J. Math. Computer Sci., 30 (2023), 352–371

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Generalized $\mathfrak{m}$-preinvexity on fractal set and related local fractional integral inequalities with applications

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Abstract

In this work, we address and explore the concept of generalized $\mathfrak{m}$-preinvex functions on fractal sets along with linked local fractional integral inequalities. Additionally, some engrossing algebraic properties are presented to facilitate the current initiated idea. Furthermore, we prove the latest variant of Hermite-Hadamard type inequality employing the proposed definition of preinvexity. We also derive several novel versions of inequalities of the Hermite-Hadamard type and Fejér-Hermite-Hadamard type for the first-order local differentiable generalized $\mathfrak{m}$-preinvex functions. Finally, some new inequalities for the generalized means and generalized random variables are established as applications.

Keywords: Generalized $\mathfrak{m}$-preinvex functions, generalized Hermite-Hadamard inequality, generalized Fejér-Hermite-Hadamard type inequality, fractal sets, local fractional integral inequalities.

2020 MSC: 26D10, 26D15, 26A33, 26A51.

1. Introduction

Theory of convex functions is not only effective in different domains of mathematics but also it provides a strong background for mathematical computations in order to handle different real world problems. In recent years, the theory of convex analysis turned out to be the center of attraction for many researchers because of its substantial applications in various areas of engineering and sciences. The notion of convexity has been studied in different frame works, e.g., quantum calculus, fractional calculus, fractal sets and fuzzy sets. In the literature, one can find different studies about extensions and generalizations of convex functions for different variations of integral inequalities that have been carried out in past few years. Significant number of studies have proven that convexity theory has a strong connection with integral inequalities. The most special inequality among all the fundamental inequalities is the Hermite-Hadamard inequality, initiated by Hadamard, see [10]. Many crucial results, which are generalizations,

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doi: 10.22436/jmcs.030.04.05

Received: 2023-01-08 *Revised:* 2023-01-15 *Accepted:* 2023-01-30

refinements, and improvements of the classical Hermite-Hadamard inequality are associated with various generalizations of convex function, we refer to [20, 23–25, 41] and references there in.

One of the notable generalizations of convex function is preinvex function. For the concept of invex set and preinvex function, see references [3, 5, 6, 12, 35]. A new class of functions, named as m-convex function, was defined by Toader, see [33]. In [14], Latif defined the class of m-preinvex and (α, m)-preinvex functions. Various authors have contributed to the study of improvements, generalizations, and properties of m-convex and m-preinvex functions with outcomes related to integral inequalities. For more investigation about m-convexity and its generalizations, we refer the reader to [7, 9, 15, 16, 19, 31, 32].

The concept of Local Fractional Calculus (LFC) has solid applications in applied sciences like physical sciences, random walk process, communication engineering, and control theory, see [4, 13, 34, 38]. Researchers have extended the concept of convexity on fractal sets to analyse Hermite-Hadamard inequalities, see [1, 8, 11, 17, 21, 22]. In particular, Du et al. [8] presented generalized m-convex functions in fractal domain and derived a number of significant results in favor of this definition and established various inequalities via such functions. Ohud Almutairi [1] studied the concept of generalized $(h-m)$-convexity in the frame work of fractal sets and established generalized Fejér-Hermite-Hadamard type inequalities through this notion of convex functions. Al-Sa'di et al. [2] introduced the γ-preinvex function and derived a number of generalized Hermite-Hadamard type inequalities. Wenbing Sun discussed various generalizations of convex functions in the fractal theory in references [26, 28–30, 39].

Inspired by the above research and ongoing work, our emphasis in this paper is to study generalized m-preinvex functions on fractal sets. Moreover, we explore some of its inspiring algebraic properties and related inequalities. In particular, we will derive some refinements of the Hermite-Hadamard inequality for differentiable generalized m-preinvex functions, as well as some latest estimates for Fejér-Hermite-Hadamard type inequality. Finally, we will discus some applications to generalized special means, and random variables.

Let us recall a few definitions of generalized convex functions that will be used in this work. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a nonempty set, $\mathcal{G} : \mathcal{U} \to \mathbb{R}^n$ be a continuous mapping, and $\gamma(.,.) : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^n$ be a bi-function. We say that the set $\mathcal{U}$ is an invex set with respect to γ if

$$\zeta + \nu\gamma(\zeta, \vartheta) \in \mathcal{U},$$

for every $\zeta, \vartheta \in \mathcal{U}$ whenever $0 \leqslant \nu \leqslant 1$, see [6]. The mapping $\mathcal{G} : \mathcal{U} \to \mathbb{R}^n$ is said to be preinvex on the invex set $\mathcal{U}$ with respect to γ if

$$\mathcal{G}(\zeta + \nu\gamma(\zeta, \vartheta)) \leqslant (1-\nu)\mathcal{G}(\zeta) + \nu\mathcal{G}(\vartheta),$$

for every $\zeta, \vartheta \in \mathcal{U}$, whenever $0 \leqslant \nu \leqslant 1$. If $-\mathcal{G}$ is a preinvex function, then $\mathcal{G}$ is preconcave function.

We recall the following condition given by Mohan and Samir for the bi-function γ, see Ref. [18] for more details.

Condition C. Let $\mathcal{U} \subseteq \mathbb{R}^n$ be an invex set. If $0 \leqslant \nu \leqslant 1$, then for every $\zeta, \vartheta \in \mathcal{U}$, the bi-function $\gamma(.,.)$ satisfies

$$\gamma(\vartheta, \vartheta + \nu\gamma(\zeta, \vartheta)) = -\rho\gamma(\zeta, \vartheta), \quad \gamma(\zeta, \vartheta + \nu\gamma(\zeta, \vartheta)) = (1-\nu)\gamma(\zeta, \vartheta).$$

Moreover, the following equality holds from Condition C:

$$\gamma(\vartheta + \nu_2\gamma(\zeta, \vartheta), \vartheta + \nu_1\gamma(\zeta, \vartheta)) = (\nu_2 - \nu_1)\gamma(\zeta, \vartheta),$$

for every $\zeta, \vartheta \in \mathcal{U}$ and $\nu_1, \nu_2 \in [0, 1]$.

We also recall the following definition.

Definition 1.1 ([14]). A function $\mathcal{G}$ on the invex set $\mathcal{U} \subseteq [0, b^*], b^* > 0$, is said to be m-preinvex with respect to γ if

$$\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta)) \leqslant (1-\nu)\mathcal{G}(\zeta)+\mathfrak{m}\nu\mathcal{G}\left(\frac{\vartheta}{\mathfrak{m}}\right),$$

holds for all $\zeta,\vartheta \in \mathcal{U}, \nu \in [0,1]$ and $\mathfrak{m} \in (0,1]$. The function $\mathcal{G}$ is said to be $\mathfrak{m}$-preconcave if and only if $-\mathcal{G}$ is $\mathfrak{m}$-preinvex.

Theorem 1.2 ([10])**.** *For a convex function $\mathcal{G} : I \subseteq \mathbb{R} \to \mathbb{R}$, if $\zeta,\vartheta \in I$ and $\zeta < \vartheta$, then*

$$\mathcal{G}\left(\frac{\zeta+\vartheta}{2}\right) \leqslant \frac{1}{\vartheta-\zeta}\int_{\zeta}^{\vartheta}\mathcal{G}(y)dy \leqslant \frac{\mathcal{G}(\zeta)+\mathcal{G}(\vartheta)}{2}.$$

We recall the theory of LFC initiated by Yang in [36, 37]. Let Ω^{ρ} be a Yang's fractional set, Ω is the base set and ρ $(0 < \rho \leqslant 1)$ be the dimension of cantor set. We define the ρ-type $\mathbb{Z}^{\rho}$ (set of integers) as

$$\mathbb{Z}^{\rho} = \{0^{\rho}, \pm 1^{\rho}, \pm 2^{\rho}, \pm 3^{\rho}, \ldots\},$$

the ρ-type Q^{ρ} is defined by

$$Q^{\rho} = \{\mathfrak{m}^{\rho} = \left(\frac{p}{q}\right)^{\rho};\ p,q \in \mathbb{Z},\ q \neq 0\},$$

the ρ-type $\mathfrak{I}^{\rho}$ is defined by

$$\mathfrak{I}^{\rho} = \{\mathfrak{m}^{\rho} \neq \left(\frac{p}{q}\right)^{\rho};\ p,q \in \mathbb{Z}, q \neq 0\},$$

and the ρ-type $\mathbb{R}^{\rho}$ can be defined by

$$\mathbb{R}^{\rho} = Q^{\rho} \cup \mathfrak{I}^{\rho}.$$

The following are some characteristics that fractal numbers satisfy.

If $\zeta^{\rho}, \vartheta^{\rho}, \nu^{\rho} \in \mathbb{R}^{\rho}$, then

- $\zeta^{\rho} + \vartheta^{\rho} \in \mathbb{R}^{\rho}, \zeta^{\rho}\vartheta^{\rho} \in \mathbb{R}^{\rho}$;
- $\zeta^{\rho} + \vartheta^{\rho} = \vartheta^{\rho} + \zeta^{\rho} = (\vartheta+\zeta)^{\rho} = (\vartheta+\zeta)^{\rho}$;
- $\zeta^{\rho} + (\vartheta^{\rho} + \nu^{\rho}) = (\zeta+\vartheta)^{\rho} + \nu^{\rho}$;
- $\zeta^{\rho}\vartheta^{\rho} = \vartheta^{\rho}\zeta^{\rho} = (\zeta\vartheta)^{\rho} = (\vartheta\zeta)^{\rho}$;
- $\zeta^{\rho}(\vartheta^{\rho}\nu^{\rho}) = (\zeta^{\rho}\vartheta^{\rho})\nu^{\rho}$;
- $\zeta^{\rho}(\vartheta^{\rho} + \nu^{\rho}) = \zeta^{\rho}\vartheta^{\rho} + \zeta^{\rho}\nu^{\rho}$;
- $\zeta^{\rho} + 0^{\rho} = 0^{\rho} + \zeta^{\rho} = \zeta^{\rho}$, and $\zeta^{\rho}1^{\rho} = 1^{\rho}\zeta^{\rho} = \zeta^{\rho}$;
- if $\zeta^{\rho} < \vartheta^{\rho}$, then $\zeta^{\rho} + \nu^{\rho} < \vartheta^{\rho} + \nu^{\rho}$;
- if $0^{\rho} < \zeta^{\rho}, 0^{\rho} < \vartheta^{\rho}$, then $0^{\rho} < \zeta^{\rho}\vartheta^{\rho}$.

It is also necessary for us to define Local Fractional Derivative (LFD) and Local Fractional Integral (LFI) on $\mathbb{R}^{\rho}$.

Definition 1.3 ([36, 37]). A non-differentiable function $\mathcal{G} : \mathbb{R} \to \mathbb{R}^{\rho}, y \to \mathcal{G}(y)$ is said to be local fractional continuous at y_0 if for any $\epsilon > 0$, there exists $\sigma > 0$ such that

$$|\mathcal{G}(y) - \mathcal{G}(y_0)| < \epsilon^{\rho},$$

holds whenever $|y - y_0| < \sigma$, with $\epsilon, \sigma \in \mathbb{R}$. We donate $\mathcal{G}(y) \in C_{\rho}(b, c)$ if $\mathcal{G}(y)$ is local continuous on (b, c).

Definition 1.4 ([36, 37]). The LFD of the function $\mathcal{G}(y)$ of order ρ $(0 < \rho \leqslant 1)$ at $y = y_0$ can be defined as

$$\mathcal{G}^{(\rho)}(y_0) = \left.\frac{d^{\rho}\mathcal{G}(y)}{dy^{\rho}}\right|_{y=y_0} = \lim_{y \to y_0} \frac{\Gamma(1+\rho)(\mathcal{G}(y) - \mathcal{G}(y_0))}{(y - y_0)^{\rho}}.$$

$D_{\rho}(b, c)$ will denote the set of ρ-local derivative functions. If there exists $\mathcal{G}^{((n+1)\rho)}(y) = \overbrace{D_y^{\rho} \cdots D_y^{\rho}}^{(n+1)\text{ times}} \mathcal{G}(y)$ for any $y \in \mathbb{I} \subseteq \mathbb{R}$, we denote $\mathcal{G} \in D_{(n+1)\rho}(\mathbb{I})$, and $n = 0, 1, 2, \ldots$.

Definition 1.5 ([36, 37]). Let $\mathcal{G}(y) \in C_{\rho}[b, c]$. The LFI of $\mathcal{G}(y)$ can be defined by

$${}_{b}I_{c}^{\rho}\mathcal{G}(y) = \frac{1}{\Gamma(1+\rho)} \int_{b}^{c} \mathcal{G}(\nu)(d\nu)^{\rho} = \frac{1}{\Gamma(1+\rho)} \lim_{\Delta\nu \to 0} \sum_{i=0}^{N-1} \mathcal{G}(\nu_i)(\Delta\nu_i)^{\rho},$$

where $b = \nu_0 < \nu_1 < \cdots < \nu_{N-1} < \nu_N = c, [\nu_i, \nu_{i+1}]$ is partition of $[b, c]$, $\Delta\nu_i = \nu_{i+1} - \nu_i, \Delta\nu = \max\{\Delta\nu_0, \Delta\nu_1 \cdots \Delta\nu_{N-1}\}$.

Note that ${}_{b}I_{b}^{\rho}\mathcal{G}(y) = 0$ and ${}_{b}I_{c}^{\rho}\mathcal{G}(y) = -{}_{c}I_{b}^{\rho}\mathcal{G}(y)$ if $b < c$. We denote $\mathcal{G}(y) \in I_{y}^{\rho}[b, c]$ if there exists ${}_{b}I_{y}^{\rho}\mathcal{G}(y)$ for any $y \in [b, c]$. Mittag-Leffler function of fractal order ρ $(0 < \rho \leqslant 1)$ on Yang's fractal sets can be defined by

$$E_{\rho}(y^{\rho}) = \sum_{k=0}^{\infty} \frac{y^{k\rho}}{\Gamma(1+k\rho)} \quad , y \in \mathbb{R}.$$

Formulas for LFC of Mittag-Leffler function are given as follows.

Lemma 1.6 ([36, 37]). *LFD and LFI of Mittag-Leffler function can be given as:*

$$\frac{d^{\rho}E_{\rho}(ky^{\rho})}{dy^{\rho}} = kE_{\rho}(ky^{\rho}), \quad k \textit{ is a constant},$$

and

$${}_{b}I_{c}^{\rho}E_{\rho}(y^{\rho}) = E_{\rho}(c^{\rho}) - E_{\rho}(b^{\rho}).$$

Lemma 1.7 ([36, 37]).

1. *Let* $\mathcal{G}_2(y) = \mathcal{G}_1^{(\rho)}(y) \in C_{\rho}[b, c]$. *Then*

$${}_{b}I_{c}^{\rho}\mathcal{G}_2(y) = \mathcal{G}(c) - \mathcal{G}(b).$$

2. *Let* $\mathcal{G}_2(y), \mathcal{G}_1(y) \in D_{\rho}[b, c]$ *and* $\mathcal{G}_2^{(\rho)}(y), \mathcal{G}_1^{(\rho)}(y) \in C_{\rho}[b, c]$. *Then*

$${}_{b}I_{c}^{\rho}\mathcal{G}_2(y)\mathcal{G}_1^{(\rho)}(y) = \mathcal{G}_2(y)\mathcal{G}_1(y)|_{b}^{c} - {}_{b}I_{c}^{\rho}\mathcal{G}_2^{(\rho)}(y)\mathcal{G}_1(y).$$

Lemma 1.8 ([36, 37]). *The local fractional derivative and integral of the function* $\mathcal{Q}(y) = y^{r\rho} \in C_{\rho}[b, c]$ *is given by*

$$\frac{d^{\rho}y^{r\rho}}{d\nu^{\rho}} = \frac{\Gamma(1+r\rho)}{\Gamma(1+(r-1)\rho)}y^{(r-1)\rho},$$

$$\frac{1}{\Gamma(1+\rho)}\int_b^c \nu^{r\rho}(d\nu)^\rho = \frac{\Gamma(1+r\rho)}{\Gamma(1+(r+1)\rho)}(c^{(r+1)\rho}-b^{(r+1)\rho}), \quad r>0.$$

In particular, if $b=0$, $c=1$, and $r=1$, in Lemma 1.8, we obtain the following formula which will be useful in the proofs (see [30]),

$$\frac{1}{\Gamma(1+\rho)}\int_0^1 \nu^\rho(d\nu)^\rho = \int_0^1(1-\nu)^\rho(d\nu)^\rho = \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}. \tag{1.1}$$

Lemma 1.9 ([36, 37]). *Let $\mathcal{Q}(y)=1$. Then by property of mean value theorem for local fractional integrals, we have*

$${}_bI_c^\rho 1^\rho = \frac{(c-b)^\rho}{\Gamma(1+\rho)}.$$

Lemma 1.10 ([36, 37]). *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and $\mathcal{G}_2(y),\mathcal{G}_1(y)\in C_\rho[b,c]$. Then*

$$\frac{1}{\Gamma(1+\rho)}\int_b^c\left|\mathcal{G}_2(y)\mathcal{G}_1(y)\right|(dy)^\rho \leqslant \left(\frac{1}{\Gamma(1+\rho)}\int_b^c\left|\mathcal{G}_2(y)\right|^p(dy)^\rho\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\rho)}\int_b^c\left|\mathcal{G}_1(y)\right|^q(dy)^\rho\right)^{\frac{1}{q}}.$$

We also recall the generalized beta function:

$$B_\rho(y,x) = \frac{1}{\Gamma(1+\rho)}\int_0^1 \nu^{(y-1)\rho}(1-\nu)^{(x-1)\rho}(d\nu)^\rho, \quad y>0, x>0.$$

2. Main results

Let us begin by introducing the generalized definition of m-preinvex mapping on fractal sets.

Definition 2.1. The mapping $\mathcal{G}:\mathcal{U}\to\mathbb{R}^\rho$ on the invex set $\mathcal{U}\subseteq[0,b^*]$, $b^*>0$, is said to be generalized m-preinvex with respect to $\gamma:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ if

$$\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta)) \leqslant (1-\nu)^\rho\mathcal{G}(\zeta)+m^\rho\nu^\rho\mathcal{G}\left(\frac{\vartheta}{m}\right)$$

holds for all $\zeta,\vartheta\in\mathcal{U}$, $\nu\in[0,1]$ and $m\in(0,1]$. If $-\mathcal{G}$ is generalized m-preinvex then $\mathcal{G}$ is said to be generalized m-preconcave function.

Remark 2.2. In Definition 2.1, if we take $\rho=1$ we get the definition of m-preinvex functions introduced in [14]. If we take $\rho=1$ and $m=1$ in Definition 2.1 we get classical definition of preinvex function introduced in [35].

Example 2.3. *Let the function $\mathcal{G}:\mathcal{U}\to\mathbb{R}^\rho$ and $\mathcal{G}(\zeta)=\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\zeta^{2\rho}$, where $\zeta\in[0,1]$ with $\gamma(\vartheta,\zeta)=\zeta-\vartheta$ and $m=1,\rho=1$. Then $\mathcal{G}(\zeta)$ is a generalized m-preinvex function.*

Definition 2.4. The mapping $\mathcal{G}:\mathcal{U}\to\mathbb{R}^\rho$ on the invex set $\mathcal{U}\subseteq[0,b^*]$, $b^*>0$, is said to be explicitly generalized m-preinvex with respect to $\gamma:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ if $\mathcal{G}(\zeta)\neq\mathcal{G}(\vartheta)$, $\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta))<(1-\nu)^\rho\mathcal{G}(\zeta)+m^\rho\nu^\rho\mathcal{G}\left(\frac{\vartheta}{m}\right)$.

Proposition 2.5. *If $\mathcal{G}_1,\mathcal{G}_2:\mathcal{U}\subseteq[0,b^*]\to\mathbb{R}^\rho$ are generalized m-preinvex functions on fractal sets, and $\Lambda^\rho>0^\rho$, then:*

1. *$\mathcal{G}_1+\mathcal{G}_2$ is a generalized m-preinvex function;*
2. *$\Lambda^\rho\mathcal{G}$ is a generalized m-preinvex function.*

Proof.

(1). Using Definition 2.1 we have

$$\mathcal{G}_1(\zeta+\nu\gamma(\vartheta,\zeta)) \leqslant (1-\nu)^\rho \mathcal{G}_1(\zeta) + m^\rho \nu^\rho \mathcal{G}1\left(\frac{\vartheta}{m}\right),$$

and

$$\mathcal{G}_2(\zeta+\nu\gamma(\vartheta,\zeta)) \leqslant (1-\nu)^\rho \mathcal{G}_2(\zeta) + m^\rho \nu^\rho \mathcal{G}_2\left(\frac{\vartheta}{m}\right),$$

$\forall \zeta,\vartheta \in \mathcal{U} \subset [0,b^*]$, $m \in (0,1)$, and $\nu \in [0,1]$. Hence,

$$\begin{aligned}(\mathcal{G}_1+\mathcal{G}_2)(\zeta+\nu\gamma(\vartheta,\zeta)) &\leqslant (1-\nu)^\rho \mathcal{G}_1(\zeta) + m^\rho\nu^\rho \mathcal{G}_1\left(\frac{\vartheta}{m}\right) + (1-\nu)^\rho \mathcal{G}_2(\zeta) + m^\rho\nu^\rho \mathcal{G}_2\left(\frac{\vartheta}{m}\right)\\
&\leqslant (1-\nu)^\rho(\mathcal{G}_1(\zeta)+\mathcal{G}_2(\zeta)) + m^\rho\nu^\rho\left(\mathcal{G}_1\left(\frac{\vartheta}{m}\right)+\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\right)\\
&\leqslant (1-\nu)^\rho((\mathcal{G}_1+\mathcal{G}_2)(\zeta)) + m^\rho\nu^\rho(\mathcal{G}_1+\mathcal{G}_2)\left(\frac{\vartheta}{m}\right).\end{aligned}$$

(2). We have,

$$\begin{aligned}\Lambda^\rho(\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta)) &\leqslant \Lambda^\rho\left((1-\nu)^\rho(\mathcal{G}(\zeta)+m^\rho\nu^\rho\mathcal{G}\left(\frac{\vartheta}{m}\right)\right)\\
&\leqslant (1-\nu)^\rho\Lambda^\rho(\mathcal{G}(\zeta)+m^\rho\nu^\rho\Lambda^\rho\mathcal{G}\left(\frac{\vartheta}{m}\right)\\
&\leqslant (1-\nu)^\rho\left(\Lambda^\rho\mathcal{G}\right)(\zeta)+m^\rho\nu^\rho\left(\Lambda^\rho\mathcal{G}\right)\left(\frac{\vartheta}{m}\right).\end{aligned}$$

□

Proposition 2.6. *Let $\mathcal{G}_1, \mathcal{G}_2 : \mathcal{U} \longrightarrow \mathbb{R}_0^\rho$ be two m-preinvex functions which are generalized similarly ordered. If $(1-\nu)^\rho + m^\rho\nu^\rho \leqslant 1$, then the product $\mathcal{G}_1\mathcal{G}_2$ is a generalized m-preinvex function with respect to γ for $m \in (0,1], \nu \in [0,1]$.*

Proof. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are generalized m-preinvex functions, then

$$\begin{aligned}&\mathcal{G}_1(\zeta+\nu\gamma(\vartheta,\zeta))\mathcal{G}_2(\zeta+\nu\gamma(\vartheta,\zeta))\\
&\quad\leqslant \left[(1-\nu)^\rho\mathcal{G}_1(\zeta)+m^\rho\nu^\rho\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\right]\left[(1-\nu)^\rho\mathcal{G}_2(\zeta)+m^\rho\nu^\rho\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\right]\\
&\quad= (1-\nu)^{2\rho}\mathcal{G}_1(\zeta)\mathcal{G}_2(\zeta)+m^\rho\nu^\rho(1-\nu)^\rho\mathcal{G}_1(\zeta)\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\\
&\qquad+m^\rho\nu^\rho(1-\nu)^\rho\mathcal{G}_2(\zeta)\mathcal{G}_1\left(\frac{\vartheta}{m}\right)+m^{2\rho}\nu^{2\rho}\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\\
&\quad\leqslant (1-\nu)^{2\rho}\mathcal{G}_1(\zeta)\mathcal{G}_2(\zeta)+m^\rho\nu^\rho(1-\nu)^\rho\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\\
&\qquad+m^\rho\nu^\rho(1-\nu)^\rho\mathcal{G}_2(\zeta)\mathcal{G}_1(\zeta)+m^{2\rho}\nu^{2\rho}\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\Big]\\
&\quad\leqslant (1-\nu)^\rho\left[(1-\nu)^\rho\mathcal{G}_1(\zeta)\mathcal{G}_2(\zeta)+m^\rho\nu^\rho\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\right]\\
&\qquad+m^\rho\nu^\rho\left[(1-\nu)^\rho\mathcal{G}_1(\zeta)\mathcal{G}_2(\zeta)+m^\rho\nu^\rho\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\right]\end{aligned}$$

$$\leqslant [(1-\nu)^{\rho}+m^{\rho}\nu^{\rho}]\left[(1-\nu)^{\rho}\mathcal{G}_1(\zeta)\mathcal{G}_2(\zeta)+m^{\rho}\nu^{\rho}\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\mathcal{G}_2\left(\frac{\vartheta}{m}\right)\right]$$
$$\leqslant (1-\nu)^{\rho}\mathcal{G}_1(\zeta)\mathcal{G}_2(\zeta)+m^{\rho}\nu^{\rho}\mathcal{G}_1\left(\frac{\vartheta}{m}\right)\mathcal{G}_2\left(\frac{\vartheta}{m}\right).$$

Hence the proof is done. □

3. Hermite-Hadamard inequalities for generalized m-preinvex functions

Theorem 3.1. *Suppose $\mathcal{U}\subseteq\mathbb{R}$ be an open invex subset with $\gamma:\mathcal{U}\times\mathcal{U}\to\mathbb{R}$, and $\zeta,\vartheta\in\mathcal{U}$, $\zeta<\zeta+\gamma(\vartheta,\zeta)$. Suppose $\mathcal{G}:\mathcal{U}\to\mathbb{R}^{\rho}$, $\rho\in(0,1]$, be a LDF with $\gamma(\vartheta,\zeta)\geqslant 0$ and $\mathcal{G}^{\rho}\in I_y^{\rho}[\zeta,\zeta+\gamma(\vartheta,\zeta)]$. If $\mathcal{G}$ is a generalized m-preinvex function on $[\zeta,\zeta+\gamma(\vartheta,\zeta)]$, and γ satisfies Condition **C**, then one has the following inequality:*

$$\frac{1^{\rho}}{\Gamma(1+\rho)}\mathcal{G}\left(\zeta+\frac{1}{2}\gamma(\vartheta,\zeta)\right)\leqslant\frac{1}{2^{\rho}\gamma^{2}(\vartheta,\zeta)}\left[{}_{\vartheta}I_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)+{}_{\frac{\vartheta}{m}}I_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\right]$$
$$\leqslant\frac{\Gamma(1+\rho)}{2^{\rho}\Gamma(1+2\rho)}\left[\mathcal{G}(\zeta)+m^{\rho}\left[\mathcal{G}\left(\frac{\zeta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m}\right)\right]+m^{2\rho}\mathcal{G}\left(\frac{\zeta}{m^{2}}\right)\right].$$

Proof. As $\mathcal{G}$ is a generalized m-preinvex function, then taking $t=\frac{1}{2}$ in Definition 2.1 we get

$$\mathcal{G}\left(a+\frac{1}{2}\gamma(b,a)\right)\leqslant\left(\frac{1}{2}\right)^{\rho}\left[\mathcal{G}(a)+m^{\rho}\mathcal{G}\left(\frac{b}{m}\right)\right].$$

Let $a=\zeta+\nu\gamma(\vartheta,\zeta)$, $b=\zeta+(1-\nu)\gamma(\vartheta,\zeta)$. Since γ satisfies Condition **C**, then we have

$$\mathcal{G}\left(\zeta+\frac{1}{2}\gamma(\vartheta,\zeta)\right)\leqslant\left(\frac{1}{2}\right)^{\rho}\left[\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta))+m^{\rho}\mathcal{G}\left(\frac{\zeta+(1-\nu)\gamma(\vartheta,\zeta)}{m}\right)\right].$$

Integrating local fractionally with respect to ν over $[0,1]$, we get

$$\begin{aligned}&\frac{2^{\rho}}{\Gamma(1+\rho)}\int_0^1\mathcal{G}\left(\zeta+\frac{1}{2}\gamma(\vartheta,\zeta)\right)(d\nu)^{\rho}\\&\leqslant\frac{1}{\Gamma(1+\rho)}\int_0^1\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta))(d\nu)^{\rho}m^{\rho}\frac{1}{\Gamma(1+\rho)}\int_0^1\mathcal{G}\left(\frac{\zeta}{m}+(1-\nu)\frac{\gamma(\vartheta,\zeta)}{m}\right)(d\nu)^{\rho}.\end{aligned}\tag{3.1}$$

It is easy to see that

$$\frac{1}{\Gamma(1+\rho)}\int_0^1\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta))(d\nu)^{\rho}=\frac{1}{\Gamma(1+\rho)}\int_{\zeta}^{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)\frac{(dx)^{\rho}}{\gamma^{\rho}(\vartheta,\zeta)}$$

and

$$\frac{1}{\Gamma(1+\rho)}\int_0^1\mathcal{G}\left(\frac{\zeta}{m}+(1-\nu)\frac{\gamma(\vartheta,\zeta)}{m}\right)(d\nu)^{\rho}=\frac{m^{\rho}}{\Gamma(1+\rho)}\int_{\frac{\zeta}{m}}^{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\frac{(dy)^{\rho}}{\gamma^{\rho}(\vartheta,\zeta)}.$$

Hence, equation (3.1) becomes

$$\frac{2^{\rho}}{\Gamma(1+\rho)}\mathcal{G}\left(\zeta+\frac{1}{2}\gamma(\vartheta,\zeta)\right)\leqslant\frac{1}{\gamma^{\rho}(\vartheta,\zeta)}\left[\frac{1}{\Gamma(1+\rho)}\int_{\zeta}^{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)(dx)^{\rho}+\frac{m^{\rho}}{\Gamma(1+\rho)}\int_{\frac{\zeta}{m}}^{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)(dy)^{\rho}\right]$$
$$\leqslant\frac{1}{\gamma^{\rho}(\vartheta,\zeta)}\left[{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)+m^{\rho}\,{}_{\frac{\zeta}{m}}I^{\rho}_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\right].$$

Thus, the first part of the inequality is proved.

To prove the second part of the inequality, we utilize the generalized m-preinvexity of $\mathcal{G}$ on fractal sets. We know that

$$\begin{aligned}
&\mathcal{G}(\zeta+\nu\gamma(\vartheta,\zeta))+m^{\rho}\mathcal{G}\left(\frac{\zeta+(1-\nu)\gamma(\vartheta,\zeta)}{m}\right)\\
&\leqslant(1-\nu)^{\rho}\mathcal{G}(\zeta)+m^{\rho}\nu^{\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right)+m^{\rho}\left[\nu^{\rho}\mathcal{G}\left(\frac{\zeta}{m}\right)+m^{\rho}(1-\nu)^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right]\\
&=(1-\nu)^{\rho}\mathcal{G}(\zeta)+m^{\rho}\nu^{\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right)+m^{\rho}\nu^{\rho}\mathcal{G}\left(\frac{\zeta}{m}\right)+m^{2\rho}(1-\nu)^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right).
\end{aligned}\tag{3.2}$$

Integrating the inequality in (3.2) local fractionally over $[0,1]$, then using (1.1) we get

$$\begin{aligned}
&\frac{1}{\gamma^{\rho}(\vartheta,\zeta)}\left[{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)+{}_{\frac{\zeta}{m}}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(y)\right]\\
&\leqslant\mathcal{G}(\zeta)\frac{1}{\Gamma(1+\rho)}\int_0^1(1-\nu)^{\rho}(d\nu)^{\rho}+m^{\rho}\left[\mathcal{G}\left(\frac{\zeta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m}\right)\right]\frac{1}{\Gamma(1+\rho)}\int_0^1\nu^{\rho}(d\nu)^{\rho}\\
&\quad+m^{2\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\frac{1}{\Gamma(1+\rho)}\int_0^1(1-\nu)^{\rho}(d\nu)^{\rho}\\
&\leqslant\mathcal{G}(\zeta)\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+m^{\rho}\left[\mathcal{G}\left(\frac{\zeta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m}\right)\right]\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+m^{2\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\\
&=\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left[\mathcal{G}(\zeta)+m^{\rho}\left[\mathcal{G}\left(\frac{\zeta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m}\right)\right]+m^{2\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right],
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1^{\rho}}{\Gamma(1+\rho)}\mathcal{G}\left(\zeta+\frac{1}{2}\gamma(\vartheta,\zeta)\right)&\leqslant\frac{1}{2^{\rho}\gamma^{\rho}(\vartheta,\zeta)}\left[{}_{\zeta}I_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)+{}_{\frac{\zeta}{m}}I_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\right]\\
&\leqslant\frac{\Gamma(1+\rho)}{2^{\rho}\Gamma(1+2\rho)}\left[\mathcal{G}(\zeta)+m^{\rho}\left[\mathcal{G}\left(\frac{\zeta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m}\right)\right]+m^{2\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right].
\end{aligned}$$

□

Corollary 3.2. *Let $\rho=1$ in Theorem 3.1. Then*

$$\begin{aligned}
\mathcal{G}\left(\zeta+\frac{1}{2}\gamma(\vartheta,\zeta)\right)&\leqslant\frac{1}{2\gamma(\vartheta,\zeta)}\left[\int_{\zeta}^{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)dx+\int_{\frac{\zeta}{m}}^{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)dy\right]\\
&\leqslant\frac{1}{4}\left[\mathcal{G}(\zeta)+m^{\rho}\left[\mathcal{G}\left(\frac{\zeta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m}\right)\right]+m^{2\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right].
\end{aligned}$$

Corollary 3.3. *Let $\gamma^{\rho}(\vartheta,\zeta)=(\vartheta-\zeta)^{\rho}$ in Theorem 3.1, then*

$$\begin{aligned}
\mathcal{G}\left(\frac{\zeta+\vartheta}{2}\right)&\leqslant\frac{1}{2(\vartheta-\zeta)^{\rho}}\left[{}_{\zeta}I_{\vartheta}\mathcal{G}(x)+{}_{\frac{\zeta}{m}}I_{\frac{\vartheta}{m}}\mathcal{G}(y)\right]\\
&\leqslant\frac{\Gamma(1+\rho)}{2^{\rho}\Gamma(1+2\rho)}\left[\mathcal{G}(\zeta)+m^{\rho}\left[\mathcal{G}\left(\frac{\zeta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m}\right)\right]+m^{2\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right].
\end{aligned}$$

Theorem 3.4. *Suppose $\mathcal{G}:\mathcal{U}\to\mathbb{R}^{\rho}$, $\rho\in(0,1]$, be a LDF with $\gamma(\vartheta,\zeta)\geqslant0$, with $\zeta,\vartheta\in\mathcal{U},\zeta<\zeta+\gamma(\vartheta,\zeta)$ and*

$\mathfrak{m} \in (0,1]$. *If* $0 \leqslant \zeta < \nu < \infty$ *and* $\mathcal{G} \in I_y^{\rho}[\zeta, \zeta + \gamma(\vartheta,\zeta)]$ *be generalized* $\mathfrak{m}$*-preinvex, then we get:*

$$\frac{1}{\gamma^{\rho}(\vartheta,\zeta)}\left[{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x) + {}_{\frac{\zeta}{m}}I^{\rho}_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\right] \leqslant \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left[\mathcal{G}(\zeta) + \mathcal{G}\left(\frac{\zeta}{m}\right) + m^{\rho}\left\{\mathcal{G}\left(\frac{\vartheta}{m}\right) + \mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right\}\right].$$

Proof. Since $\mathcal{G}$ is generalized $\mathfrak{m}$-preinvex, then

$$\begin{aligned}
\mathcal{G}(\zeta + \nu\gamma(\vartheta,\zeta)) &\leqslant (1-\nu)^{\rho}\mathcal{G}(\zeta) + m^{\rho}\nu^{\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right),\\
\mathcal{G}(\zeta + (1-\nu)\gamma(\vartheta,\zeta)) &\leqslant \nu^{\rho}\mathcal{G}(\zeta) + m^{\rho}(1-\nu)^{\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right),\\
\mathcal{G}\left(\frac{\zeta}{m} + \nu\frac{\gamma(\vartheta,\zeta)}{m}\right) &\leqslant (1-\nu)^{\rho}\mathcal{G}\left(\frac{\zeta}{m}\right) + m^{\rho}\nu^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right),\\
\mathcal{G}\left(\frac{\zeta}{m} + (1-\nu)\frac{\gamma(\vartheta,\zeta)}{m}\right) &\leqslant \nu^{\rho}\mathcal{G}\left(\frac{\zeta}{m}\right) + m^{\rho}(1-\nu)^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right).
\end{aligned}$$

Adding up the above four inequalities we get

$$\begin{aligned}
&\mathcal{G}(\zeta + \nu\gamma(\vartheta,\zeta)) + \mathcal{G}(\zeta + (1-\nu)\gamma(\vartheta,\zeta)) + \mathcal{G}\left(\frac{\zeta}{m} + \nu\frac{\gamma(\vartheta,\zeta)}{m}\right) + \mathcal{G}\left(\frac{\zeta}{m} + (1-\nu)\frac{\gamma(\vartheta,\zeta)}{m}\right)\\
&\leqslant (1-\nu)^{\rho}\mathcal{G}(\zeta) + m^{\rho}\nu^{\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right) + \nu^{\rho}\mathcal{G}(\zeta) + m^{\rho}(1-\nu)^{\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right)\\
&\quad + (1-\nu)^{\rho}\mathcal{G}\left(\frac{\zeta}{m}\right) + m^{\rho}\nu^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right) + \nu^{\rho}\mathcal{G}\left(\frac{\zeta}{m}\right) + m^{\rho}(1-\nu)^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{\Gamma(1+\rho)}\int_0^1 \mathcal{G}(\zeta + \nu\gamma(\vartheta,\zeta))(d\nu)^{\rho} &= \frac{1}{\Gamma(1+\rho)}\int_0^1 \mathcal{G}\left(\zeta + (1-\nu)\gamma(\vartheta,\zeta)\right)(d\nu)^{\rho}\\
&= \frac{1}{\Gamma(1+\rho)}\int_{\zeta}^{\zeta+\gamma(\vartheta,\zeta)} \mathcal{G}(x)\left(\frac{dx}{\gamma(\vartheta,\zeta)}\right)^{\rho}\\
&= \frac{1}{\gamma^{\rho}(\vartheta,\zeta)\Gamma(1+\rho)}\int_{\zeta}^{\zeta+\gamma(\vartheta,\zeta)} \mathcal{G}(x)(dx)^{\rho} = \frac{1}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\zeta}I_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x),
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\Gamma(1+\rho)}\int_0^1 \mathcal{G}\left(\frac{\zeta}{m} + \nu\frac{\gamma(\vartheta,\zeta)}{m}\right)(d\nu)^{\rho} &= \frac{1}{\Gamma(1+\rho)}\int_0^1 \mathcal{G}\left(\frac{\zeta}{m} + (1-\nu)\frac{\gamma(\vartheta,\zeta)}{m}\right)(d\nu)^{\rho}\\
&= \frac{1}{\Gamma(1+\rho)}\int_{\frac{\zeta}{m}}^{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}} \mathcal{G}(y)\left(\frac{dy}{\gamma(\vartheta,\zeta)}\right)^{\rho} = \frac{1}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\frac{\zeta}{m}}I_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{G}(\vartheta + \nu\gamma(\vartheta,\zeta)) + \mathcal{G}(\zeta + (1-\nu)\gamma(\vartheta,\zeta)) + \mathcal{G}\left(\frac{\zeta}{m} + \nu\frac{\gamma(\vartheta,\zeta)}{m}\right) + \mathcal{G}\left(\frac{\zeta}{m} + (1-\nu)\frac{\gamma(\vartheta,\zeta)}{m}\right)\\
&\leqslant \left[\mathcal{G}(\zeta) + m^{2\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right) + \mathcal{G}\left(\frac{\zeta}{m}\right) + m^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right](1-\nu)^{\rho} \qquad (3.3)\\
&\quad + \left[m^{\rho}\mathcal{G}\left(\frac{\vartheta}{m}\right) + \mathcal{G}(\zeta) + m^{\rho}\mathcal{G}\left(\frac{\vartheta}{m^2}\right) + \mathcal{G}\left(\frac{\zeta}{m}\right)\right]\nu^{\rho}.
\end{aligned}$$

Integrating inequality (3.3) local fractionally with respect to ν over $[0,1]$, we get

$$2^{\rho}\left[\frac{1}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\zeta}I_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)+\frac{1}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\frac{\zeta}{m}}I_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\right]$$
$$\leqslant\left[\mathcal{G}(\zeta)+\mathcal{G}\left(\frac{\zeta}{m}\right)+m^{\rho}\left\{\mathcal{G}\left(\frac{\vartheta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right\}\right]\left(\frac{1}{\Gamma(1+\rho)}\int_0^1\nu^{\rho}(d\nu)^{\rho}+\frac{1}{\Gamma(1+\rho)}\int_0^1(1-\nu)^{\rho}(d\nu)^{\rho}\right),$$
$$\frac{2^{\rho}}{\gamma^{\rho}(\vartheta,\zeta)}\left[{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)+{}_{\frac{\zeta}{m}}I^{\rho}_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\right]$$
$$\leqslant\left[\mathcal{G}(\zeta)+\mathcal{G}\left(\frac{\zeta}{m}\right)+m^{\rho}\left\{\mathcal{G}\left(\frac{\vartheta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right\}\right]\left[\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right].$$

Hence,

$$\frac{2^{\rho}}{\gamma^{\rho}(\vartheta,\zeta)}\left[{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)+{}_{\frac{\zeta}{m}}I^{\rho}_{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)\right]\leqslant 2^{\rho}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left[\mathcal{G}(\zeta)+\mathcal{G}\left(\frac{\zeta}{m}\right)+m^{\rho}\left\{\mathcal{G}\left(\frac{\vartheta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right\}\right],$$

which proves the theorem. □

Corollary 3.5. *If $\rho=1$ in Theorem 3.4, then*

$$\frac{1}{\gamma(\vartheta,\zeta)}\left[\int_{\zeta}^{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)dx+\int_{\frac{\zeta}{m}}^{\frac{\zeta+\gamma(\vartheta,\zeta)}{m}}\mathcal{G}(y)dy\right]\leqslant\frac{1}{2}\left[\mathcal{G}(\zeta)+\mathcal{G}\left(\frac{\zeta}{m}\right)+m^{\rho}\left\{\mathcal{G}\left(\frac{\vartheta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right\}\right].$$

Corollary 3.6. *If $\gamma^{\rho}(\vartheta,\zeta)=(\vartheta-\zeta)^{\rho}$ in Theorem 3.4, then*

$$\frac{1}{(\vartheta-\zeta)^{\rho}}\left[{}_{\zeta}I^{\rho}_{\vartheta}\mathcal{G}(x)+{}_{\frac{\zeta}{m}}I^{\rho}_{\frac{\vartheta}{m}}\mathcal{G}(y)\right]\leqslant\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left[\mathcal{G}(\zeta)+\mathcal{G}\left(\frac{\zeta}{m}\right)+m^{\rho}\left\{\mathcal{G}\left(\frac{\vartheta}{m}\right)+\mathcal{G}\left(\frac{\vartheta}{m^2}\right)\right\}\right].$$

4. Inequalities for differentiable generalized m-preinvex functions

This section is dedicated to establishing some inequalities for differentiable generalized m-Preinvex Functions. We first recall the following identity obtained by Sun in [27].

Lemma 4.1. *Let $\mathcal{U}\subseteq\mathbb{R}$ be an open invex subset with $\gamma:\mathcal{U}\times\mathcal{U}\longrightarrow\mathbb{R}$, $\zeta,\vartheta\in\mathbb{R}$, $\zeta<\zeta+\gamma(\vartheta,\zeta)$. Suppose that $\mathcal{G}:\mathcal{U}\to\mathbb{R}^{\rho}$, $\rho\in(0,1]$, such that $\mathcal{G}\in D_{\rho}(\mathbb{R})$ and $\mathcal{G}^{\rho}\in C_{\rho}[\zeta,\zeta+\gamma(\vartheta,\zeta)]$. Then the following equality holds:*

$$-\frac{\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))+\mathcal{G}(\zeta)}{2^{\rho}}+\frac{\Gamma(1+\rho)}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)=\frac{\gamma^{\rho}(\vartheta,\zeta)}{2^{\rho}\Gamma(1+\rho)}\int_0^1(1-2\nu)^{\rho}\mathcal{G}^{\rho}(\zeta+\nu\gamma(\vartheta,\zeta))(d\nu)^{\rho}.$$

Theorem 4.2. *Assume the hypothesis of Lemma 4.1. If $\left|\mathcal{G}^{\rho}\right|$ is generalized m-preinvex on $\mathcal{U}$, then we get:*

$$|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}-\frac{\Gamma(1+\rho)}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)|$$
$$\leqslant\frac{\gamma^{\rho}(\vartheta,\zeta)}{4^{\rho}}\left[\left(3^{\rho}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}-\frac{\Gamma(1+\rho)}{\Gamma(1+3\rho)}\right)|\mathcal{G}^{\rho}(\zeta)|+m^{\rho}\left(3^{\rho}\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)\left|\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right|\right].$$

Proof. From Lemma 4.1 we have

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}-\frac{1}{\gamma^{\rho}(\vartheta,\zeta)}\int_{\zeta}^{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)dx\right|\leqslant\frac{\gamma^{\rho}(\vartheta,\zeta)}{2^{\rho}\Gamma(1+\rho)}\int_0^1|1-2\nu|^{\rho}\,|\mathcal{G}^{\rho}(\zeta+\nu\gamma(\vartheta,\zeta))|(d\nu)^{\rho}.$$

As $|\mathcal{G}^{\rho}|$ is generalized m-preinvex on $\mathcal{U}$, then for every $\zeta, \vartheta \in \mathcal{U}$, $\zeta < \zeta + \gamma(\vartheta, \zeta)$ and $\nu \in [0, 1]$, we have

$$
\begin{aligned}
&|\mathcal{G}^{\rho}(\zeta + \nu\gamma(\vartheta, \zeta))| \\
&\quad \leqslant (1-\nu)^{\rho} |\mathcal{G}^{\rho}(\zeta)| + m^{\rho}\nu^{\rho} \left|\mathcal{G}(\frac{\vartheta}{m})\right| \\
&\quad \leqslant \frac{\gamma^{\rho}(\vartheta,\zeta)}{2^{\rho}\Gamma(1+\rho)} \left[|\mathcal{G}^{\rho}(\zeta)| \int_0^1 |1-2\nu|^{\rho}(1-\nu)^{\rho}(d\nu)^{\rho} + m^{\rho} \left|\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right| \int_0^1 |1-2\nu|^{\rho}\nu^{\rho}(d\nu)^{\rho} \right] \\
&\quad \leqslant \frac{\gamma^{\rho}(\vartheta,\zeta)}{4^{\rho}} \left[3^{\rho}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} - \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right) |\mathcal{G}(\zeta)| + m^{\rho}\left(3^{\rho}\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right) \left|\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right| \right],
\end{aligned}
$$

where the last inequality follows by the fact that,

$$
\begin{aligned}
\frac{1}{\Gamma(1+\rho)}\int_0^1 |1-2\nu|^{\rho}(1-\nu)^{\rho}(d\nu)^{\rho} &= \frac{3^{\rho}}{2^{\rho}}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} - \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right), \\
\frac{1}{\Gamma(1+\rho)}\int_0^1 |1-2\nu|^{\rho}\nu^{\rho}(d\nu)^{\rho} &= \left(\frac{3^{\rho}\Gamma(1+2\rho)}{2^{\rho}\Gamma(1+3\rho)} - \frac{1}{2^{\rho}}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right),
\end{aligned}
$$

which completes the proof of the theorem. □

Remark 4.3. If we take $\rho = 1$ in Theorem 4.2, we get Theorem 4 in [14].

Remark 4.4. If we take $m = 1$ in Theorem 4.2, we get Theorem 6 in [27].

Corollary 4.5. *If $\gamma^{\rho}(\vartheta,\zeta) = (\vartheta-\zeta)^{\rho}$ in Theorem 4.2, then*

$$
\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\vartheta)}{2^{\rho}} - \frac{\Gamma(1+\rho)}{(\vartheta-\zeta)^{\rho}}\int_{\zeta}^{\vartheta}\mathcal{G}(x)dx\right| \\
&\quad \leqslant \frac{(\vartheta-\zeta)^{\rho}}{4^{\rho}}\left[3^{\rho}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} - \frac{\Gamma(1+\rho)}{\Gamma(1+3\rho)}\right)|\mathcal{G}^{\rho}(\zeta)| + m^{\rho}\left(3^{\rho}\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)\left|\mathcal{G}\left(\frac{\vartheta}{m}\right)\right|\right].
\end{aligned}
$$

Theorem 4.6. *Assume the hypothesis of Lemma 4.1. If $|\mathcal{G}^{\rho}|^{q}$ is generalized m-preinvex on $\mathcal{U}$, for $q > 1$, then we get:*

$$
\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}} - \frac{\Gamma(1+\rho)}{\gamma^{\rho}(\vartheta,\zeta)}\, {}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)\right| \\
&\quad \leqslant \frac{\gamma^{\rho}(\vartheta,\zeta)}{2^{\rho}}\left[\left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(1+p)\rho)}\right)^{1/p}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(|\mathcal{G}^{\rho}(\zeta)|^{q} + m^{\rho}\left|\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right|^{q}\right)\right)^{1/q}\right],
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 4.1 and the famous Hölder's Inequality, we obtain

$$
\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2} - \frac{\Gamma(1+\rho)}{\gamma^{\rho}(\vartheta,\zeta)} I_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)\right| \\
&\quad \leqslant \frac{\gamma^{\rho}(\vartheta,\zeta)}{2^{\rho}}\left(\frac{1}{\Gamma(1+\rho)}\int_0^1 |(1-2\nu)^{\rho}|^{p}\,(d\nu)^{\rho}\right)\left(\frac{1}{\Gamma(1+\rho)}\int_0^1 |\mathcal{G}^{\rho}(\zeta+\nu\gamma(\vartheta,\zeta))|^{q}\,(d\nu)^{\rho}\right)^{1/q}.
\end{aligned}
\tag{4.1}
$$

Since $|\mathcal{G}^{\rho}|^{q}$ is generalized m-preinvex on $\mathcal{U}$ for every $\zeta, \vartheta \in \mathcal{U}$, and $\zeta < \zeta + \gamma(\vartheta,\zeta)$, $m \in (0, 1]$, we have

$$
|\mathcal{G}^{\rho}(\zeta+\nu\gamma(\vartheta,\zeta))|^{q} \leqslant (1-\nu)^{\rho}|\mathcal{G}^{\rho}(\zeta)|^{q} + m^{\rho}\nu^{\rho}\left|+\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right|^{q}
$$

and
$$\frac{1}{\Gamma(1+\rho)}\int_0^1 |(1-2\nu)^{\rho}|^{p}\,(d\nu)^{\rho}=\frac{\Gamma(1+p\rho)}{\Gamma(1+(p+1)\rho)}. \tag{4.2}$$

Consequently,

$$\begin{aligned}
&\frac{1}{\Gamma(1+\rho)}\int_0^1 |\mathcal{G}^{\rho}\left(\zeta+\nu\gamma(\vartheta,\zeta)\right)|^{q}(d\nu)^{\rho}\\
&\leqslant |\mathcal{G}^{\rho}(\zeta)|^{q}\left(\frac{1}{\Gamma(1+\rho)}\int_0^1(1-\nu)^{\rho}(d\nu)^{\rho}\right)+m^{\rho}\left|\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right|^{q}\left(\frac{1}{\Gamma(1+\rho)}\int_0^1\nu^{\rho}(d\nu)^{\rho}\right) \qquad (4.3)\\
&\leqslant |\mathcal{G}^{\rho}(\zeta)|^{q}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+m^{\rho}\left|\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right|^{q}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}.
\end{aligned}$$

Substituting (4.2) and (4.3) in (4.1) gives the desired inequality. □

Remark 4.7. If we take $\rho=1$ in Theorem 4.6, we get Theorem 5 in [14].

Remark 4.8. If we take $m=1$ in Theorem 4.6, we get Theorem 7 in [27].

Corollary 4.9. *If $\gamma^{\rho}(\vartheta,\zeta)=(\vartheta-\zeta)^{\rho}$ in Theorem 4.6, then*

$$\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\vartheta)}{2^{\rho}}-\frac{\Gamma(1+\rho)}{(\vartheta-\zeta)^{\rho}}{}_{\zeta}I^{\rho}_{\vartheta}\mathcal{G}(x)\right|\\
&\leqslant \frac{|(\vartheta-\zeta)^{\rho}|}{2^{\rho}}\left[\left(\frac{\Gamma(1+p\rho)}{\Gamma\left(1+(1+p)\rho\right)}\right)^{1/p}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\left(|\mathcal{G}^{\rho}(\zeta)|^{q}+m^{\rho}\left|\mathcal{G}^{\rho}\left(\frac{\vartheta}{m}\right)\right|^{q}\right)\right)^{1/q}\right].
\end{aligned}$$

Theorem 4.10. *Assume the hypothesis of Lemma 4.1. If $|\mathcal{G}^{\rho}|^{q}$ is generalized m-preinvex on $\mathcal{U}$ for $q\geqslant 1$, then*

$$\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}-\frac{\Gamma(1+\rho)}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)\right|\\
&\leqslant \frac{|\gamma^{\rho}(\vartheta,\zeta)|}{2^{\rho}}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-1/q}\left[|\mathcal{G}'(\zeta)|^{q}\frac{3^{\rho}}{2^{\rho}}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}-\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right)\right.\\
&\qquad\left.+m^{\rho}|\mathcal{G}^{\rho}(\vartheta/m)|^{q}\left(\frac{3^{\rho}}{2^{\rho}}\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}-\frac{1}{2^{\rho}}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)\right]^{1/q}.
\end{aligned} \tag{4.4}$$

Proof. By using the Lemma 4.1 and power mean integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(1+\rho)}{\gamma^{\rho}(\vartheta,\zeta)}{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)-\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}\right|\\
&\leqslant \frac{|\gamma^{\rho}(\vartheta,\zeta)|}{2^{\rho}}\left(\frac{1}{\Gamma(1+\rho)}\int_0^1|1-2\nu|(d\nu)^{\rho}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\rho)}\int_0^1|1-2\nu|\,|\mathcal{G}^{\rho}(\zeta+\nu\gamma(\vartheta,\zeta))|^{q}(d\nu)^{\rho}\right)^{1/q}.
\end{aligned} \tag{4.5}$$

Now, since
$$\frac{1}{\Gamma(1+\rho)}\int_0^1 |(1-2\nu)^{\rho}|\,(d\nu)^{\rho}=\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}, \tag{4.6}$$

and

$$\frac{1}{\Gamma(1+\rho)}\int_0^1 |(1-2\nu)^{\rho}|\,|\mathcal{G}^{\rho}(\zeta+\nu\gamma(\vartheta,\zeta))(d\nu)^{\rho}$$

$$\leqslant \frac{1}{\Gamma(1+\rho)}\int_0^1 |(1-2\nu)^\rho| \left[(1-\nu)^\rho |\mathcal{G}^\rho(\zeta)|^q + m^\rho\nu^\rho|\mathcal{G}^\rho(\vartheta/m)|^q\right](d\nu)^\rho \tag{4.7}$$
$$= |\mathcal{G}^\rho(\zeta)|^q \frac{1}{\Gamma(1+\rho)}\int_0^1 |(1-2\nu)^\rho|\ (1-\nu)^\rho d\nu + m^\rho |\mathcal{G}^\rho(\vartheta/m)|^q \frac{1}{\Gamma(1+\rho)}\int_0^1 \nu^\rho |(1-2\nu)^\rho|\ (d\nu)^\rho$$
$$= |\mathcal{G}^\rho(\zeta)|^q \frac{3^\rho}{2^\rho}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} - \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right) + m^\rho |\mathcal{G}^\rho(\vartheta/m)|^q \left(\frac{3^\rho}{2^2}\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \frac{\Gamma(1+\rho)}{2^\rho\Gamma(1+2\rho)}\right),$$

we get (4.4) by using (4.6) and (4.7) in (4.5). □

Remark 4.11. If we take $\rho = 1$ in Theorem 4.10, we get Theorem 6 in [14].

Remark 4.12. If we take $m = 1$ in Theorem 4.10, we get Theorem 8 in [27].

Corollary 4.13. *Choosing $\gamma^\rho(\vartheta,\zeta) = (\vartheta-\zeta)^\rho$ in Theorem 4.10, we get*

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\vartheta)}{2^\rho} - \frac{\Gamma(1+\rho)}{(\vartheta-\zeta)^\rho}{}_\zeta I_\vartheta^\rho \mathcal{G}(x)\right|$$
$$\leqslant \frac{|(\vartheta-\zeta)^\rho|}{2^\rho}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-\frac{1}{q}} \Bigg[|\mathcal{G}^\rho(\zeta)|^q \frac{3^\rho}{2^\rho}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} - \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right)$$
$$+ m^\rho \left|\mathcal{G}^\rho\left(\frac{\vartheta}{m}\right)\right|^q \left(\frac{3^\rho}{2^\rho}\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} - \frac{\Gamma(1+\rho)}{2^\rho\Gamma(1+2\rho)}\right)\Bigg]^{1/q}.$$

5. Some estimation type results for Fejér-Hermite-Hadamard type inequalities

Lemma 5.1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respet to $\gamma : K\times K \to \mathbb{R}$ and $\zeta,\vartheta \in K$. Let $\mathcal{G} : K \to R^\rho$ be a local continuous on K, and suppose $\omega : [\zeta,\zeta+\gamma(\vartheta,\zeta)] \to R^\rho; w \geqslant 0^\rho$ be a local continuous and symmetric about $\frac{2\zeta+\gamma(\vartheta,\zeta)}{2}$. Let $\mathcal{G}^\rho \in \mathbb{C}_\rho([\zeta,\zeta+\gamma(\vartheta,\zeta)])$ for $\zeta,\vartheta \in I$ and $\zeta < \zeta+\gamma(\vartheta,\zeta)$. Then*

$$\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^\rho}{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x) - {}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)$$
$$= \left(\frac{\gamma(\vartheta,\zeta)}{4}\right)^\rho \frac{1}{\Gamma(1+\rho)}\int_0^1 \left[\frac{1}{\Gamma(1+\rho)}\int_{l(\nu)}^{m(\nu)} \omega(x)(dx)^\rho\right]\left[\mathcal{G}^\rho(m(\nu)) - \mathcal{G}^\rho(l(\nu))\right](d\nu)^\rho,$$

where $l(\nu) = \zeta + \frac{1}{2}\nu\gamma(\vartheta,\zeta)$, and $m(\nu) = \zeta + \left(1-\frac{1}{2}\nu\right)\gamma(\vartheta,\zeta)$. In particular,

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^\rho}{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x) - {}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)\right|$$
$$\leqslant \left(\frac{\gamma(\vartheta,\zeta)}{4}\right)^\rho \frac{1}{\Gamma(1+\rho)}\int_0^1 \left|\left[\frac{1}{\Gamma(1+\rho)}\int_{l(\nu)}^{m(\nu)} \omega(x)(dx)^\rho\right]\right|\left[\left|\mathcal{G}^\rho(m(\nu))\right| + \left|\mathcal{G}^\rho(l(\nu))\right|\right](d\nu)^\rho,$$

and

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^\rho}{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x) - {}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)\right|$$
$$\leqslant \left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^\rho \frac{|\omega|_\infty}{\Gamma(1+\rho)}\left(\frac{1}{\Gamma(1+\rho)}\int_0^1 (1-\nu)^\rho\left(\left|\mathcal{G}^\rho(m(\nu))\right| + \left|\mathcal{G}^\rho(l(\nu))\right|\right)(d\nu)^\rho\right),$$

where $|\omega|_\infty = \sup_{x\in[\zeta,\zeta+\gamma(\vartheta,\zeta)]} \omega(x)$, and $\nu \in [0,1]$.

Proof. Since $\omega(x)$ is symmetric about $\frac{2\zeta+\gamma(\vartheta,\zeta)}{2}$, then $\omega(l(\nu))=\omega(m(\nu))$ for all $\nu\in[0,1]$, hence, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\rho)}\int_0^1\left[\frac{1}{\Gamma(1+\rho)}\int_{l(\nu)}^{m(\nu)}\omega(x)(dx)^\rho\right]\mathcal{G}^{\rho}(l(\nu))(d\nu)^\rho\\
&=\left(\frac{2}{\gamma(\vartheta,\zeta)}\right)^\rho\left[\frac{1}{\Gamma(1+\rho)}\int_{l(\nu)}^{m(\nu)}\omega(x)(dx)^\rho\right]\mathcal{G}(l(\nu))\Big|_0^1\\
&\quad-\left(\frac{2}{\gamma(\vartheta,\zeta)}\right)^\rho\left(\frac{\gamma(\vartheta,\zeta)}{2}\right)^\rho {}_0I_1^\rho\Big[\omega(m(\nu))+\omega(l(\nu))\Big]\mathcal{G}(l(\nu))\\
&=\left(\frac{2}{\gamma(\vartheta,\zeta)}\right)^\rho\mathcal{G}(\zeta)\,{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)-2^\rho{}_0I_1^\rho\omega(l(\nu))\mathcal{G}(l(\nu))\\
&=\left(\frac{2}{\gamma(\vartheta,\zeta)}\right)^\rho\mathcal{G}(\zeta)\,{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)-\left(\frac{4}{\gamma(\vartheta,\zeta)}\right)^\rho{}_\zeta I^\rho_{\frac{2\zeta+\gamma(\vartheta,\zeta)}{2}}\mathcal{G}(x)\omega(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&-\frac{1}{\Gamma(1+\rho)}\int_0^1\left[\frac{1}{\Gamma(1+\rho)}\int_{l(\nu)}^{m(\nu)}\omega(x)(dx)^\rho\right]\mathcal{G}^{\rho}(m(\nu))(d\nu)^\rho\\
&=\left(\frac{2}{\gamma(\vartheta,\zeta)}\right)^\rho\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))\,{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)-\left(\frac{4}{\gamma(\vartheta,\zeta)}\right)^\rho{}_{\frac{2\zeta+\gamma(\vartheta,\zeta)}{2}}I^\rho_{\zeta}\mathcal{G}(x)\omega(x).
\end{aligned}$$

After making suitable adjustments and operations we get the desired identity and inequalities. □

Remark 5.2. If we take $\rho=1$ in Lemma 5.1 we get Lemma 1 in [40].

Theorem 5.3. *Assume the hypothesis of Lemma 5.1. If $|\mathcal{G}^{\rho}|^q$ is generalized m-preinvex on $[\zeta,\zeta+\gamma(\vartheta,\zeta)]$ for $q\geqslant 1$, $\rho\in[0,1]$, and $m\in[0,1]$, then we get:*

$$\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^\rho}\,{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)-{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)\right|\\
&\leqslant\left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^\rho\frac{|\omega|_\infty}{\Gamma(1+\rho)}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\frac{1}{\Gamma(1+\rho)}-\frac{3^\rho}{2^\rho}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right)|\mathcal{G}^{\rho}(\zeta)|^q+\frac{m^\rho B_\rho(2,2)}{2^\rho}|\mathcal{G}^{\rho}(\vartheta/m)|^\rho\right]^{1/q}\\
&+\left[\frac{B_\rho(2,2)}{2^\rho}|\mathcal{G}^{\rho}(\zeta)|^q+m^\rho\left(\frac{1}{\Gamma(1+\rho)}-\frac{3^\rho}{2^\rho}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right)|\mathcal{G}^{\rho}(\vartheta/m)|^q\right]^{1/q},
\end{aligned}$$

where $|\omega|_\infty=\sup_{x\in[\zeta,\zeta+\gamma(\vartheta,\zeta)]}\omega(x)$.

Proof. By using Lemma 5.1 and Power Mean Inequality, we obtain

$$\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^\rho}\,{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)-{}_\zeta I^\rho_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)\right|\\
&\leqslant\left(\frac{\gamma(\vartheta,\zeta)}{4}\right)^\rho\frac{1}{\Gamma(1+\rho)}\int_0^1\left|\frac{1}{\Gamma(1+\rho)}\int_{l(\nu)}^{m(\nu)}\omega(x)(dx)^\rho\right|\Big|\mathcal{G}^{\rho}(m(\nu))-\mathcal{G}^{\rho}(l(\nu))\Big|(d\nu)^\rho
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\frac{\gamma(\vartheta,\zeta)}{4}\right)^{\rho} \frac{1}{\Gamma(1+\rho)} \int_0^1 \left|\frac{1}{\Gamma(1+\rho)} \int\limits_{l(\nu)}^{m(\nu)} \omega(x)(dx)^{\rho}\right| \left(\left|\mathcal{G}^{\rho}(m(\nu))\right| + \left|\mathcal{G}^{\rho}(l(\nu))\right|\right)(d\nu)^{\rho} \\
&\leqslant \left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^{\rho} \frac{|\omega|_{\infty}}{\Gamma(1+\rho)} \left(\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left(\left|\mathcal{G}^{\rho}(m(\nu))\right| + \left|\mathcal{G}^{\rho}(l(\nu))\right|\right)(d\nu)^{\rho}\right) \\
&\leqslant \left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^{\rho} \frac{|\omega|_{\infty}}{\Gamma(1+\rho)} \left(\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho}(d\nu)^{\rho}\right)^{1-1/q} \\
&\quad \times \left[\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left|\mathcal{G}^{\rho}(\zeta + \frac{1}{2}\nu\gamma(\vartheta,\zeta))\right|^{q} (d\nu)^{\rho}\right]^{1/q} \\
&\quad + \left[\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left|\mathcal{G}^{\rho}(\zeta + (1-\frac{1}{2}\nu)\gamma(\vartheta,\zeta))\right|^{q} (d\nu)^{\rho}\right]^{1/q} \\
&= \left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^{\rho} \frac{|\omega|_{\infty}}{\Gamma(1+\rho)} \left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-1/q} \left(\left[\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left|\mathcal{G}^{\rho}(\zeta + \frac{1}{2}\nu\gamma(\vartheta,\zeta))\right|^{q} (d\nu)^{\rho}\right]^{1/q}\right. \\
&\quad \left. + \left[\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left|\mathcal{G}^{\rho}(\zeta + (1-\frac{1}{2}\nu)\gamma(\vartheta,\zeta))\right|^{q} (d\nu)^{\rho}\right]^{1/q}\right).
\end{aligned}
\tag{5.1}
$$

Since $|\mathcal{G}^{\rho}|^{q}$ is a generalized m-preinvex function, then for any $\nu \in [0,1]$ we get

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left|\mathcal{G}^{\rho}(\zeta + \frac{1}{2}\nu\gamma(\vartheta,\zeta))\right|^{q} (d\nu)^{\rho} \\
&\quad \leqslant \frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} |(1-\frac{1}{2}\nu)^{\rho} \mathcal{G}^{\rho}(\zeta) + m^{\rho}\left(\frac{\nu}{2}\right)^{\rho} \mathcal{G}^{\rho}(\vartheta/m)|^{q} (d\nu)^{\rho} \\
&\quad \leqslant |\mathcal{G}^{\rho}(\zeta)|^{q} \frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho}(1-\frac{1}{2}\nu)^{\rho}(d\nu)^{\rho} + |\mathcal{G}^{\rho}(\vartheta/m)|^{q} \frac{m^{\rho}}{2^{\rho}\Gamma(1+\rho)} \int_0^1 \nu^{\rho}(1-\nu)^{\rho}(d\nu)^{\rho} \\
&\quad \leqslant \left(\frac{1}{\Gamma(1+\rho)} - \frac{3^{\rho}}{2^{\rho}} \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} + \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right) |\mathcal{G}^{\rho}(\zeta)|^{q} + \frac{m^{\rho} B_{\rho}(2,2)}{2^{\rho}} |\mathcal{G}^{\rho}(\vartheta/m)|^{q},
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left|\mathcal{G}^{\rho}(\zeta + (1-\frac{1}{2}\nu))\gamma(\vartheta,\zeta))\right|^{q} (d\nu)^{\rho} \\
&\quad \leqslant \frac{1}{\Gamma(1+\rho)} \int_0^1 (1-\nu)^{\rho} \left|(\frac{\nu}{2})^{\rho} \mathcal{G}^{\rho}(\zeta) + m^{\rho}(1-\frac{\nu}{2})^{\rho} \mathcal{G}^{\rho}(\vartheta/m)\right|^{q} (d\nu)^{\rho} \\
&\quad \leqslant \frac{1}{2^{\rho}} |\mathcal{G}^{\rho}(\zeta)|^{q} B_{\rho}(2,2) + m^{\rho} \left|\mathcal{G}^{\rho}(\vartheta/m)\right|^{q} \left[\frac{1}{\Gamma(1+\rho)} - \frac{3^{\rho}}{2^{\rho}} \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} + \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right].
\end{aligned}
\tag{5.3}
$$

Substituting (5.2) and (5.3) in (5.1) we get the desired result. □

Corollary 5.4. *If* $q = 1$ *in Theorem 5.3, then*

$$
\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta) + \mathcal{G}(\zeta + \gamma(\vartheta,\zeta))}{2^{\rho}} {}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)} \omega(x) - {}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)} \omega(x)\mathcal{G}(x)\right| \\
&\quad \leqslant \left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^{\rho} \frac{|\omega|_{\infty}}{\Gamma(1+\rho)} \left[|\mathcal{G}^{\rho}(\zeta)| + m^{\rho}|\mathcal{G}^{\rho}(\vartheta/m)|\right] \left[\frac{1}{\Gamma(1+\rho)} - \frac{3^{\rho}}{2^{\rho}} \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} + \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} + \frac{1}{2^{\rho}} B_{\rho}(2,2)\right].
\end{aligned}
$$

Corollary 5.5. *If $\gamma(\vartheta,\zeta)=\vartheta-\zeta$ in Corollary 5.4, then*

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\vartheta)}{2^{\rho}}{}_{\zeta}I^{\rho}_{\vartheta}\omega(x)-{}_{\zeta}I^{\rho}_{\vartheta}\omega(x)\mathcal{G}(x)\right|$$
$$\leqslant\left(\frac{(\vartheta-\zeta)^2}{4}\right)^{\rho}\frac{|\omega|_{\infty}}{\Gamma(1+\rho)}\left[|\mathcal{G}^{\rho}(\zeta)|+m^{\rho}|\mathcal{G}^{\rho}(\vartheta/m)|\right]\left[\frac{1}{\Gamma(1+\rho)}-\frac{3^{\rho}}{2^{\rho}}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}+\frac{1}{2^{\rho}}B_{\rho}(2,2)\right].$$

Corollary 5.6. *If $\omega(x)=1$ in Theorem 5.3, then*

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}-{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)\right|$$
$$\leqslant\left(\frac{\gamma(\vartheta,\zeta)}{4}\right)^{\rho}\frac{1}{\Gamma(1+\rho)}\left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-\frac{1}{q}}\left[\frac{1}{\Gamma(1+\rho)}-\frac{3^{\rho}}{2^{\rho}}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}+\frac{1}{2^{\rho}}B_{\rho}(2,2)\right].$$

Corollary 5.7. *If $\omega(x)=1$, $q=1$, and $m=1$ in Theorem 5.3, then*

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}-{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\mathcal{G}(x)\right|$$
$$\leqslant\left(\frac{\gamma(\vartheta,\zeta)}{4}\right)^{\rho}\frac{1}{\Gamma(1+\rho)}\left[|\mathcal{G}^{\rho}(\zeta)|+|\mathcal{G}^{\rho}(\vartheta)|\right]\left[\frac{1}{\Gamma(1+\rho)}-\frac{3^{\rho}}{2^{\rho}}\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}+\frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}+\frac{1}{2^{\rho}}B_{\rho}(2,2)\right].$$

Theorem 5.8. *Assume the hypothesis of Theorem 5.3. If $|\mathcal{G}^{\rho}|^{q}$ is generalized m-preinvex for $q>1$, on $[\zeta,\zeta+\gamma(\vartheta,\zeta)]$, $\rho\in[0,1]$ and $m\in(0,1]$, then*

$$\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)-{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)\right|$$
$$\leqslant\left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^{\rho}\frac{|\omega|_{\infty}}{\Gamma(1+\rho)}\left(\frac{\Gamma\left(1+\left(\frac{q}{q-1}\right)\rho\right)}{\Gamma\left(1+\left(\frac{2q-1}{q}\right)\rho\right)}\right)^{\frac{q-1}{q}}\frac{1}{\Gamma(1+\rho)}\left[|\mathcal{G}^{\rho}(\zeta)|^{q}+m^{\rho}\,|\mathcal{G}^{\rho}(\vartheta/m)|^{q}\right],$$

where $|\omega|_{\infty}=\sup_{x\in[\zeta,\zeta+\gamma(\vartheta,\zeta)]}\omega(x)$.

Proof. By Lemma 5.1 and Holder's Inequality we have

$$\begin{aligned}
&\left|\frac{\mathcal{G}(\zeta)+\mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}}{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)-{}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)\right|\\
&\leqslant\left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^{\rho}\left(\frac{1}{\Gamma(1+\rho)}\int_0^1(1-\nu)^{\rho\frac{q}{q-1}}d\nu\right)^{1-\frac{1}{q}}\frac{|\omega|_{\infty}}{\Gamma(1+\rho)}\\
&\quad\times\left(\left[\frac{1}{\Gamma(1+\rho)}\int_0^1\left|\mathcal{G}'\left(\zeta+\frac{\nu}{2}\gamma(\vartheta,\zeta)\right)\right|^{q}dt\right]^{1/q}+\left[\frac{1}{\Gamma(1+\rho)}\int_0^1\left|\mathcal{G}'\left(\zeta+(1-\frac{\nu}{2}\right)\gamma(\vartheta,\zeta)\right|^{q}d\nu\right]^{1/q}\right) \qquad (5.4)\\
&\leqslant\left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)\frac{|\omega|_{\infty}}{\Gamma(1+\rho)}\left(\frac{\Gamma(1+\frac{q}{q-1}\rho)}{\Gamma\left(1+\left(\frac{2q-1}{q-1}\right)\rho\right)}\right)^{\frac{q}{q-1}}\left(\left[\frac{1}{\Gamma(1+\rho)}\int_0^1\left|\mathcal{G}^{\rho}\left(\zeta+\frac{\nu}{2}\gamma(\vartheta,\zeta)\right)\right|^{q}(d\nu)^{\rho}\right]^{\frac{1}{q}}\right.\\
&\quad\left.+\left[\frac{1}{\Gamma(1+\rho)}\int_0^1\left|\mathcal{G}^{\rho}\left(\zeta+\left(1-\frac{\nu}{2}\right)\gamma(\vartheta,\zeta)\right)\right|^{q}(d\nu)^{\rho}\right]^{\frac{1}{q}}\right).
\end{aligned}$$

Since $|\mathcal{G}^{\rho}|^q$ is generalized m-preinvex then for any $\nu \in [0,1]$ we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\rho)} \int_0^1 \left| \mathcal{G}^{\rho}\left(\zeta + \frac{\nu}{2}\gamma(\vartheta,\zeta)\right)\right|^q (d\nu)^{\rho} \\
&\leqslant |\mathcal{G}^{\rho}(\zeta)|^q \frac{1}{\Gamma(1+\rho)} \int_0^1 \left(1-\frac{\nu}{2}\right)^{\rho} (d\nu)^{\rho} + |\mathcal{G}^{\rho}(\vartheta/m)|^q \frac{m^{\rho}}{\Gamma(1+\rho)} \int_0^1 \left(\frac{\nu}{2}\right)^{\rho} (d\nu)^{\rho} \\
&\leqslant |\mathcal{G}^{\rho}(\zeta)|^q \left[\frac{1}{\Gamma(1+\rho)} - \frac{\Gamma(1+\rho)}{2^{\rho}\Gamma(1+2\rho)}\right] + |\mathcal{G}^{\rho}(\vartheta/m)|^q \frac{m^{\rho}}{2^{\rho}} \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(1+\rho)} \int_0^1 \left| \mathcal{G}^{\rho}\left(\zeta + \left(1-\frac{\nu}{2}\right)\gamma(\vartheta,\zeta)\right)\right|^q (d\nu)^{\rho} \\
&\leqslant |\mathcal{G}^{\rho}(\zeta)|^q \frac{1}{\Gamma(1+\rho)} \int_0^1 \left(\frac{\nu}{2}\right)^{\rho} (d\nu)^{\rho} + |\mathcal{G}^{\rho}(\vartheta/m)|^q \frac{m^{\rho}}{\Gamma(1+\rho)} \int_0^1 \left(1-\frac{\nu}{2}\right)^{\rho} (d\nu)^{\rho} \\
&\leqslant |\mathcal{G}^{\rho}(\zeta)|^q \frac{\Gamma(1+\rho)}{2^{\rho}\Gamma(1+2\rho)} + |\mathcal{G}^{\rho}(\vartheta/m)|^q m^{\rho} \left[\frac{1}{\Gamma(1+\rho)} - \frac{\Gamma(1+\rho)}{2^{\rho}\Gamma(1+2\rho)}\right].
\end{aligned}$$

Substituting these values in inequality (5.4), we get

$$\begin{aligned}
&\left| \frac{\mathcal{G}(\zeta) + \mathcal{G}(\zeta+\gamma(\vartheta,\zeta))}{2^{\rho}} {}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\omega(x) - {}_{\zeta}I^{\rho}_{\zeta+\gamma(\vartheta,\zeta)}\omega(x)\mathcal{G}(x)\right| \\
&\leqslant \left(\frac{\gamma^2(\vartheta,\zeta)}{4}\right)^{\rho} \frac{|\omega|_{\infty}}{\Gamma(1+\rho)} \left(\frac{\Gamma\left(1+\left(\frac{q}{q-1}\right)\rho\right)}{\Gamma\left(1+\left(\frac{2q-1}{q}\right)\rho\right)}\right)^{\frac{q-1}{q}} \frac{1}{\Gamma(1+\rho)} \left[|\mathcal{G}^{\rho}(\zeta)|^q + m^{\rho}|\mathcal{G}^{\rho}(\vartheta/m)|^q\right].
\end{aligned}$$

□

Corollary 5.9. *If $\gamma(\vartheta,\zeta) = \vartheta - \zeta$ and $m = 1$ in Theorem 5.8, we get*

$$\begin{aligned}
&\left| \frac{\mathcal{G}(\zeta) + \mathcal{G}(\vartheta)}{2^{\rho}} {}_{\zeta}I^{\rho}_{\vartheta}\omega(x) - {}_{\zeta}I^{\rho}_{\vartheta}\omega(x)\mathcal{G}(x)\right| \\
&\leqslant \left(\frac{(\vartheta-\zeta)^2}{4}\right)^{\rho} \frac{|\omega|_{\infty}}{\Gamma(1+\rho)} \left(\frac{\Gamma(1+(\frac{q}{q-1})\rho)}{\Gamma(1+(\frac{2q}{q-1})\rho)}\right)^{\frac{q-1}{q}} \frac{1}{\Gamma(1+\rho)} \left[|\mathcal{G}^{\rho}(\zeta)|^q + |\mathcal{G}^{\rho}(\vartheta)|^q\right].
\end{aligned}$$

6. Applications to generalized special means

Consider the ρ-type special means for $\mu_1, \mu_2 \in \mathbb{R}^{\rho}, \mu_1 < \mu_2$.

(1) The *generalized arithmetic mean* is defined as

$$A_{\rho}(\mu_1,\mu_2) := \left(\frac{\mu_1+\mu_2}{2}\right)^{\rho} = \frac{\mu_1^{\rho}+\mu_2^{\rho}}{2^{\rho}}.$$

(2) The *generalized logarithmic mean* is defined as

$$L_{r^{\rho}}(\mu_1,\mu_2) := \left[\frac{\Gamma(1+r\rho)}{\Gamma(1+(r+1)\rho)} \frac{\mu_2^{(r+1)\rho} - \mu_1^{(r+1)\rho}}{(\mu_2-\mu_1)^{\rho}}\right]^{\frac{1}{r}},$$

$\forall r \in \mathbb{Z}\backslash\{-1,0\}$.

For $\rho = 1$, we get the following special means:

$$A(\mu_1, \mu_2) = \frac{\mu_1 + \mu_2}{2} \quad \text{and} \quad L_r(\mu_1, \mu_2) = \left[\frac{\mu_2^{r+1} - \mu_1^{r+1}}{(\mu_2 - \mu_1)(r+1)}\right]^{\frac{1}{r}},$$

$\forall r \in \mathbb{Z} \backslash \{-1, 0\}$.

If we choose $\mathcal{G}(x) = x^{r\rho}$ with $r \in \mathbb{R}$, $|r| \geqslant 2$ in Corollary 4.13 and in Corollary 5.5, respectively, then we get following.

Proposition 6.1. *For $0 < \zeta < \vartheta$, $p, q > 1$, and $\rho \in (0, 1]$, we have*

$$\begin{aligned} |A_\rho(\zeta^r, \vartheta^r) - \Gamma(1+\rho)\, L_{r^\rho}^r(\zeta^r, \vartheta^r)| \leqslant \frac{|(\vartheta - \zeta)|^\rho}{2^\rho} \left(\frac{\Gamma(1+p\rho)}{\Gamma(1+(1+p)\rho)}\right)^{1/p} \left[\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} \left|\frac{\Gamma(1+r\rho)}{\Gamma(1+(r-1)\rho)} \zeta^{(r-1)\rho}\right|^q \right. \\ \left. + m^\rho \left|\frac{\Gamma(1+r\rho)}{\Gamma(1+(r-1)^\rho)} \left(\frac{\vartheta}{m}\right)^{(r-1)\rho}\right|^q \right]^{1/q}. \end{aligned}$$

Proposition 6.2. *For $0 < \zeta < \vartheta$ and $q > 1$, $\rho \in (0, 1]$, we have*

$$\begin{aligned} & |A_\rho(\zeta^r, \vartheta^r) - \Gamma(1+\rho) L_{r^\rho}^r(\zeta^r, \vartheta^r)| \\ & \leqslant \frac{|(\vartheta - \zeta)^\rho|}{2^\rho} \left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)}\right)^{1-1/q} \left[\frac{3^\rho}{2^\rho} \left(\frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} - \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)}\right)\right. \\ & \quad \left. \times \left|\frac{\Gamma(1+r\rho)}{\Gamma(1+(r-1)\rho)} \zeta^{(r-1)\rho}\right|^q + m^\rho \left|\frac{\Gamma(1+r\rho)}{\Gamma(1+(r-1)\rho)} \left(\frac{\vartheta}{m}\right)^{(r-1)\rho}\right|^q \right]^{1/q}. \end{aligned}$$

7. Applications to generalized random variables

Suppose X is a generalized random variable with a probability density function (P.D.F.) $\Theta : [\zeta, \vartheta] \to \mathbb{R}^\rho$. The generalized P.D.F. Θ has upper and lower bounds, i.e., ρ-type real number, x, y satisfying $0^\rho \leqslant x \leqslant \Theta(\nu) \leqslant y \leqslant 1^\rho$.

For all $\nu \in [\zeta, \vartheta]$, the generalized r^{th} moment of X is given as

$$E_r^\rho(X) = \frac{1}{\Gamma(1+\rho)} \int_\zeta^\vartheta \nu^{r\rho} \Theta(\nu) (d\nu)^\rho; \; r \geqslant 0.$$

Let $0 < \zeta < \vartheta$, $r \geqslant 1$ and $\omega(x) \in C_\rho[\zeta, \vartheta]$ be a generalized prabability density of X that is symmetric about $\frac{\zeta+\vartheta}{2}$. If $\mathcal{G}(x) = x^{r\rho}$ in Corollary 5.5, then we have the following result.

Proposition 7.1. *For $0 < \zeta < \vartheta$ and $q = 1$, $\rho \in (0, 1]$, we have*

$$\begin{aligned} & \left|\frac{\zeta^{r\rho} + \vartheta^{r\rho}}{2^\rho} {}_\zeta I_\vartheta^\rho \omega(x) - E_r^\rho(X)\right| \\ & \leqslant \left(\frac{(\vartheta - \zeta)^2}{4}\right)^\rho \frac{|\omega|_\infty}{\Gamma(1+\rho)} \left[\zeta^{(r-1)\rho} + m^\rho \left(\frac{\vartheta}{m}\right)^{(r-1)\rho}\right] \\ & \quad \times \left[\frac{1}{\Gamma(1+\rho)} - \frac{3^\rho}{2^\rho} \frac{\Gamma(1+\rho)}{\Gamma(1+2\rho)} + \frac{\Gamma(1+2\rho)}{\Gamma(1+3\rho)} + \frac{1}{2^\rho} \beta_\rho(2, 2)\right]. \end{aligned}$$

8. Conclusions

In this paper, we have investigated a new generalized class of preinvex functions on fractal sets, i.e., generalized m-preinvex functions, and addressed some properties of this generalized class. Inequalities of Hermite-Hadamard-type have been established for this class of functions. Additionally, we derived a number of new integral inequalities for differentiable generalized m-preinvex functions. Moreover, by establishing a weighted identity for symmetric first order differentiable generalized m-preinvex mappings, we derived a right-sided Fejér-Hermite-Hadamard type inequalities. Many existing results in the literature are special cases for our findings, as mentioned in the remarks. Finally, we established some new inequalities for the generalized special mean and random variables as applications.

Acknowledgements

The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions, which helped us to improve the manuscript.

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