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Generalised Hermite-Hadamard type inequalities for (s, r)-convex functions in mixed kind with applications

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Abstract

In this article, some generalized inequalities of the Hermite-Hadamard type for functions whose modulus of the derivatives are (s, r)-convex in mixed kind and applications for probability theory and numerical integration are given. Various established results of different articles would be recaptured as special cases. We also provide special cases of the class of (s, r)-convex function on several choices of s, r.

Keywords: Convex function, Hermite-Hadamard inequality, Hölder inequality, power-mean inequality, Beta function, Gamma function, numerical integration, probability density function.

2020 MSC: 26A33, 26A51, 26D15, 26D99, 47A30, 33B10.

1. Introduction

To generalize the Hermite-Hadamard inequality, we require to generalize the concept of convex functions, by using this way we may easily see generalizations and its particular cases. From the history, we remind some definitions for several convex functions [2].

Definition 1.1. Any function $h : K \subseteq \mathbb{R} \to \mathbb{R}$ is called the convex, if

$$h(\zeta y + (1-\zeta)z) \leqslant \zeta h(y) + (1-\zeta)h(z), \quad \forall\, y, z \in K, \zeta \in [0,1].$$

Here we have P-convex function which is extracted from [5].

Definition 1.2. Any function $h : K \subseteq \mathbb{R} \to \mathbb{R}$ is called the P-convex, if

$$h(\zeta y + (1-\zeta)z) \leqslant h(y) + h(z), \quad h \geqslant 0, \quad \forall\, y, z \in K, \zeta \in [0,1].$$

Here we also have definition of quasi-convex function [7].

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Definition 1.3. Any function $h : K \subseteq \mathbb{R} \to \mathbb{R}$ is called the quasi-convex, if

$$h(\zeta y + (1-\zeta)z) \leqslant \max\{h(y), h(z)\}, \quad \forall\, y, z \in K, \zeta \in [0,1].$$

Here we provide definition of s-convex functions in the 1st kind as follows (see [15]).

Definition 1.4. Suppose $s \in (0,1]$. Any function $h : K \subseteq [0,\infty) \to [0,\infty)$ is known as s-convex in the 1st kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant \zeta^s h(y) + (1-\zeta^s)h(z), \quad \forall\, y, z \in K, \zeta \in [0,1].$$

Remark 1.5. Note, this definition we also include $s = 0$. Moreover if select $s = 0$, we acquire quasi-convexity (see Definition 1.3).

For 2nd kind convexity we remind definition from [15].

Definition 1.6. Suppose $s \in (0,1]$. Any function $h : K \subseteq [0,\infty) \to [0,\infty)$ is known as s-convex in the 2nd kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant \zeta^s h(y) + (1-\zeta)^s h(z), \quad \forall\, y, z \in K, \zeta \in [0,1].$$

Remark 1.7. In the same way, we have slightly improved definition of 2nd kind convexity by including $s = 0$. Moreover if select $s = 0$, we easily acquire P-convexity (see Definition 1.2).

A new class of function which will be called class of (s,r)-convex functions in the mixed kind is extracted from [8].

Definition 1.8. Suppose $(s,r) \in [0,1]^2$. Any function $h : K \subseteq [0,\infty) \to [0,\infty)$, is called the (s,r)-convex in the mixed kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant \zeta^{rs} h(y) + (1-\zeta^r)^s h(z), \ \forall\, y, z \in K, \zeta \in [0,1].$$

Remark 1.9. In Definition 1.8, we have the following cases.

(i) If we select $s = 1$ and $r = s$, where $s \in (0,1]$ in Definition 1.8, we acquire s-convex function in the 1st kind.
(ii) If we select $s = 1$ and $r \in [0,1]$ in Definition 1.8, we acquire quasi-convex function.
(iii) If we select $r = 1$, where $s \in (0,1]$ in Definition 1.8, we acquire s-convex function in the 2nd kind.
(iv) If we select $s = 0$, where $r \in [0,1]$ in Definition 1.8, we acquire P-convex function.
(v) If we select $s = r = 1$ in Definition 1.8, it gives us ordinary convex function.

At this stage, our main focus is on Hermite-Hadamard type inequalities, which was acquired by Hermite and Hadamard independently and stated as: A function $h : K \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function, $j, k \in K$, $j < k$, iff,

$$h\left(\frac{j+k}{2}\right) \leqslant \frac{1}{k-j}\int_j^k h(y)\,dy \leqslant \frac{h(j)+h(k)}{2}, \tag{1.1}$$

called the Hermite-Hadamard inequality. Moreover, the equation (1.1) has been generalized or refined for convex, s-convex, and quasi-convex functions and other types of functions by a lot of researchers.

In [4], the below result is given by Dragomir et al. having the Hermite-Hadamard type inequality.

Proposition 1.10. *Suppose* $h : K^0 \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping in* K^0, $j, k \in K^0$ *with* $j < k$. *If* $|h'|$ *is convex in the* $[j,k]$, *then*

$$\left|\frac{h(j)+h(k)}{2} - \frac{1}{k-j}\int_j^k h(u)\,du\right| \leqslant \frac{(k-j)(|h'(j)|+|h'(k)|)}{8} \tag{1.2}$$

holds.

For some recent consequences about Hermite-Hadamard type inequalities by using several classes of convex functions, see [1, 3, 9, 10, 12–14, 19].

In 2011, Kavurmaci et al. [9] gave few new inequalities of Hermite-Hadamard type for convex functions with applications. In 2016, Liu et al. [13] established few Hermite-Hadamard type inequalities for MT-convex functions

The main aim of this article is to generalize some Hermite-Hadamard type inequalities for (s, r)-convex functions in mixed kind through classical integrals using Hölder inequality and power-mean inequality and applications are also provided for probability theory and numerical integration. We would recapture several results of different articles [4, 9] and also provide special cases of class of (s, r)-convex functions on different choices of s, r as remarks.

2. Generalized Hermite-Hadamard type inequalities

Derivation of our main theorems, we require following lemma which is extracted from [9].

Lemma 2.1. *Suppose* $h : K \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping in the* $K^0 \subset \mathbb{R}$, *where* $j, k \in K$ *with* $k > j$. *If* $h' \in L[j, k]$, *then*

$$\begin{aligned}&\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\\&=\frac{(y-j)^2}{k-j}\int_0^1(\zeta-1)h'(\zeta y+(1-\zeta)j)d\zeta+\frac{(k-y)^2}{k-j}\int_0^1(1-\zeta)h'(\zeta y+(1-\zeta)k)d\zeta.\end{aligned}$$

The following results can be obtained by using Lemma 2.1.

Theorem 2.2. *Suppose* $h : K \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping in the* $K^0 \subset \mathbb{R}$ *provided that* $h' \in L[j, k]$, *where* $k, j \in K$ *with* $k > j$. *If* $|h'|$ *is* (s, r)*-convex in the* $[j, k]$, *then*

$$\begin{aligned}&\left|\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\right|\\&\leqslant\frac{(y-j)^2}{k-j}\left[\frac{|h'(y)|}{(rs+1)(rs+2)}+\frac{|h'(j)|}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right]\\&\quad+\frac{(k-y)^2}{k-j}\left[\frac{|h'(y)|}{(rs+1)(rs+2)}+\frac{|h'(k)|}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right]\end{aligned}$$

for every $h \in [j, k]$ *and* $r > 0$.

Proof. By Lemma 2.1 and using the modulus, we have

$$\begin{aligned}&\left|\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\right|\\&\leqslant\frac{(y-j)^2}{k-j}\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)j)|d\zeta+\frac{(k-y)^2}{k-j}\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)k)|d\zeta.\end{aligned}$$

Since $|f'|$ is (s, r)-convex in mixed kind, then we get

$$\begin{aligned}&\left|\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\right|\\&\leqslant\frac{(y-j)^2}{k-j}\int_0^1(1-\zeta)\left[\zeta^{rs}|h'(y)|+(1-\zeta^r)^s|h'(j)|\right]d\zeta\end{aligned}$$

$$+\frac{(k-y)^2}{k-j}\int_0^1(1-\zeta)\left[\zeta^{rs}|h'(y)|+(1-\zeta^r)^s|h'(k)|\right]d\zeta$$
$$=\frac{(y-j)^2}{k-j}\left[\frac{|h'(y)|}{(sr+1)(sr+2)}+\frac{|h'(j)|}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right]$$
$$+\frac{(k-y)^2}{k-j}\left[\frac{|h'(y)|}{(sr+1)(sr+2)}+\frac{|h'(k)|}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right],$$

which completes the proof. □

Note: Here B is Beta function and it is stated as $B(l,m)=\int_0^1\zeta^{l-1}(1-\zeta)^{m-1}d\zeta=\dfrac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$. Since $\Gamma(l)=\int_0^\infty e^{-u}u^{l-1}du$.

Remark 2.3. Some remarks about Theorem 2.2 are following as special cases.

(i) If we select $s=1$ and $r=s$, where $s\in(0,1]$ in Theorem 2.2, we acquire inequality for s-convex function in the 1st kind.
(ii) If we select $s=1$ and $r\in[0,1]$ in Theorem 2.2, we acquire inequality for quasi-convex function.
(iii) If we select $r=1$, where $s\in(0,1]$ in Theorem 2.2, we acquire inequality for s-convex function in the 2nd kind.
(iv) If we select $s=0$, where $r\in[0,1]$ in Theorem 2.2, we acquire inequality for P-convex function.

Remark 2.4. If we select $s=r=1$ in Theorem 2.2, we capture the main Theorem 4 of [9].

Corollary 2.5. *If we select* $y=\frac{j+k}{2}$ *in Theorem* 2.2, *then*

$$\left|\frac{h(j)+h(k)}{2}-\frac{1}{k-j}\int_j^k h(u)du\right|$$
$$\leqslant\frac{k-j}{4}\left[\frac{2|h'(\frac{j+k}{2})|}{(rs+1)(rs+2)}+(|h'(j)|+|h'(k)|)\left(\frac{B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)}{r}\right)\right].$$

Remark 2.6. Some remarks about Corollary 2.5 are following as special cases.

(i) In Corollary 2.5, using the convexity of $|h'|$, then we can acquire established inequality (1.2) (recapture Theorem 2.2 of [4]).
(ii) If we select $s=1$ and $r=s$, where $s\in(0,1]$ in Corollary 2.5, we acquire inequality for s-convex function in the 1st kind.
(iii) If we select $s=1$, and $r\in[0,1]$ in Corollary 2.5, we acquire inequality for quasi-convex function.
(iv) If we select $r=1$, where $s\in(0,1]$ in Corollary 2.5, we acquire inequality for s-convex function in the 2nd kind.
(v) If we select $s=0$, where $r\in[0,1]$ in Corollary 2.5, we acquire inequality for P-convex function.

Remark 2.7. If acquire $s=r=1$ in Corollary 2.5, we recapture the Corollary 2 of [9].

Theorem 2.8. *Suppose* $h:K\subseteq\mathbb{R}\to\mathbb{R}$ *is a differentiable mapping in the* $K^0\subset\mathbb{R}$ *provided that* $h'\in L[j,k]$, *where* $k,j\in K$ *with* $k>j$. *If* $|h'|^{\frac{p}{p-1}}$ *is* (s,r)*-convex in* $[j,k]$ *and for some fixed* $q>1$, *then*

$$\left|\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\right|$$
$$\leqslant\frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-j)^2\left(\frac{|h'(y)|^q}{rs+1}+\frac{|h'(j)|^q}{r}B\left(\frac{1}{r},s+1\right)\right)^{\frac{1}{q}}\right.$$

$$+(k-y)^2\left(\frac{|h'(y)|^q}{rs+1}+\frac{|h'(k)|^q}{r}B\left(\frac{1}{r},s+1\right)\right)^{\frac{1}{q}}\Bigg]$$

for every $h\in[j,k]$ *and* $r>0$.

Proof. By Lemma 2.1 and (s,r)-convexity of $|h'|$, then applying the well-known Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\right|\\
&\leqslant \frac{(y-j)^2}{k-j}\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)j)|d\zeta+\frac{(k-y)^2}{k-j}\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)k)|d\zeta\\
&\leqslant \frac{(y-j)^2}{k-j}\int_0^1(1-\zeta)\left[\zeta^{rs}|h'(y)|+(1-\zeta^r)^s|h'(j)|\right]d\zeta\\
&\quad+\frac{(k-y)^2}{k-j}\int_0^1(1-\zeta)\left[\zeta^{rs}|h'(y)|+(1-\zeta^r)^s|h'(k)|\right]d\zeta\\
&\leqslant \frac{(y-j)^2}{k-j}\left(\int_0^1(1-\zeta)^p d\zeta\right)^{\frac{1}{p}}\left[\int_0^1\left(\zeta^{rs}|h'(y)|+(1-\zeta^r)^s|h'(j)|\right)^q d\zeta\right]^{\frac{1}{p}}\\
&\quad+\frac{(k-y)^2}{k-j}\left(\int_0^1(1-\zeta)^p d\zeta\right)^{\frac{1}{p}}\left[\int_0^1\left(\zeta^{rs}|h'(y)|+(1-\zeta^r)^s|h'(k)|\right)^q d\zeta\right]^{\frac{1}{p}}\\
&\leqslant \frac{(y-j)^2}{k-j}\left(\int_0^1(1-\zeta)^p d\zeta\right)^{\frac{1}{p}}\left[\int_0^1\zeta^{rs}|h'(y)|^q d\zeta+\int_0^1(1-\zeta^r)^s|h'(j)|^q d\zeta\right]^{\frac{1}{p}}\\
&\quad+\frac{(k-y)^2}{k-j}\left(\int_0^1(1-\zeta)^p d\zeta\right)^{\frac{1}{p}}\left[\int_0^1\zeta^{rs}|h'(y)|^q d\zeta+\int_0^1(1-\zeta^r)^s|h'(k)|^q d\zeta\right]^{\frac{1}{p}}\\
&\leqslant \frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\Bigg[(y-j)^2\left(\frac{|h'(y)|^q}{rs+1}+\frac{|h'(j)|^q}{r}B\left(\frac{1}{r},s+1\right)\right)^{\frac{1}{q}}\\
&\quad+(k-y)^2\left(\frac{|h'(y)|^q}{rs+1}+\frac{|h'(k)|^q}{r}B\left(\frac{1}{r},s+1\right)\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

□

Remark 2.9. All remarks hold for Theorem 2.8 as we have given remarks (i) to (iv) for Theorem 2.2.

Remark 2.10. If we select $s=r=1$ in Theorem 2.8, we capture the Theorem 5 of [9].

Corollary 2.11. *If we select* $y=\frac{j+k}{2}$ *in Theorem 2.8, we get*

$$\begin{aligned}
\left|\frac{h(j)+h(k)}{2}-\frac{1}{k-j}\int_j^k h(u)du\right|\leqslant \frac{k-j}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}&\Bigg[\left(\frac{|h'(j)|^q}{r}B\left(\frac{1}{r},s+1\right)\right.\\
&\left.+\frac{|h'(\frac{j+k}{2})|^q}{rs+1}\right)^{\frac{1}{q}}+\left(\frac{|h'(k)|^q}{r}B\left(\frac{1}{r},s+1\right)+\frac{|h'(\frac{j+k}{2})|^q}{rs+1}\right)^{\frac{1}{q}}\Bigg].
\end{aligned}$$

Remark 2.12. All remarks hold for Corollary 2.11 as we have given remarks (ii) to (v) for Corollary 2.5.

Remark 2.13. If we select $s=r=1$ in Corollary 2.11, we recapture the Corollary 3 of [9].

Theorem 2.14. *Suppose* $h : K \subseteq \mathbb{R} \to \mathbb{R}$ *is a differentiable mapping in the* $K^0 \subset \mathbb{R}$ *provided that* $h' \in L[j,k]$, *where* $k, j \in K$ *with* $k > j$. *If* $|h'|^q$ *is* (s,r)*-convex in* $[j,k]$ *and for some fixed* $q \geqslant 1$, *then*

$$\begin{aligned}&\left|\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\right|\\&\leqslant \frac{1}{2^{1-\frac{1}{q}}(k-j)}\left[(y-j)^2\left(\frac{|h'(y)|^q}{(rs+1)(rs+2)}+\frac{|h'(j)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right.\\&\left.\quad+(k-y)^2\left(\frac{|h'(y)|^q}{(rs+1)(rs+2)}+\frac{|h'(k)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right]\end{aligned}$$

for every $h \in [j,k]$, $q = \frac{p}{p-1}$ *and* $r > 0$.

Proof. Consider $q \geqslant 1$ and from Lemma 2.1 and applying well-known power-mean inequality, we have

$$\begin{aligned}&\left|\frac{(k-y)h(k)+(y-j)h(j)}{k-j}-\frac{1}{k-j}\int_j^k h(u)du\right|\\&\leqslant \frac{(y-j)^2}{k-j}\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)j)|d\zeta+\frac{(k-y)^2}{k-j}\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)k)|d\zeta\\&\leqslant \frac{(y-j)^2}{k-j}\left(\int_0^1(1-\zeta)d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)j)|^q d\zeta\right)^{\frac{1}{q}}\\&\quad+\frac{(k-y)^2}{k-j}\left(\int_0^1(1-\zeta)d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)k)|^q d\zeta\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|$ is (s,r)-convex in mixed kind, then we consider

$$\begin{aligned}\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)j)|^q d\zeta &\leqslant \int_0^1(1-\zeta)\left[\zeta^{rs}|h'(y)|^q+(1-\zeta^r)^s|h'(j)|^q\right]d\zeta\\&=\frac{|h'(y)|^q}{(rs+1)(rs+2)}+\frac{|h'(j)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right).\end{aligned}$$

Analogously,

$$\int_0^1(1-\zeta)|h'(\zeta y+(1-\zeta)k)|^q d\zeta \leqslant \frac{|h'(y)|^q}{(rs+1)(rs+2)}+\frac{|h'(k)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right).$$

We get the required result by collecting all above inequalities. □

Remark 2.15. All remarks hold for Theorem 2.14 as we have given remarks (i) to (iv) for Theorem 2.2.

Remark 2.16. If we select $s = r = 1$ in Theorem 2.14, we capture the Theorem 7 of [9].

Corollary 2.17. *If we select* $y = \frac{j+k}{2}$ *in Theorem* 2.14, *we get*

$$\begin{aligned}&\left|\frac{h(j)+h(k)}{2}-\frac{1}{k-j}\int_j^k h(u)du\right|\\&\leqslant 2^{\frac{1}{q}-3}(k-j)\left[\left(\frac{|h'(\frac{j+k}{2})|^q}{(sr+1)(sr+2)}+\frac{|h'(j)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right.\\&\left.\quad+\left(\frac{|h'(\frac{j+k}{2})|^q}{(rs+1)(rs+2)}+\frac{|h'(k)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Remark 2.18. All remarks hold for Corollary 2.17 as we have given remarks (ii) to (v) for Corollary 2.5.

Remark 2.19. If we select $s = r = 1$ in Corollary 2.17, we recapture the Corollary 4 of [9].

3. Application to numerical integration

3.1. *The trapezoidal formula*

Suppose $d : j = \theta_0 < \theta_1 < \cdots < \theta_n = k$ is a division of $[j, k]$ and $h_i = \theta_{i+1} - \theta_i$, $(i = 0, 1, \cdots , n-1)$ and consider the quadrature formula

$$\int_j^k h(u)du = Q(h, d) + R(h, d), \tag{3.1}$$

where

$$Q(h, d) = \sum_{i=0}^{n-1} \Big((\theta_{i+1} - y)h(\theta_{i+1}) + (y - \theta_i)h(\theta_i) \Big)$$

for trapezoidal version and $R(h, d)$ represents the associated approximation error.

Theorem 3.1. *Suppose the assumptions of Theorem 2.2 is true for each division* $'d'$ *of* $[j, k]$. *Then in* (3.1), *the trapezoidal error estimate satisfies*

$$\begin{aligned}\Big|R(h, d)\Big| \leqslant{}& \sum_{i=0}^{n-1}(y - \theta_i)^2\left[\frac{|h'(y)|}{(sr+1)(sr+2)} + \frac{|h'(\theta_i)|}{r}\left(B\left(\frac{1}{r}, s+1\right) - B\left(\frac{2}{r}, s+1\right)\right)\right]\\ &+ \sum_{i=0}^{n-1}(\theta_{i+1} - y)^2\left[\frac{|h'(y)|}{(rs+1)(rs+2)} + \frac{|h'(\theta_{i+1})|}{r}\left(B\left(\frac{1}{r}, s+1\right) - B\left(\frac{2}{r}, s+1\right)\right)\right]\end{aligned}$$

for every $h \in [j, k]$ *and* $r > 0$.

Proof. Using Theorem 2.2 on the sub-interval $[\theta_i, \theta_{i+1}]$ $(i = 0, 1, \ldots , n-1)$, then

$$\begin{aligned}&\left|\frac{(\theta_{i+1} - y)h(\theta_{i+1}) + (y - \theta_i)h(\theta_i)}{h_i} - \frac{1}{h_i}\int_{\theta_i}^{\theta_{i+1}} h(u)du\right|\\ &\leqslant \frac{(y - \theta_i)^2}{h_i}\left[\frac{|h'(y)|}{(sr+1)(sr+2)} + \frac{|h'(\theta_i)|}{r}\left(B\left(\frac{1}{r}, s+1\right) - B\left(\frac{2}{r}, s+1\right)\right)\right]\\ &\quad+ \frac{(\theta_{i+1} - y)^2}{h_i}\left[\frac{|h'(y)|}{(sr+1)(sr+2)} + \frac{|h'(\theta_{i+1})|}{r}\left(B\left(\frac{1}{r}, s+1\right) - B\left(\frac{2}{r}, s+1\right)\right)\right].\end{aligned}$$

Hence in (3.1) we have to sum over i from 0 to $n-1$, then

$$\begin{aligned}\left|\int_j^k h(u)du - Q(h, d)\right| &= \left|\sum_{i=0}^{n-1}\left[\int_{\theta_i}^{\theta_{i+1}} h(u)du - \Big((\theta_{i+1} - y)h(\theta_{i+1}) + (y - \theta_i)h(\theta_i)\Big)\right]\right|\\ &\leqslant \sum_{i=0}^{n-1}\left|\int_{\theta_i}^{\theta_{i+1}} h(u)du - \Big((\theta_{i+1} - y)h(\theta_{i+1}) + (y - \theta_i)h(\theta_i)\Big)\right|\\ &\leqslant \sum_{i=0}^{n-1}(y - \theta_i)^2\left[\frac{|h'(y)|}{(rs+1)(rs+2)} + \frac{|h'(\theta_i)|}{r}\left(B\left(\frac{1}{r}, s+1\right) - B\left(\frac{2}{r}, s+1\right)\right)\right]\\ &\quad+ \sum_{i=0}^{n-1}(\theta_{i+1} - y)^2\left[\frac{|h'(y)|}{(sr+1)(sr+2)} + \frac{|h'(\theta_{i+1})|}{r}\left(B\left(\frac{1}{r}, s+1\right) - B\left(\frac{2}{r}, s+1\right)\right)\right],\end{aligned}$$

which completes the proof. □

Remark 3.2. We can get similar results as above for Corollary 2.5, Theorem 2.8, Corollary 2.11, Theorem 2.14, and Corollary 2.17.

Remark 3.3. All remarks of Section 2 are also hold for Theorem 3.1 and for Remark 3.2.

4. Applications to probability theory

Suppose Y is a random variable choosing values in finite interval $[j,k]$, with the probability density function $h:[j,k]\to[0,1]$ and with the cumulative distribution function $G(y)=P(Y\leqslant y)=\int_j^k h(u)du$.

Theorem 4.1. *Suppose the assumptions of Theorem 2.2 are true, then*

$$\begin{aligned}&\left|\frac{(k-y)G(k)+(y-j)G(j)}{k-j}-\frac{k-E(Y)}{k-j}\right|\\&\leqslant\frac{(y-j)^2}{k-j}\left[\frac{|G'(y)|}{(sr+1)(sr+2)}+\frac{|G'(j)|}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right]\\&+\frac{(k-y)^2}{k-j}\left[\frac{|G'(y)|}{(sr+1)(sr+2)}+\frac{|G'(k)|}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right]\end{aligned}\tag{4.1}$$

for every $h\in[j,k]$ *and* $r>0$*, where* $E(Y)$ *is the expectation of* Y.

Proof. Select $h=G$, we obtain (4.1), by applying the identity

$$E(Y)=\int_j^k uG(u)du=k-\int_j^k G(u)du.$$

Since $G(j)=0$ and $G(k)=1$. We left the details to research scholars. □

Remark 4.2. With all the assumptions of Corollary 2.5, Theorem 2.8, Corollary 2.11, Theorem 2.14, and Corollary 2.17, then the following equalities hold respectively:

$$\left|\frac{G(j)+G(k)}{2}-\frac{k-E(Y)}{k-j}\right|\leqslant\frac{k-j}{4}\left[\frac{2|G'(\frac{j+k}{2})|}{(rs+1)(rs+2)}+(|G'(j)|+|G'(k)|)\left(\frac{B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)}{r}\right)\right],$$

$$\begin{aligned}\left|\frac{(k-y)G(k)+(y-j)G(j)}{k-j}-\frac{k-E(Y)}{k-j}\right|&\leqslant\frac{1}{k-j}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[(y-j)^2\left(\frac{|G'(y)|^q}{(rs+1)}+\frac{|G'(j)|^q}{r}B\left(\frac{1}{r},s+1\right)\right)^{\frac{1}{q}}\right.\\&\left.+(k-y)^2\left(\frac{|G'(y)|^q}{(rs+1)}+\frac{|G'(k)|^q}{r}B\left(\frac{1}{r},s+1\right)\right)^{\frac{1}{q}}\right],\end{aligned}$$

$$\begin{aligned}\left|\frac{G(j)+G(k)}{2}-\frac{k-E(Y)}{k-j}\right|&\leqslant\frac{k-j}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|G'(j)|^q}{r}B\left(\frac{1}{r},s+1\right)+\frac{|G'(\frac{j+k}{2})|^q}{rs+1}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{|G'(k)|^q}{r}B\left(\frac{1}{r},s+1\right)+\frac{|G'(\frac{j+k}{2})|^q}{rs+1}\right)^{\frac{1}{q}}\right],\end{aligned}$$

$$\begin{aligned}\left|\frac{(k-y)G(k)+(y-j)G(j)}{k-j}-\frac{k-E(Y)}{k-j}\right|&\leqslant\frac{1}{2^{1-\frac{1}{q}}(k-j)}\left[(y-j)^2\left(\frac{|G'(y)|^q}{(sr+1)(sr+2)}+\frac{|G'(j)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right.\\&\left.+(k-y)^2\left(\frac{|G'(y)|^q}{(sr+1)(sr+2)}+\frac{|G'(k)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right],\end{aligned}$$

$$\begin{aligned}\left|\frac{G(j)+G(k)}{2}-\frac{k-E(Y)}{k-j}\right|&\leqslant\frac{2^{\frac{1}{q}}}{8(k-j)}\left[\left(\frac{|G'(\frac{j+k}{2})|^q}{(sr+1)(sr+2)}+\frac{|G'(j)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{|G'(\frac{j+k}{2})|^q}{(sr+1)(sr+2)}+\frac{|G'(k)|^q}{r}\left(B\left(\frac{1}{r},s+1\right)-B\left(\frac{2}{r},s+1\right)\right)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof of all above results are same as we have given in the Proof of Theorem 4.1.

Remark 4.3. All remarks of Section 2 are also hold for Theorem 4.1 and Remark 4.2.

5. Conclusion

In this article, we have generalized some results about famous Hermite-Hadamard type inequalities for (s, r)-convax functions in mixed kind via classical integrals using Hölder and Power-mean inequality and applications for probability theory and numerical integration are also provided. We have recaptured various results of different articles [4, 9] and also provided special cases of class of (s, r)-convex functions on several choices of s, r as remarks.

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