|f - S_n\| &= \| \sum_{i=1}^{\infty} g_i \otimes \delta_i - \sum_{k=1}^{n} \sum_{i=1}^{\infty} f_{k_i} \otimes \delta_i\| \\ &= \| \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} f_{k_i} \otimes \delta_i - \sum_{k=1}^{n} \sum_{i=1}^{\infty} f_{k_i} \otimes \delta_i\| \\ &= \| \sum_{i=1}^{\infty} (\sum_{k=1}^{\infty} f_{k_i} \otimes \delta_i - \sum_{k=1}^{n} f_{k_i} \otimes \delta_i)\| \\ &= \| \sum_{i=1}^{\infty} \sum_{k=n+1}^{\infty} f_{k_i} \otimes \delta_i\| \\ &= \sum_{i=1}^{\infty} \| \sum_{k=n+1}^{\infty} f_{k_i}\|_\infty + \| \sum_{k=n+1}^{\infty} f_{k_i}^{(\alpha)}\|_\infty \\ &\leqslant \sum_{i=1}^{\infty} \sum_{k=n+1}^{\infty} \|f_{k_i}\|_\infty + \sum_{k=n+1}^{\infty} \|f_{k_i}^{(\alpha)}\|_\infty < \infty.\end{aligned}$$

This implies $\lim_{n\to\infty} \|f - S_n\| = 0$. Hence S_n converges to f in Σ. □

3. Main results

Let u be α-differentiable on $I = [0, 1]$ with values in the Hilbert space $X = \ell^2$, where $\ell^2 = \{(x_n) : \sum_{n=1}^{\infty} | x_n |^2 < \infty\}$. In ℓ^2 we write $[x_1, x_2, \ldots]$ to denote the span of $\{x_1, x_2, \ldots, \}$. Let $A : Dom(A) \subseteq \ell^2 \to \ell^2$ be a densely defined closed linear operator on ℓ^2, where domain of A contains the elements of the natural basis of ℓ^2.

Consider the abstract Cauchy problem

$$\begin{cases} u^{(\alpha)}(t) = Au(t) + f(t)z, & 0 < t \leqslant 1, \\ u(0) = x. \end{cases} \tag{P1}$$

where $u(t) \in Dom(A)$, $f \in C(I)$ and $z \in \ell^2$.

In this section, we look for a solution to problem (P1) among infinite rank functions of the form $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i \in \Sigma$, where $u_i^{(\alpha)} \in C(I), i = 1, 2, \ldots$. To prove problem (P1) has a unique solution in Σ, we need to assume the following conditions:

(H1) $(|c_j|) \in \ell^1$ and $(|\gamma_j|) \in \ell^1$, where $c_j = \langle x, \delta_j \rangle$ and $\gamma_j = \langle z, \delta_j \rangle$;
(H2) $(|a_{jj} c_j|) \in \ell^1$, where $a_{ij} = \langle A\delta_i, \delta_j \rangle$;
(H3) $\mathrm{Re}(a_{ii}) \leqslant \omega$ for all $i = 1, 2, \ldots$;
(H4) $\sum_{j=1}^{\infty} (\sum_{i=1, i\neq j}^{\infty} |a_{ij}|^2)^{\frac{1}{2}} \leqslant C$.

Theorem 3.1. *Assume that conditions* (H1), (H2), (H3), *and* (H4) *are satisfied. Then problem* (P1) *has unique solution, provided*

$$L = C\left(\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega}\right) < 1 .$$

Proof. Since $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i \in \Sigma$, we have $\sum_{i=1}^{\infty} u_i(t)\delta_i$ and $\sum_{i=1}^{\infty} u_i^{(\alpha)}(t)\delta_i$ converges uniformly in $C(I, \ell^2)$. By Lemma 2.5, we get $u^{(\alpha)}(t) = \sum_{i=1}^{\infty} u_i^{(\alpha)}(t)\delta_i$. We can write problem (P1) in the form

$$\begin{cases} u_j^{(\alpha)}(t) = \sum_{i=1}^{\infty} u_i(t)\langle A\delta_i, \delta_j \rangle + f(t)\langle z, \delta_j \rangle, \\ u_j(0) = \langle x, \delta_j \rangle = c_j. \end{cases} \tag{3.1}$$

Since $a_{ij} = \langle A\delta_i, \delta_j \rangle$ and $\gamma_j = \langle z, \delta_j \rangle$, we can rewrite (3.1) as

$$\begin{cases} u_j^{(\alpha)}(t) = a_{jj}u_j(t) + \sum_{i=1, i\neq j}^{\infty} a_{ij}u_i(t) + f(t)\gamma_j, \\ u_j(0) = c_j. \end{cases} \tag{3.2}$$

Multiplying (3.2) by the integrating factor $e^{I_\alpha(-a_{jj})}$ and integrating, we obtain

$$u_j(t) = e^{a_{jj}\frac{t^\alpha}{\alpha}} c_j + \int_0^t e^{a_{jj}(\frac{t^\alpha - s^\alpha}{\alpha})} s^{\alpha-1} \left(\sum_{i=1, i\neq j}^{\infty} a_{ij}u_i(s) + f(s)\gamma_j\right) ds. \tag{3.3}$$

We define $Du(t) = (a_{11}u_1(t), a_{22}u_2(t), \ldots)^T$, where $D = \mathrm{diag}(a_{11}, a_{22}, \ldots)$ and

$$Mu(t) = \left(\sum_{i=2, j\neq 1}^{\infty} a_{ij}u_i(t), \sum_{i=1, j\neq 2}^{\infty} a_{ij}u_i(t), \ldots\right)^T,$$

where $M = [a_{ij}]^T - D$ and $[a_{ij}]$ is an infinite matrix. By assumptions (H3) and (H4) we conclude that

$$\|e^{\frac{t^\alpha}{\alpha} D}\| = \sup_{i=1,2,\ldots} |e^{\frac{t^\alpha}{\alpha} a_{ii}}| \leqslant e^{\frac{t^\alpha}{\alpha}\omega} \ , \ \ t \in [0, 1],$$

and

$$\begin{aligned} \|Mu(t)\|_2 &= \left(\sum_{j=1}^{\infty} |\sum_{i=1, i\neq j}^{\infty} a_{ij}u_i(t)|^2\right)^{\frac{1}{2}} \\ &\leqslant \sum_{j=1}^{\infty} \left(|\sum_{i=1, i\neq j}^{\infty} a_{ij}u_i(t)|^2\right)^{\frac{1}{2}} \\ &= \sum_{j=1}^{\infty} |\sum_{i=1, i\neq j}^{\infty} a_{ij}u_i(t)| \\ &\leqslant \sum_{j=1}^{\infty} \sum_{i=1, i\neq j}^{\infty} |a_{ij}u_i(t)| \\ &\leqslant \sum_{j=1}^{\infty} \left(\sum_{i=1, i\neq j}^{\infty} |a_{ij}|^2\right)^{\frac{1}{2}} \left(\sum_{i=1, i\neq j}^{\infty} |u_i|^2\right)^{\frac{1}{2}} \end{aligned}$$

$$\leqslant \sum_{j=1}^{\infty}(\sum_{i=1,i\neq j}^{\infty} |a_{ij}|^2)^{\frac{1}{2}}\|u(t)\|_2 \leqslant C\|u(t)\|_2.$$

Hence

$$\|M\| = \sup_{\|u(t)\|_2\leqslant 1} \|Mu(t)\|_2 \leqslant C.$$

So, we can write (3.3) as

$$u(t) = e^{\frac{t^\alpha}{\alpha}D}x + \int_0^t e^{(\frac{t^\alpha - s^\alpha}{\alpha})D} s^{\alpha-1}(Mu(s) + f(s)z)ds. \tag{3.4}$$

To prove that problem (P1) has a unique solution, it is enough to show that the integral equation (3.4) has a unique solution. Let us define the operator $T : C(I, \ell^2) \to C(I, \ell^2)$ by

$$Tu(t) = e^{\frac{t^\alpha}{\alpha}D}x + \int_0^t e^{(\frac{t^\alpha - s^\alpha}{\alpha})D} s^{\alpha-1}(Mu(s) + f(s)z)ds.$$

Then, by the Banach's fixed point theorem, we obtain

$$\begin{aligned}\|Tu(t) - Tv(t)\|_2 &\leqslant \int_0^t e^{(\frac{t^\alpha - s^\alpha}{\alpha})\omega} s^{\alpha-1}\|M\|\|u(s) - v(s)\|_2 ds\\ &\leqslant \|u - v\|_\infty\|M\| \int_0^t s^{\alpha-1}e^{(\frac{t^\alpha - s^\alpha}{\alpha})\omega} ds \leqslant C\|u - v\|_\infty(\frac{e^{\frac{t^\alpha}{\alpha}\omega} - 1}{\omega}).\end{aligned}$$

Taking supremum over $t \in [0, 1]$, we have

$$\|Tu - Tv\|_\infty \leqslant C\|u - v\|_\infty(\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega}).$$

Since $L = C(\frac{e^{\frac{\omega}{\alpha}}-1}{\omega}) < 1$, then T is contraction and by the Banach fixed point theorem, T has a unique fixed point. Now, we check $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i \in \Sigma$.

$$\begin{aligned}|u_j(t)| &\leqslant e^{\frac{t^\alpha}{\alpha}\omega}|c_j| + \|f\|_\infty|\gamma_j|\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega} + \int_0^t s^{\alpha-1}e^{(\frac{t^\alpha - s^\alpha}{\alpha})\omega} \sum_{i=1,i\neq j}^{\infty} |a_{ij}u_i(s)|ds\\
&\leqslant e^{\frac{t^\alpha}{\alpha}\omega}|c_j| + \|f\|_\infty|\gamma_j|\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega} + \int_0^t s^{\alpha-1}e^{(\frac{t^\alpha - s^\alpha}{\alpha})\omega}(\sum_{i=1,i\neq j}^{\infty} |a_{ij}|^2)^{\frac{1}{2}}(\sum_{i=1,i\neq j}^{\infty} |u_i(s)|^2)^{\frac{1}{2}} ds\\
&\leqslant e^{\frac{t^\alpha}{\alpha}\omega}|c_j| + \|f\|_\infty|\gamma_j|\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega} + (\sum_{i=1,i\neq j}^{\infty} |a_{ij}|^2)^{\frac{1}{2}} \int_0^t s^{\alpha-1}e^{(\frac{t^\alpha - s^\alpha}{\alpha})\omega} \sum_{i=1,i\neq j}^{\infty} |u_i(s)|ds\\
&\leqslant e^{\frac{t^\alpha}{\alpha}\omega}|c_j| + \|f\|_\infty|\gamma_j|\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega} + (\sum_{i=1,i\neq j}^{\infty} |a_{ij}|^2)^{\frac{1}{2}} \int_0^t s^{\alpha-1}e^{(\frac{t^\alpha - s^\alpha}{\alpha})\omega} \sum_{i=1,i\neq j}^{\infty} \|u_i\|_\infty ds\\
&\leqslant e^{\frac{t^\alpha}{\alpha}\omega}|c_j| + \|f\|_\infty|\gamma_j|\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega} + (\sum_{i=1,i\neq j}^{\infty} |a_{ij}|^2)^{\frac{1}{2}} \sum_{i=1}^{\infty} \|u_i\|_\infty\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega}.\end{aligned}$$

Taking supremum over $t \in [0, 1]$, we have

$$\|u_j\|_\infty \leqslant \sup_{t\in[0,1]} (e^{\frac{t^\alpha}{\alpha}\omega})|c_j| + \|f\|_\infty|\gamma_j|\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega} + (\sum_{i=1,i\neq j}^{\infty} |a_{ij}|^2)^{\frac{1}{2}} \sum_{i=1}^{\infty} \|u_i\|_\infty\frac{e^{\frac{\omega}{\alpha}} - 1}{\omega},$$

and thus

$$\sum_{j=1}^{\infty}\|u_j\|_\infty \leqslant \sup_{t\in[0,1]} e^{\frac{t^\alpha}{\alpha}\omega}\sum_{j=1}^{\infty}|c_j| + \|f\|_\infty \frac{e^{\frac{\omega}{\alpha}}-1}{\omega}\sum_{j=1}^{\infty}|\gamma_j| + \sum_{j=1}^{\infty}(\sum_{i=1,i\neq j}^{\infty}|a_{ij}|^2)^{\frac{1}{2}}\frac{e^{\frac{\omega}{\alpha}}-1}{\omega}\sum_{i=1}^{\infty}\|u_i\|_\infty$$
$$\leqslant \sup_{t\in[0,1]} e^{\frac{t^\alpha}{\alpha}\omega}\sum_{j=1}^{\infty}|c_j| + \|f\|_\infty \frac{e^{\frac{\omega}{\alpha}}-1}{\omega}\sum_{j=1}^{\infty}|\gamma_j| + C\frac{e^{\frac{\omega}{\alpha}}-1}{\omega}\sum_{j=1}^{\infty}\|u_j\|_\infty.$$

This implies

$$(1-L)\sum_{j=1}^{\infty}\|u_j\|_\infty \leqslant \sup_{t\in[0,1]} e^{\frac{t^\alpha}{\alpha}\omega}\sum_{j=1}^{\infty}|c_j| + \|f\|_\infty \frac{e^{\frac{\omega}{\alpha}}-1}{\omega}\sum_{j=1}^{\infty}|\gamma_j|,$$
$$\sum_{j=1}^{\infty}\|u_j\|_\infty \leqslant \frac{1}{1-L}(\sup_{t\in[0,1]} e^{\frac{t^\alpha}{\alpha}\omega}\sum_{j=1}^{\infty}|c_j| + \|f\|_\infty \frac{e^{\frac{\omega}{\alpha}}-1}{\omega}\sum_{j=1}^{\infty}|\gamma_j|).$$

By assumption (H1), $\sum_{j=1}^{\infty}\|u_j\|_\infty < \infty$. On other hand, we see that

$$|u_j^{(\alpha)}(t)| \leqslant e^{\frac{t^\alpha}{\alpha}\omega}|a_{jj}c_j| + \|f\|_\infty|\gamma_j| + (\sum_{i=1,i\neq j}^{\infty}|a_{ij}|^2)^{\frac{1}{2}}\sum_{i=1}^{\infty}\|u_i\|_\infty.$$

Hence

$$\sum_{j=1}^{\infty}\|u_j^{(\alpha)}\|_\infty \leqslant \sup_{t\in[0,1]} e^{\frac{t^\alpha}{\alpha}\omega}\sum_{j=1}^{\infty}|a_{jj}c_j| + \|f\|_\infty\sum_{j=1}^{\infty}|\gamma_j| + \sum_{j=1}^{\infty}(\sum_{i=1,i\neq j}^{\infty}|a_{ij}|^2)^{\frac{1}{2}}\sum_{i=1}^{\infty}\|u_i\|_\infty$$
$$\leqslant \sup_{t\in[0,1]} e^{\frac{t^\alpha}{\alpha}\omega}\sum_{j=1}^{\infty}|a_{jj}c_j| + \|f\|_\infty\sum_{j=1}^{\infty}|\gamma_j| + C\sum_{i=1}^{\infty}\|u_i\|_\infty.$$

By assumptions (H1) and (H2), and since $\sum_{i=1}^{\infty}\|u_i\|_\infty < \infty$, then we get $\sum_{j=1}^{\infty}\|u_j^{(\alpha)}\|_\infty < \infty$. Consequently $u \in \Sigma$. □

Remark 3.2. As a special case, if A is diagonal operator, i.e., $A\delta_i = \lambda_i\delta_i$, then the solution of problem (P1) writes in the form

$$u(t) = \sum_{i=1}^{\infty}(e^{\lambda_i\frac{t^\alpha}{\alpha}}c_i + \int_0^t e^{\lambda_i(\frac{t^\alpha - s^\alpha}{\alpha})}s^{\alpha-1}f(s)\gamma_i ds)\delta_i.$$

Now, let us prove the existence and uniqueness of the solution to problem (P1) using a different strategy. For this, we begin with the following definition.

Definition 3.3. A linear operator A defined on a Hilbert space H is called semi-diagonal, if there exist orthogonal subspaces $\{V_j\}_{j=1}^{\infty}$ such that

(i) $\dim V_j < \infty$;
(ii) $A(V_j) \subseteq V_j, \forall j$;
(iii) $H = \overset{\infty}{\underset{j=1}{\oplus}} V_j$.

Theorem 3.4. *Assume A is a semi-diagonal linear operator and assumption* (H1) *is satisfied. Then Problem* (P1) *has a unique solution.*

Proof. Since A is semi-diagonal, there exist orthogonal subspaces $\{V_j\}_{j=1}^{\infty}$ such that for each j, $\dim V_j < \infty$, $A(V_j) \subseteq V_j$, and $l^2 = \bigoplus_{j=1}^{\infty} V_j$. We may assume that $\{\delta_{n_{j-1}+1}, \ldots, \delta_{n_j}\}$, where $n_0 = 0$, $j = 1, 2, \ldots$, is the corresponding basis for V_j. Now, $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i = \sum_{j=1}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} u_i(t)\delta_i$. Put $v_j(t) = \sum_{i=n_{j-1}+1}^{n_j} u_i(t)\delta_i \in V_j$ and write $z = \sum_{j=1}^{\infty} z_j$, where $z_j = \sum_{i=n_{j-1}+1}^{n_j} \gamma_i \delta_i \in V_j$. Since $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i \in \Sigma$, we have $\sum_{i=1}^{\infty} u_i(t)\delta_i$ and $\sum_{i=1}^{\infty} u_i^{(\alpha)}(t)\delta_i$ converge uniformly in $C(I, \ell^2)$. By Lemma 2.5, we get $u^{(\alpha)}(t) = \sum_{i=1}^{\infty} u_i^{(\alpha)}(t)\delta_i$. Then we can write (P1) as

$$\begin{cases} \sum_{j=1}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} u_i^{(\alpha)}(t)\delta_i = \sum_{j=1}^{\infty} \sum_{i=n_{j-1}+1}^{n_j} u_i(t)A\delta_i + f(t)\sum_{j=1}^{\infty} z_j, \\ u(0) = \sum_{j=1}^{\infty} x_j, \end{cases}$$

where $x_j = \sum_{i=n_{j-1}+1}^{n_j} c_i\delta_i \in V_j$. Since the subspaces $\{V_j\}_{j=1}^{\infty}$ are orthogonal, $A(V_j) \subseteq V_j$, and $\sum_{i=n_{j-1}+1}^{n_j} u_i(t)\delta_i \in V_j$, we obtain

$$\sum_{i=n_{j-1}+1}^{n_j} u_i^{(\alpha)}(t)\delta_i = \sum_{i=n_{j-1}+1}^{n_j} u_i(t)A\delta_i + f(t)z_j. \tag{3.5}$$

Taking the inner product of both sides of (3.5) with δ_k $(n_{j-1}+1 \leqslant k \leqslant n_j)$, we have

$$u_k^{(\alpha)}(t) = \sum_{i=n_{j-1}+1}^{n_j} u_i(t)\langle A\delta_i, \delta_k\rangle + f(t)\langle z_j, \delta_k\rangle. \tag{3.6}$$

Let $A_j = A\,|_{V_j}$ be the restriction of A on V_j. So A_j has a matrix representation given by $A_j = [a_{ik}]$, such that $a_{ik} = \langle A\delta_k, \delta_i\rangle$. Thus equation (3.6) can be written as

$$v_j^{(\alpha)}(t) = A_j v_j(t) + f(t)z_j.$$

This system has a unique solution of the form (see [20])

$$v_j(t) = e^{\frac{t^\alpha}{\alpha}A_j}x_j + \int_0^t s^{\alpha-1}f(s)e^{(\frac{t^\alpha - s^\alpha}{\alpha})A_j}z_j\,ds.$$

Now, we check that $u \in \Sigma$.

$$\|v_j(t)\| \leqslant e^{\frac{\|A_j\|}{\alpha}}\|x_j\| + \|f\|_\infty e^{\frac{\|A_j\|}{\alpha}}\frac{\|z_j\|}{\alpha} \quad \text{and} \quad \|v_j^{(\alpha)}(t)\| \leqslant \|A_j\|e^{\frac{\|A_j\|}{\alpha}}\|x_j\| + \|f\|_\infty\|z_j\|,$$

we put $m = \sup_{j=1,2,\ldots}\|A_j\|$ and since the norms on finite dimensional vector spaces are equivalent, we have

$$\|x_j\| \leqslant K\sum_{i=n_{j-1}+1}^{n_j}|c_i|,$$

and

$$\sum_{i=n_{j-1}+1}^{n_j}\|u_i\|_\infty + \|u_i^{(\alpha)}\|_\infty \leqslant M(\|v_j\|_\infty + \|v_j^{(\alpha)}\|_\infty).$$

Hence we obtain

$$\sum_{j=1}^{\infty}\|u_j\|_\infty + \|u_j^{(\alpha)}\|_\infty = \sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j}\|u_i\|_\infty + \|u_i^{(\alpha)}\|_\infty$$

$$\leqslant MKe^{\frac{m}{\alpha}}(1+m)\sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j}|c_i| + MK\|f\|_\infty(1+e^{\frac{m}{\alpha}})\sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j}|\gamma_i|.$$

By assumption (H1), we get $\|u\| < \infty$ and therefore $u \in \Sigma$. □

Now, we will study the following degenerate abstract Cauchy problem

$$\begin{cases} Bu^{(\alpha)}(t) = Au(t) + f(t)z, & 0 < t \leqslant 1, \\ u(0) = x, \end{cases} \tag{P2}$$

where $A : \mathrm{Dom}(A) \subseteq \ell^2 \to \ell^2, B : \mathrm{Dom}(B) \subseteq \ell^2 \to \ell^2$ are two densely defined linear operators on ℓ^2, such that the domains of A and B contain the elements of the natural basis of ℓ^2. Also $u(t) \in \mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, $u^{(\alpha)}(t) \in \mathrm{Dom}(B)$, $f \in C(I)$ and $z \in \ell^2$. We assume that $A(\sum_{i=1}^{\infty} u_i(t)\delta_i) = \sum_{i=1}^{\infty} u_i(t)A\delta_i$ and $B(\sum_{i=1}^{\infty} u_i(t)\delta_i) = \sum_{i=1}^{\infty} u_i(t)B\delta_i$.

We look for a solution to problem (P2) among infinite rank functions of the form $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i \in \Sigma$. So, we need the following assumptions.

Assumption 3.5. A and B are semi-diagonal with the same decomposition, i.e., there exist orthogonal subspaces $\{V_j\}_{j=1}^{\infty}$ such that

(i) $\dim V_j < \infty$;
(ii) $A(V_j) \subseteq V_j$ and $B(V_j) \subseteq V_j, \forall j$;
(iii) $l^2 = \overset{\infty}{\underset{j=1}{\oplus}} V_j$.

Assumption 3.6. $A_j \mid_{\ker(B_j)}$ is invertible for every j, where $A_j = A\mid_{V_j}$.

Theorem 3.7. *Under Assumptions 3.5 and 3.6, and* (H1), *with* $f \in C^{(\alpha)}(I)$, *problem* (P2) *has a unique solution.*

Proof. Since A and B are semi-diagonal, there exist orthogonal subspaces $\{V_j\}_{j=1}^{\infty}$ such that for each j, $\dim V_j < \infty$, $A(V_j) \subseteq V_j$, $B(V_j) \subseteq V_j$, and $l^2 = \overset{\infty}{\underset{j=1}{\oplus}} V_j$. We may assume that $\{\delta_{n_{j-1}+1}, \dots, \delta_{n_j}\}$, where $n_0 = 0$, $j = 1, 2, \dots$, is the corresponding basis for V_j. Now, $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i = \sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j} u_i(t)\delta_i$. Put $v_j(t) = \sum_{i=n_{j-1}+1}^{n_j} u_i(t)\delta_i \in V_j$ and write $z = \sum_{j=1}^{\infty} z_j$, where $z_j = \sum_{i=n_{j-1}+1}^{n_j} \gamma_i\delta_i \in V_j$. Since $u(t) = \sum_{i=1}^{\infty} u_i(t)\delta_i \in \Sigma$, we have $\sum_{i=1}^{\infty} u_i(t)\delta_i$ and $\sum_{i=1}^{\infty} u_i^{(\alpha)}(t)\delta_i$ converge uniformly in $C(I, \ell^2)$, and we get $u^{(\alpha)}(t) = \sum_{i=1}^{\infty} u_i^{(\alpha)}(t)\delta_i$. Then we can write (P2) as

$$\begin{cases} \sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j} u_i^{(\alpha)}(t)B\delta_i = \sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j} u_i(t)A\delta_i + f(t)\sum_{j=1}^{\infty} z_j, \\ u(0) = \sum_{j=1}^{\infty} x_j, \end{cases}$$

where $x_j = \sum_{i=n_{j-1}+1}^{n_j} c_i\delta_i \in V_j$. Since the subspaces $\{V_j\}_{j=1}^{\infty}$ are orthogonal, $A(V_j) \subseteq V_j$,$B(V_j) \subseteq V_j$ and $\sum_{i=n_{j-1}+1}^{n_j} u_i(t)\delta_i \in V_j$, we get

$$\sum_{i=n_{j-1}+1}^{n_j} u_i^{(\alpha)}(t)B\delta_i = \sum_{i=n_{j-1}+1}^{n_j} u_i(t)A\delta_i + f(t)z_j. \tag{3.7}$$

Taking the inner product of both sides of (3.7) with δ_k $(n_{j-1}+1 \leqslant k \leqslant n_j)$, we have

$$\sum_{i=n_{j-1}+1}^{n_j} u_i^{(\alpha)}(t)\langle B\delta_i, \delta_k\rangle = \sum_{i=n_{j-1}+1}^{n_j} u_i(t)\langle A\delta_i, \delta_k\rangle + f(t)\langle z_j, \delta_k\rangle. \tag{3.8}$$

Let $B_j = B\mid_{V_j}$ be the restriction of B on V_j. So B_j has a matrix representation which is $B_j = [b_{ik}]$, such that $b_{ik} = \langle B\delta_k, \delta_i\rangle$. Thus equation (3.8) can be written as

$$B_j v_j^{(\alpha)}(t) = A_j v_j(t) + f(t) z_j. \tag{3.9}$$

Since $\dim V_j < \infty$, then $V_j = \ker B_j \oplus \ker B_j^{\perp}$ so $v_j(t) = y_j(t) + w_j(t)$, $x_j = x_j^1 + x_j^2$, and $z_j = z_j^1 + z_j^2$, where $y_j(t), x_j^1, z_j^1 \in \ker B_j$ and $w_j(t), x_j^2, z_j^2 \in (\ker B_j)^{\perp}$.

Define $\tilde{B_j} = B_j \mid_{(\ker B_j)^{\perp}}$ and $\tilde{A_j} = A_j \mid_{(\ker B_j)^{\perp}}$. Then $\tilde{B_j}$ is invertible (since $\tilde{B_j}$ is one to one in a finite dimensional space), so we can restrict equation (3.9) on $(\ker B_j)^{\perp}$ to obtain

$$\tilde{B_j} w_j^{(\alpha)}(t) = \tilde{A_j} w_j(t) + f(t) z_j^2.$$

This equation has a unique solution of the form

$$w_j(t) = e^{\tilde{B_j}^{-1}\tilde{A_j}\frac{t^{\alpha}}{\alpha}} x_j^2 + \int_0^t s^{\alpha-1} f(s) e^{\tilde{B_j}^{-1}\tilde{A_j}\frac{t^{\alpha}-s^{\alpha}}{\alpha}} \tilde{B_j}^{-1} z_j^2 ds.$$

Now, let us restrict equation (3.9) on $\ker B_j$. Suppose $C_j = A_j\mid_{\ker B_j}$ which is invertible by Assumption 3.6. So equation (3.9) becomes

$$0 = C_j y_j(t) + f(t) z_j^1.$$

This equation has a unique solution given by:

$$y_j(t) = -f(t) C_j^{-1} z_j^1.$$

Hence $v_j(t) = y_j(t) + w_j(t)$ is a unique solution of equation (3.9). Now, it remains to show $u \in \Sigma$.

$$\begin{aligned}
\|v_j\|_\infty &\leqslant \|w_j\|_\infty + \|y_j\|_\infty \\
&\leqslant e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|x_j^2\| + e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|\tilde{B_j}^{-1}\| \|z_j^2\| \frac{\|f\|_\infty}{\alpha} + \|C_j^{-1}\| \|z_j^1\| \|f\|_\infty \\
&\leqslant e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} (\|x_j^1\|^2 + \|x_j^2\|^2)^{\frac{1}{2}} + e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|\tilde{B_j}^{-1}\| \frac{\|f\|_\infty}{\alpha} (\|z_j^1\|^2 + \|z_j^2\|^2)^{\frac{1}{2}} + \|C_j^{-1}\| \|f\|_\infty (\|z_j^1\|^2 + \|z_j^2\|^2)^{\frac{1}{2}} \\
&= e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|x_j\| + e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|\tilde{B_j}^{-1}\| \frac{\|f\|_\infty}{\alpha} \|z_j\| + \|C_j^{-1}\| \|f\|_\infty \|z_j\|, \\
\|v_j\|_\infty &\leqslant e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|x_j\| + e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|\tilde{B_j}^{-1}\| \frac{\|f\|_\infty}{\alpha} \|z_j\| + \|C_j^{-1}\| \|f\|_\infty \|z_j\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|v_j^{(\alpha)}\|_\infty &\leqslant \|w_j^{(\alpha)}\|_\infty + \|y_j^{(\alpha)}\|_\infty \\
&\leqslant \|\tilde{B_j}^{-1}\tilde{A_j}\| e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|x_j^2\| + \|\tilde{B_j}^{-1}\| \|z_j^2\| \frac{\|f\|_\infty}{\alpha} + \|C_j^{-1}\| \|z_j^1\| \|f^{(\alpha)}\|_\infty \\
&\leqslant \|\tilde{B_j}^{-1}\tilde{A_j}\| e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} (\|x_j^1\|^2 + \|x_j^2\|^2)^{\frac{1}{2}} + \|\tilde{B_j}^{-1}\| \frac{\|f\|_\infty}{\alpha} (\|z_j^1\|^2 + \|z_j^2\|^2)^{\frac{1}{2}} \\
&\quad + \|C_j^{-1}\| \|f^{(\alpha)}\|_\infty (\|z_j^1\|^2 + \|z_j^2\|^2)^{\frac{1}{2}} \\
&= \|\tilde{B_j}^{-1}\tilde{A_j}\| e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|x_j\| + \|\tilde{B_j}^{-1}\| \frac{\|f\|_\infty}{\alpha} \|z_j\| + \|C_j^{-1}\| \|f^{(\alpha)}\|_\infty \|z_j\|, \\
\|v_j^{(\alpha)}\|_\infty &\leqslant \|\tilde{B_j}^{-1}\tilde{A_j}\| e^{\frac{\|\tilde{B_j}^{-1}\tilde{A_j}\|}{\alpha}} \|x_j\| + \|\tilde{B_j}^{-1}\| \frac{\|f\|_\infty}{\alpha} \|z_j\| + \|C_j^{-1}\| \|f^{(\alpha)}\|_\infty \|z_j\|.
\end{aligned}$$

We put

$$m_1 = \sup_{j=1,2,..} \|\tilde{B_j}^{-1}\tilde{A_j}\|, \quad m_2 = \sup_{j=1,2,..} \|\tilde{B_j}^{-1}\|, \quad m_3 = \sup_{j=1,2,..} \|C_j^{-1}\|.$$

Then we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\|u_j\|_\infty+\|u_j^{(\alpha)}\|_\infty &= \sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j}\|u_i\|_\infty+\|u_i^{(\alpha)}\|_\infty\\ &\leqslant M\sum_{j=1}^{\infty}\|v_j\|_\infty+\|v_j^{(\alpha)}\|_\infty\\ &\leqslant MKe^{\frac{m_1}{\alpha}}(1+m_1)\sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j}|c_i|+MKm_2\frac{\|f\|_\infty}{\alpha}(e^{\frac{m_1}{\alpha}}+1)\sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j}|\gamma_i|\\ &\quad+MKm_3(\|f\|_\infty+\|f^{(\alpha)}\|_\infty)\sum_{j=1}^{\infty}\sum_{i=n_{j-1}+1}^{n_j}|\gamma_i|.\end{aligned}$$

By (H1), $\sum_{j=1}^{\infty}\|u_j\|_\infty+\|u_j^{(\alpha)}\|_\infty<\infty$. Consequently $u\in\Sigma$. □

Corollary 3.8. *Under Assumption 3.5, condition* (H1)*, and the assumption that* B *is one to one, problem* (P2) *has a unique solution.*

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