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A note on partially degenerate Hermite-Bernoulli polynomials of the first kind

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Abstract

In this paper, we introduce a new class of partially degenerate Hermite-Bernoulli polynomials of the first kind and generalized Gould-Hopper-partially degenerate Bernoulli polynomials of the first kind and present some properties and identities of these polynomials. A new class of polynomials generalizing different classes of Hermite polynomials such as the real Gould-Hopper, as well as the 1-d and 2-d holomorphic, ternary and polyanalytic complex Hermite polynomials and their relationship to the partially degenerate Hermite-Bernoulli polynomials of the first kind are also discussed.

Keywords: Hermite polynomials, partially degenerate Bernoulli polynomials of the first kind, partially degenerate Hermite-Bernoulli polynomials of the first kind, symmetry identities.

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1. Introduction

Recently, by using different methods and techniques, multiple sums of special functions involving products of power series and multivariate generating functions for special polynomials and numbers of higher-order have been introduced and investigated systematically by many authors. Therefore, modification, unification, and generalization of various special polynomials and numbers including Bernoulli and Euler-type polynomials and numbers have been studied systematically. Among others, Bell derived many new and interesting generating functions by applying symbolic methods (see [1–31]). By using multiple sums of special functions involving trigonometric functions and Bessel functions, Brychkov [4] derived many identities and relations for not only Bernoulli and Euler polynomials but also some orthogonal polynomials. Using (1.3)-(1.12), in this paper, we derive several new and novel formulas and relations among the Bernoulli numbers and polynomials, the Hermite numbers and polynomials, the degenerate type polynomials, other special polynomials, the Daehee numbers, the Stirling numbers, and the combinatorial sums.

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Let p be a fixed odd prime number. Throughout the article, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The normalized p-adic is given by $|p|_p = \frac{1}{p}$. For $f : \mathbb{Z}_p \to \mathbb{C}_p$ (f being a continuous function), the bosonic p-adic integral of f is defined by:

$$\int_{\mathbb{Z}_p} f(\xi) d\mu_0(\xi) = \lim_{N\to\infty} \sum_{\xi=0}^{p^N-1} f(\xi)\mu_0(\xi + p^N \mathbb{Z}_p) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{\xi=0}^{p^N-1} f(\xi). \tag{1.1}$$

It is apparent from (1.1) that

$$\int_{\mathbb{Z}_p} f(\xi+1) d\mu_0(\xi) - \int_{\mathbb{Z}_p} f(\xi) d\mu_0(\xi) = 2\sum_{l=0}^{j-1} f'(l), (j \in \mathbb{N}),$$

where $f_j(\xi) = f(\xi + j)$ (see [17, 21, 22]).

The Bernoulli polynomials are defined by the generating function

$$\int_{\mathbb{Z}_p} e^{(\xi+\eta)z} d\mu_0(\eta) = \frac{z}{e^z - 1} e^{\xi z} = \sum_{j=0}^{\infty} B_j(\xi) \frac{z^j}{j!}, \quad \text{(see [12, 14])}.$$

When $\xi = 0$, $B_j = B_j(0)$ are called the Bernoulli numbers.

The two variable Hermite Kampé de Fériet polynomials $H_j(\xi, \eta)$ [3, 6] are defined by

$$H_j(\xi, \eta) = j! \sum_{r=0}^{[\frac{j}{2}]} \frac{\eta^r \xi^{j-2r}}{r!(j-2r)!}. \tag{1.2}$$

These polynomials are usually defined by the generating function

$$e^{\xi z + \eta z^2} = \sum_{j=0}^{\infty} H_j(\xi, \eta) \frac{z^j}{j!}, \tag{1.3}$$

and reduce to the ordinary Hermite polynomials $H_j(\xi)$ when $\eta = -1$ and ξ is replaced by 2ξ.

Jedda et al. [13] introduced a class of two-index real Hermite polynomials of degree $p + j$ by

$$h_{p,j}(\xi) = (-\frac{d}{d\xi} + 2\xi)^p (\xi)^j = p!j! \sum_{k=0}^{\min(p,j)} \frac{(-1)^k \xi^{p-k} H_{p-k}(\xi)}{k!(j-k)!(p-k)}. \tag{1.4}$$

Note that $h_{p,0}(\xi) = H_p\xi(\xi)$, $h_{0,j\xi}(\xi) = \xi^j$ and $h_{p,j}(0) = 0$, if $p < j$. The generating function of $h_{p,j}$ is given by

$$\sum_{p=0}^{\infty}\sum_{j=0}^{\infty} h_{p,j}(\xi) \frac{u^p v^j}{p!j!} = e^{-u^2 + (2u+v)\xi - uv}. \tag{1.5}$$

Furthermore, for $\eta = u = -v$, Jedda et al. [13] proved that

$$e^{\xi\eta} = \sum_{p,j=0}^{\infty} (-1)^j h_{p,j}(\xi) \frac{\eta^{p+j}}{p!j!}. \tag{1.6}$$

The generating function of Gould-Hopper polynomials $G_m^{(q)}(w|\gamma)$ introduced by Dattoli et al. [6, p. 72]) are given by

$$e^{wv + \gamma v^q} = \sum_{m=0}^{\infty} G_m^{(q)}(w|\gamma) \frac{v^m}{m!}, \tag{1.7}$$

so that for every complex numbers u, v, z and w, we have (see [6, pages 5 and 6]):

$$\sum_{j=0}^{\infty} H_{j,m}^{(p,q)}(z,w|\gamma)\frac{u^n}{n!} = G_m^{(q)}(w|u^p\gamma)e^{zu}, \tag{1.8}$$

and

$$\sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(z,w|\gamma)\frac{v^m}{m!} = G_j^{(q)}(w|v^p\gamma)e^{zv}, \tag{1.9}$$

where the polynomials $H_{j,m}^{(p,q)}(z,w|\gamma)$ contain all the classes given above. Moreover, they give rise to new classes of polynomials of Hermite type. The concrete study of this polynomial is presented in [6] in a unified way and includes the connection to Gould-Hopper polynomials [9], operational representations and connection to hypergeometric function, generating functions, addition formulas of Runge type, multiplication formulas, Nielson formulas and higher order differential equations they obey.

The Daehee polynomials are defined by the generating function

$$\frac{\log(1+z)}{z}(1+z)^{\xi} = \sum_{j=0}^{\infty} D_j(\xi)\frac{z^j}{j!}, \quad \text{(see [25])}. \tag{1.10}$$

In the case $\xi = 0$, $D_j = D_j(0)$ are the Daehee numbers.

The Bernoulli polynomials of the second kind are defined by the generating function

$$\frac{z}{\log(1+z)}(1+z)^{\xi} = \sum_{j=0}^{\infty} b_j(\xi)\frac{z^j}{j!}, \quad \text{(see [12])}. \tag{1.11}$$

At the point $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind.

For $\lambda, z \in \mathbb{C}_p$ with $|\lambda z|_p < p^{-\frac{1}{p-1}}$, the partially degenerate Bernoulli polynomials of the first kind are defined by the generating function (see [23])

$$\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z - 1}e^{\xi z} = \sum_{j=0}^{\infty} B_j(\xi|\lambda)\frac{z^j}{j!}. \tag{1.12}$$

When $\xi = 0$, $B_j(\lambda) = B_j(0|\lambda)$ are called the partially degenerate Bernoulli numbers of the first kind.

Kwon et al. [25] proved that

$$D_j^{(r)}(\xi|\lambda) = \sum_{l=0}^{j}\binom{j}{l}D_l^{(r)}(\lambda)\xi^{j-l},$$

where

$$\left(\frac{\log(1+t)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\right)^r e^{xt} = \sum_{n=0}^{\infty} D_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \tag{1.13}$$

The falling factorial sequence is defined by

$$(\xi)_0 = 1, \quad (\xi)_j = \xi(\xi-1)\cdots(\xi-j+1), (j \geqslant 1).$$

The first kind of Stirling numbers is defined by

$$(\xi)_j = \sum_{k=0}^{j} S_1(j,k)\xi^k, (j \geqslant 0), \text{ (see [1–10])}. \tag{1.14}$$

and as an inversion formula of (1.13), the Stirling numbers of the the second kind is given by (see [3, 5–11]):

$$\xi^j = \sum_{k=0}^{j} S_2(j,k)(\xi)_k. \tag{1.15}$$

From (1.14) and (1.15), we note that the generating function of Stirling numbers of the first kind and that of the second kind are respectively given by (see [12–31]):

$$\frac{1}{k!}(\log(1+z))^k = \sum_{j=k}^{\infty} S_1(j,k)\frac{z^j}{j!},$$

and

$$\frac{1}{k!}(e^z-1)^k = \sum_{j=k}^{\infty} S_2(j,k)\frac{z^j}{j!}, (k \geqslant 0).$$

For each $p \geqslant 0$, $S_p(j)$ [28] defined by

$$S_p(j) = \sum_{l=0}^{j} l^p,$$

is called the sum of integer power sum or simply powers sum. The exponential generating function for $S_p(j)$ is

$$\sum_{p=0}^{\infty} S_p(j)\frac{z^p}{p!} = 1 + e^z + e^{2z} + \cdots + e^{jz} = \frac{e^{(j+1)z}-1}{e^z-1}. \tag{1.16}$$

In this paper, we introduce partially degenerate Hermite-Bernoulli polynomials of the first kind and deduce some relevant properties by using a preliminary study of these polynomials. We derive some theorems on implicit summation formulae for partially degenerate Hermite-Bernoulli polynomials of the first kind ${}_HB_j^{(r)}(\xi,\eta|\lambda)$. We derive a definition of the two-index real partially degenerate Hermite-Bernoulli polynomials of the first kind and generalized Gould-Hopper-partially degenerate Hermite-Bernoulli polynomials of the first kind combined with their properties and special cases. Finally, we derive some symmetry identities for partially degenerate Hermite-Bernoulli polynomials of the first kind.

2. Partially degenerate Hermite-Bernoulli polynomials of the first kind

In this section, we introduce partially degenerate Hermite-Bernoulli polynomials and investigate some properties of these polynomials. We begin with the following definition.

Definition 2.1. Let us assume that $\lambda, z \in \mathbb{C}_p$ such that $|\lambda z|_p < p^{-\frac{1}{p-1}}$. Then, we consider the partially degenerate Hermite-Bernoulli polynomials of the first kind which are given by the generating function:

$$\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{\xi z+\eta z^2} = \sum_{j=0}^{\infty} {}_HB_j(\xi,\eta|\lambda)\frac{z^j}{j!}. \tag{2.1}$$

At the point, $\xi = \eta = 0$ in (2.1), ${}_HB_j(0,0|\lambda) = {}_HB_j(\lambda)$ are called the partially degenerate Hermite-Bernoulli numbers of the first kind. From (2.1), we note that

$$\sum_{j=0}^{\infty} \lim_{\lambda\to 0} {}_HB_j(\xi,\eta|\lambda)\frac{z^j}{j!} = \lim_{\lambda\to 0}\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{\xi z+\eta z^2} = \frac{z}{e^z-1}e^{\xi z+\eta z^2} = \sum_{j=0}^{\infty} {}_HB_j(\xi,\eta)\frac{z^j}{j!}. \tag{2.2}$$

Thus, we get

$$\lim_{\lambda\to 0} {}_H B_j(\xi,\eta|\lambda) = {}_H B_j(\xi,\eta), (j \geqslant 0),$$

where ${}_H B_j(\xi,\eta)$ are called the Hermite-Bernoulli polynomials.

Theorem 2.2. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j(\xi,\eta|\lambda) = \sum_{l=0}^{j} \binom{j}{l} B_{l,\lambda} H_{j-l}(\xi,\eta). \tag{2.3}$$

Proof. From (2.1), we have

$$\begin{aligned}\sum_{j=0}^{\infty} {}_H B_j(\xi,\eta|\lambda)\frac{z^j}{j!} &= \frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1} e^{\xi z+\eta z^2} \\ &= \left(\sum_{l=0}^{\infty} B_{l,\lambda}\frac{z^l}{l!}\right)\left(\sum_{j=0}^{\infty} H_j(\xi,\eta)\frac{z^j}{j!}\right) = \left(\sum_{l=0}^{j}\binom{j}{l} B_{l,\lambda}H_{j-l}(\xi,\eta)\right)\frac{z^j}{j!}.\end{aligned}$$

Comparing the coefficients of $\frac{z^j}{j!}$, we get (2.3). □

Corollary 2.3. *For* $\eta = 0$ *in* (2.1), *we have*

$$B_{j,\lambda}(\xi) = \sum_{l=0}^{j}\binom{j}{l} B_{j-l,\lambda}\xi^l.$$

Theorem 2.4. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j(\xi,\eta|\lambda) = \sum_{l=0}^{j}\binom{j}{l}\frac{(-1)^l l!}{l+1}\lambda^l {}_H B_{j-l}(\xi,\eta|\lambda). \tag{2.4}$$

Proof. Form (2.1), we have

$$\begin{aligned}\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{\xi z+\eta z^2} &= \left(\frac{\log(1+\lambda z)}{\lambda z}\right)\left(\frac{z}{e^z-1}\right)e^{\xi z+\eta z^2} \\ &= \left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l+1}\lambda^l z^m\right)\left(\sum_{j=0}^{\infty} {}_H B_j(\xi,\eta)\frac{z^j}{j!}\right) \\ &= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}\frac{(-1)^l l!}{(l+1)l!}\lambda^l {}_H B_{j-l}(\xi,\eta)\right)\frac{z^j}{j!}.\end{aligned} \tag{2.5}$$

In view of (2.1) and (2.5), we get the result (2.4). □

Corollary 2.5. *For* $\eta = 0$ *in Theorem* 2.4, *we get*

$$B_{j,\lambda}(\xi) = \sum_{l=0}^{j}\binom{j}{l}\frac{(-1)^l l!}{l+1}\lambda^l B_{j-l}(\xi).$$

Theorem 2.6. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j(\xi,\eta|\lambda) = \sum_{l=0}^{j}\binom{j}{l} {}_H B_{j-l}(\xi,\eta)\lambda^l D_l(0). \tag{2.6}$$

Proof. Since

$$\begin{aligned}\sum_{j=0}^{\infty} {}_{H}B_j(\xi,\eta|\lambda)\frac{z^j}{j!} &= \frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{\xi z+\eta z^2}\\ &= \left(\frac{z}{e^z-1}\right)e^{\xi z+\eta z^2}\left(\frac{\log(1+\lambda z)}{\lambda z}\right)\\ &= \left(\sum_{j=0}^{\infty} {}_{H}B_j(\xi,\eta)\frac{z^j}{j!}\right)\left(\sum_{l=0}^{\infty} D_l(0)\frac{(\lambda z)^l}{l!}\right)\\ &= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l} {}_{H}B_{j-l}(\xi,\eta)\lambda^l D_l(0)\right)\frac{z^j}{j!}.\end{aligned} \tag{2.7}$$

In view of (2.1) and (2.7), we get the result (2.6). □

Corollary 2.7. *On taking* $\eta = 0$ *in Theorem 2.6, we acquire*

$$B_{j,\lambda}(\xi) = \sum_{l=0}^{j}\binom{j}{l}B_{j-l}(\xi)\lambda^l D_l.$$

Theorem 2.8. *Let* $j \geqslant 0$*. Then*

$$\sum_{l=0}^{j}\binom{j}{l}(-\xi)_l(-1)^l {}_{H}B_{j-l}(\eta,\zeta|\lambda) = \sum_{l=0}^{j}\binom{j}{l}D_l(x)\lambda^l {}_{H}B_{j-l}(\eta,\xi). \tag{2.8}$$

Proof. Consider equation (2.1), we have

$$\begin{aligned}\sum_{j=0}^{\infty} {}_{H}B_j(\eta,\zeta|\lambda)\frac{z^j}{j!} &= \frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{\eta z+\zeta z^2},\\ (1+z)^{\xi}\sum_{j=0}^{\infty} {}_{H}B_j(\eta,\zeta|\lambda)\frac{z^j}{j!} &= \left(\frac{z}{e^z-1}\right)\left(\frac{\log(1+\lambda z)}{\lambda z}\right)(1+z)^{\xi}e^{\eta z+\zeta z^2}\\ &= \sum_{l=0}^{\infty} D_l(\xi)\frac{(\lambda t)^l}{l!}\sum_{j=0}^{\infty} {}_{H}B_j(\eta,\zeta)\frac{z^j}{j!}\\ &= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l} {}_{H}B_{j-l}(\eta,\zeta)D_l(\xi)\lambda^l\right)\frac{z^j}{j!}.\end{aligned} \tag{2.9}$$

On the other hand, we have

$$\sum_{l=0}^{\infty}(x)_l\frac{z^l}{l!}\sum_{j=0}^{\infty} {}_{H}B_j(\eta,\xi)\frac{z^j}{j!} = \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l} {}_{H}B_{j-l}(\eta,\zeta)(x)_l\right)\frac{z^j}{j!}. \tag{2.10}$$

Therefore, by (2.9) and (2.10), we get the result (2.8). □

Theorem 2.9. *Let* $j \geqslant 0$*. Then*

$${}_{H}B_j(\xi,\eta) = \sum_{l=0}^{j}\binom{j}{l}\lambda^l b_{l,\lambda}(0) {}_{H}B_{j-l}(\xi,\eta|\lambda). \tag{2.11}$$

Proof. In (2.1), we note that

$$\begin{aligned}\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{\xi z+\eta z^2} &= \sum_{j=0}^{\infty} {}_HB_j(\xi,\eta|\lambda)\frac{z^j}{j!},\\ \frac{z}{e^z-1}e^{\xi z+\eta z^2} &= \left(\frac{\lambda z}{\log(1+\lambda z)}\right)\left(\sum_{j=0}^{\infty} {}_HB_j(\xi,\eta|\lambda)\frac{z^j}{j!}\right),\\ \frac{z}{e^z-1}e^{\xi z+\eta z^2} &= \left(\sum_{l=0}^{\infty} b_{l,\lambda}(0)\frac{\lambda^l z^l}{l!}\right)\left(\sum_{j=0}^{\infty} {}_HB_j(\xi,\eta|\lambda)\frac{z^j}{j!}\right)\\ &= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}\lambda^l b_{l,\lambda}(0)\,{}_HB_{j-l}(\xi,\eta|\lambda)\right)\frac{z^j}{j!}.\end{aligned} \tag{2.12}$$

Therefore, by (2.1) and (2.12), we get the result (2.11). □

Theorem 2.10. *Let* $j \geqslant 0$. *Then*

$${}_HB_j(\xi,\eta|\lambda) = \sum_{l=0}^{j}\binom{j}{l}{}_HB_{j-l}(p\xi,q\eta|\lambda)H_l((1-p)\xi,(1-q)\eta). \tag{2.13}$$

Proof. Rewrite (2.1), we have

$$\begin{aligned}\sum_{j=0}^{\infty} {}_HB_j(\xi,\eta|\lambda)\frac{z^j}{j!} &= \frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{(1-p)\xi z+(1-q)\eta z^2}e^{p\xi z+q\eta z^2}\\ &= \sum_{j=0}^{\infty} {}_HB_{j,\lambda}(p\xi,q\eta)\frac{z^j}{j!}\sum_{l=0}^{\infty} H_l((1-p)\xi,(1-q)\eta)\frac{z^l}{l!}.\end{aligned}$$

Replacing j by $j-l$ and comparing the coefficients of z^j, we get the result (2.13). □

Corollary 2.11. *For* $p, q \in \mathbb{R}$, *we have*

$${}_HB_j(\xi,\eta|\lambda) = \sum_{l=0}^{j}\binom{n}{l}{}_HB_{j-l}\left((1-p)\xi,(1-q)\eta|\lambda\right)H_l\left(p\xi,q\eta\right).$$

Theorem 2.12. *Let* $j \geqslant 0$ *and* $d \in \mathbb{N}$. *Then*

$${}_HB_j(\xi,\eta|\lambda) = d^{j-1}\sum_{l=0}^{d-1} {}_HB_{j,\lambda/d}\left(\frac{l+\xi}{d},\eta\right).$$

Proof. From (2.1) in the form

$$\begin{aligned}\sum_{j=0}^{\infty} {}_HB_j(\xi,\eta|\lambda)\frac{z^j}{j!} = \frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}e^{\xi z+\eta z^2} &= \frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^{dz}-1}\sum_{l=0}^{d-1} e^{(l+\xi)z+\eta z^2}\\ &= \sum_{j=0}^{\infty}\left(d^{j-1}\sum_{l=0}^{d-1} {}_HB_{j,\lambda/d}\left(\frac{l+\xi}{d},\eta\right)\right)\frac{z^j}{j!},\end{aligned} \tag{2.14}$$

by (2.1) and (2.2), we get (2.14). □

Corollary 2.13. *Let* $j \geqslant 0$ *and* $d \in \mathbb{N}$, *we have*

$$B_{n,\lambda}(x) = d^{n-1}\sum_{l=0}^{d-1} B_{n,\lambda/d}\left(\frac{l+x}{d}\right).$$

3. Higher-order partially degenerate Hermite-Bernoulli polynomials of the first kind

This section incorporates the definition of higher-order partially degenerate Hermite-Bernoulli polynomials of the first kind and a preliminary study of these polynomials.

Definition 3.1. Let us assume that $\lambda, z \in \mathbb{C}_p$ such that $|\lambda z|_p < p^{-1/p-1}$, we define higher-order partially degenerate Hermite polynomials are as follows

$$\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{\xi z+\eta z^2} = \sum_{j=0}^{\infty} {}_H B_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!}. \tag{3.1}$$

When $\eta = 0$, then ${}_H B_j^{(r)}(\xi, 0|\lambda) = {}_H B_j^{(r)}(\xi|\lambda)$ and higher-order partially degenerate Hermite-Bernoulli polynomials of the first kind reduce to higher-order partially degenerate Bernoulli polynomials of the first kind. When $\xi = \eta = 0$, we write ${}_H B_j^{(r)}(\lambda) = {}_H B_j^{(r)}(0,0|\lambda)$, the higher-order partially degenerate Bernoulli numbers of the first kind for $j \geqslant 0$.

Theorem 3.2. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j^{(r)}(\xi,\eta|\lambda) = \sum_{l=0}^{[\frac{j}{2}]} \binom{j}{2l} B_{j-2l}^{(r)}(\xi|\lambda)\eta^l.$$

Proof. In (3.1), we expand $e^{\eta z^2}$ in series, use (1.2) and then compare the coefficients of z on both the sides to get the result. □

Theorem 3.3. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j^{(r)}(\xi,\eta|\lambda) = \sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_l S_2(k,l)\, {}_H B_{j-k}(0,\eta|\lambda). \tag{3.2}$$

Proof. From (3.1), we have

$$\begin{aligned}
\sum_{j=0}^{\infty} {}_H B_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!} &= \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{\xi z+\eta z^2} \\
&= \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{\eta z^2}(e^z-1+1)^{\xi} \\
&= \sum_{j=0}^{\infty} {}_H B_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{l=0}^{\infty}(\xi)_l\frac{1}{l!}(e^z-1)^l \\
&= \sum_{j=0}^{\infty} {}_H B_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{l=0}^{\infty}(\xi)_l\sum_{k=l}^{\infty} S_2(k,l)\frac{z^k}{k!} \\
&= \sum_{j=0}^{\infty} {}_H B_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{l=0}^{k}(\xi)_l S_2(k,l)\frac{z^k}{k!} \\
&= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_l S_2(k,l)\, {}_H B_{j-k}^{(r)}(0,\eta|\lambda)\right)\frac{z^j}{j!}.
\end{aligned}$$

Comparing the coefficients of z, we get (3.2). □

Theorem 3.4. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j^{(r)}(\xi+\alpha,\eta|\lambda) = \sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_l S_2(k+\alpha,l+\alpha)\,{}_H B_{j-k}(0,\eta|\lambda). \tag{3.3}$$

Proof. By changing ξ by $\xi+\alpha$ in (3.1), we see that

$$\begin{aligned}
\sum_{j=0}^{\infty} {}_H B_j^{(r)}(\xi+\alpha,\eta|\lambda)\frac{z^j}{j!} &= \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{\xi z+\eta z^2} e^{\alpha t}\\
&= \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{\eta z^2} e^{\alpha z}(e^z-1+1)^{\xi}\\
&= \sum_{j=0}^{\infty} {}_H B_j(0,\eta|\lambda)\frac{z^j}{j!} e^{\alpha z}\sum_{l=0}^{\infty}(\xi)_l \frac{1}{l!}(e^z-1)^l\\
&= \sum_{j=0}^{\infty} {}_H B_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!} e^{\alpha z}\sum_{l=0}^{\infty}(\xi)_l \sum_{k=l}^{\infty} S_2(k,l)\frac{z^k}{k!}\\
&= \sum_{j=0}^{\infty} {}_H B_j^{(r)}(0,\eta|\lambda)\frac{z^j}{j!}\sum_{k=0}^{\infty}\sum_{l=0}^{k}(\xi)_l S_2(k+\alpha,l+\alpha)\frac{z^k}{k!}\\
&= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{j}{k}(\xi)_l S_2(k+\alpha,l+\alpha)\,{}_H B_{j-k}^{(r)}(0,\eta|\lambda)\right)\frac{z^j}{j!}.
\end{aligned}\tag{3.4}$$

In view of (3.1) and (3.4), we obtain the result (3.3). □

Theorem 3.5. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j^{(r)}(\xi,\eta|\lambda) = \sum_{l=0}^{j}\binom{j}{l}\,{}_H B_{j-l}^{(r-k)}(\xi,\eta|\lambda)\,{}_H B_l^{(k)}(0,0|\lambda).$$

Proof. We observe that

$$\begin{aligned}
\sum_{j=0}^{\infty} {}_H B_j^{(r)}(\xi,\eta|\lambda)\frac{z^j}{j!} &= \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{\xi z+\eta z^2}\\
&= \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^{r-k}\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^{k} e^{\xi z+\eta z^2}\\
&= \sum_{j=0}^{\infty} {}_H B_j^{(r-k)}(\xi,\eta|\lambda)\frac{z^j}{j!}\sum_{l=0}^{\infty} {}_H B_l^{(k)}(0,0|\lambda)\frac{z^l}{l!}\\
&= \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}\,{}_H B_{j-l}^{(r-k)}(\xi,\eta|\lambda)\,{}_H B_l^{(k)}(0,0|\lambda)\right)\frac{z^j}{j!}.
\end{aligned}$$

Now comparing the coefficients of z, we get the required result. □

Theorem 3.6. *Let* $j \geqslant 0$. *Then*

$$ {}_H B_j^{(r)}(\zeta,u|\lambda) = \sum_{k=0}^{j}\binom{j}{k} H_k(\alpha-\xi+\zeta,\beta-\eta+u)\,{}_H B_{j-k}^{(r)}(\xi-\alpha,\eta-\beta|\lambda).$$

Proof. In (3.1), we have

$$\begin{aligned}\sum_{j=0}^{\infty} {}_HB_j^{(r)}(\zeta,u|\lambda)\frac{z^j}{j!} &= \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{(\xi-\alpha)z+(\eta-\beta)z^2}e^{-(\xi-\zeta-\alpha)z-(\eta-u-\beta)z^2}\\ &= e^{-(\xi-\zeta-\alpha)z-(\eta-u-\beta)z^2}\sum_{j=0}^{\infty} {}_HB_j^{(r)}(\xi-\alpha,\eta-\beta|\lambda)\frac{z^j}{j!}\\ &= \sum_{k=0}^{\infty} H_k(\alpha-\xi+\zeta,\beta-\eta+u)\frac{z^k}{k!}\sum_{j=0}^{\infty} {}_HB_j^{(r)}(\xi-\alpha,\eta-\beta|\lambda)\frac{z^j}{j!}\\ &= \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi+\zeta,\beta-\eta+u)_HB_{j-k}^{(r)}(\xi-\alpha,\eta-\beta|\lambda)\right)\frac{z^j}{j!}.\end{aligned} \tag{3.5}$$

Therefore, by (3.1) and (3.5), we arrive at the desired result. □

Remark 3.7. Letting $u=\zeta=0$ in Theorem 3.6, we get following.

Corollary 3.8. *Let* $j \geqslant 0$. *Then*

$$B_j^{(r)}(\lambda)=\sum_{k=0}^{j}\binom{j}{k}H_k(\alpha-\xi,\beta-\eta)_HB_{n-k}^{(r)}(\xi-\alpha,\eta-\beta|\lambda).$$

Theorem 3.9. *Let* $j \geqslant 0$. *Then*

$$\sum_{q=0}^{j}\sum_{l=0}^{[\frac{j-q}{2}]}\left(\frac{\xi}{\eta^2}-\frac{\eta}{\xi^2}\right)^l\frac{{}_HB_{j-2l-q}^{(k)}(\xi,\eta|\lambda)B_q^{(k)}(\lambda)}{l!q!(j-q-2l)!\eta^q\xi^{n-q-2l}}=\sum_{l=0}^{j}\frac{B_l^{(k)}(\lambda)_HB_{j-l}^{(k)}(\eta,\xi|\lambda)}{(j-l)!l!\xi^l\eta^{j-l}}. \tag{3.6}$$

Proof. By changing z with $\frac{z}{\xi}$ and r by k, we can write (3.1) as

$$\sum_{j=0}^{\infty} {}_HB_j^{(k)}(\xi,\eta|\lambda)\frac{z^j}{\xi^j j!}=\left(\frac{\log(1+\lambda\frac{z}{\xi})^{\frac{1}{\lambda}}}{e^{\frac{z}{\xi}}-1}\right)^k e^{z+\eta\frac{z^2}{\xi^2}}. \tag{3.7}$$

Now interchanging ξ by η, we have

$$\sum_{j=0}^{\infty} {}_HB_j^{(k)}(\eta,\xi|\lambda)\frac{z^j}{\eta^j j!}=\left(\frac{\log(1+\lambda\frac{z}{\eta})^{\frac{1}{\lambda}}}{e^{\frac{z}{\eta}}-1}\right)^k e^{z+\xi\frac{z^2}{\eta^2}}. \tag{3.8}$$

Comparison of (3.7) and (3.8) yields

$$\begin{aligned} &e^{\xi\frac{z^2}{\eta^2}-\eta\frac{z^2}{\xi^2}}\left(\frac{2}{(1+\lambda\frac{z}{\eta})^{\frac{1}{\lambda}}+1}\right)^k\sum_{j=0}^{\infty} {}_HB_j^{(k)}(\xi,\eta|\lambda)\frac{z^j}{\xi^j j!}\\ &=\left(\frac{\log(1+\lambda\frac{z}{\eta})^{\frac{1}{\lambda}}}{e^{\frac{z}{\eta}}-1}\right)^k\sum_{j=0}^{\infty} {}_HB_j^{(k)}(\eta,\xi|\lambda)\frac{z^j}{\eta^j j!}\\ &=\sum_{l=0}^{\infty}\frac{(\frac{\xi}{\eta^2}-\frac{\eta}{\xi^2})^l}{l!}z^{2l}\sum_{q=0}^{\infty}B_q^{(k)}(\lambda)\frac{z^q}{\eta^q q!}\sum_{j=0}^{\infty} {}_HB_j^{(k)}(\xi,\eta|\lambda)\frac{z^j}{\xi^j j!}\end{aligned} \tag{3.9}$$

$$= \sum_{l=0}^{\infty} B_l^{(k)}(\lambda)\frac{z^l}{\xi^l l!}\sum_{j=0}^{\infty} {}_HB_j^{(k)}(\eta,\xi|\lambda)\frac{z^j}{\eta^j j!},$$

$$\sum_{j=0}^{\infty}\left(\sum_{q=0}^{j}\sum_{l=0}^{[\frac{j-q}{2}]}\left(\frac{\xi}{\eta^2}-\frac{\eta}{\xi^2}\right)^l \frac{{}_HB_{j-2l-q}^{(k)}(\xi,\eta|\lambda)B_q^{(k)}(\lambda)}{l!q!(j-q-2l)!\eta^q\xi^{j-q-2l}}\right)z^j = \sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\frac{B_l^{(k)}(\lambda){}_HB_{j-l}^{(k)}(\eta,\xi|\lambda)}{(j-l)!l!\xi^l\eta^{j-l}}\right)z^j.$$

Thus, by (3.9), we obtain the result (3.6). □

Theorem 3.10. *Let* $j \geqslant 0$. *Then*

$$ {}_HB_{q+l}^{(r)}(\zeta,\eta|\lambda) = \sum_{j,p=0}^{q,l}\binom{q}{j}\binom{l}{p}(\zeta-\xi)^{j+p}\,{}_HB_{q+l-p-j}^{(r)}(\xi,\eta|\lambda).$$

Proof. Replacing z with $z+u$ in (3.1), we get

$$\left(\frac{\log(1+\lambda(z+u))^{\frac{1}{\lambda}}}{e^{z+u}-1}\right)^r e^{\eta(z+u)^2} = e^{-\xi(z+u)}\sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\eta,\xi|\lambda)\frac{z^q}{q!}\frac{u^l}{l!}, \quad \text{(see [21])}.$$

Changing ξ by ζ in the above equation, we get

$$e^{(\zeta-\xi)(z+u)}\sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\xi,\eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\zeta,\eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!},$$

$$\sum_{N=0}^{\infty}\frac{[(\zeta-\xi)(z+u)]^N}{N!}\sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\xi,\eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\zeta,\eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!},$$

$$\sum_{N=0}^{\infty} f(N)\frac{(\xi+\eta)^N}{N!} = \sum_{j,m=0}^{\infty} f(j+m)\frac{\xi^j}{j!}\frac{\eta^m}{m!},$$

$$\sum_{j,p=0}^{\infty}\frac{(\zeta-\xi)^{j+p}z^j u^p}{j!p!}\sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\xi,\eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\zeta,\eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!},$$

$$\sum_{q,l=0}^{\infty}\sum_{j,p=0}^{q,l}\frac{(\zeta-\xi)^{j+p}}{j!p!}{}_HB_{q+l-j-p}^{(r)}(\xi,\eta|\lambda)\frac{z^q}{(q-j)!}\frac{u^l}{(l-p)!} = \sum_{q,l=0}^{\infty} {}_HB_{q+l}^{(r)}(\zeta,\eta|\lambda)\frac{z^q}{q!}\frac{u^l}{l!}.$$

Equating the like powers of z and u in the above equation, we get the required result. □

4. A class of two-index real Hermite polynomials and partially degenerate Bernoulli polynomials of the first kind

This section is a consequence of the definition of the two-index real Hermite partially degenerate Bernoulli polynomials of the first kind and generalized partially degenerate Gould-Hopper-Bernoulli polynomials of the first kind combined with their properties and special cases.

We define partially degenerate two-index real Hermite-Bernoulli polynomials of the first kind by the generating function

$$\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{-u^2z^2+(2uz+v)\xi-uvz} = \sum_{j=0}^{\infty} {}_hB_j^{(r)}(\xi,u,v|\lambda)\frac{z^j}{j!}. \tag{4.1}$$

For $\nu = 0$, (4.1) reduces to

$$\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{-u^2z^2+(2uz)\xi} = \sum_{j=0}^{\infty} {}_{h}B_j^{(r)}(\xi,u,0|\lambda)\frac{z^j}{j!} = \sum_{j=0}^{\infty} B_j^{(r)}(\xi|\lambda)\frac{z^j}{j!}\sum_{m=0}^{\infty} H_m(\xi)\frac{u^m z^m}{m!}.$$

Replacing j by $j-m$ and comparing the coefficients of z^j, we get

$${}_{h}B_j^{(r)}(\xi,u,0|\lambda) = \sum_{m=0}^{j}\binom{j}{m} u^m B_{j-m}^{(r)}(\xi|\lambda)H_m(\xi),$$

where $H_m(\xi)$ is ordinary Hermite polynomials.

Note that the above result for $u = 1$ is a special case of (3.1) because when ξ is replaced by 2ξ and $\eta = -1$, then we have

$$\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{2\xi z - z^2} = \sum_{j=0}^{\infty} {}_{H}B_j^{(r)}(2\xi,-1|\lambda)\frac{z^j}{j!}.$$

In other words

$${}_{h}B_j^{(r)}(\xi,1,0|\lambda) = {}_{H}B_j^{(r)}(2\xi,-1|\lambda).$$

Theorem 4.1. *Let* $j \geqslant 0$. *Then*

$${}_{h}B_m^{(r)}(\xi,u,\nu|\lambda) = \sum_{j=0}^{\infty}\sum_{s=0}^{m} B_s^{(r)}(\lambda)h_{m-s,j}(\xi)\frac{u^{m-s}\nu^j}{(m-s)!j!}.$$

Proof. On replacing u by uz in (1.6), we have

$$\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} h_{m,j}(\xi)\frac{(uz)^m\nu^j}{m!j!} = e^{-u^2z^2+(2uz+\nu)\xi-u\nu z}.$$

Then using (4.1), we can write

$$\sum_{s=0}^{\infty} {}_{h}B_s^{(r)}(\xi,u,\nu|\lambda)\frac{z^s}{s!} = \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{-u^2z^2+(2uz+\nu)\xi-u\nu z} = \sum_{s=0}^{\infty} B_s^{(r)}(\lambda)\frac{z^s}{s!}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} h_{m,j}(\xi)\frac{(uz)^m\nu^j}{m!j!}.$$

Now replacing m by $m-s$ and comparing the coefficients of z^s, we get the required result. □

Theorem 4.2. *Let* $j \geqslant 0$. *Then*

$$B_m^{(r)}(\xi\eta|\lambda) = \sum_{j=0}^{\infty}\sum_{s=0}^{m} B_s^{(r)}(\lambda)h_{m-s,j}(\xi)\frac{\eta^{m-s}}{(m-s)!j!}.$$

Proof. We multiply both sides of (1.7) by

$$\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r,$$

and replace η by ηz to get

$$\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{\xi\eta z} = \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r \sum_{m,j=0}^{\infty}(-1)^j h_{m,j}(\xi)\frac{(\eta z)^{m+j}}{m!j!}$$

$$= \sum_{s=0}^{\infty} B_s^{(r)}(\lambda)\frac{z^s}{s!} \sum_{m,j=0}^{\infty} (-1)^j h_{m,j}(\xi)\frac{(\eta z)^{m+j}}{m!j!}.$$

Thus we have

$$\sum_{s=0}^{\infty} B_s^{(r)}(\xi\eta|\lambda)\frac{z^s}{s!} = \sum_{s=0}^{\infty} B_s^{(r)}(\lambda)\frac{z^s}{s!} \sum_{m,j=0}^{\infty} (-1)^j h_{m,j}(\xi)\frac{(\eta z)^{m+j}}{m!j!}.$$

Now replacing m by $m-s$ and comparing the coefficients of z^s, we get the required result. □

Ghanmi and Lamsaf [9] analyzed a new class of polynomials generalizing different classes of Hermite polynomials such as the real Gould-Hopper, as well as the 1-d and 2-d holomorphic, ternary and polyanalytic complex Hermite polynomials. In the following theorem, we are concerned with a special and unified generalization. More precisely, we deal with the generalized Gould-Hopper polynomials and partially degenerate Bernoulli polynomials of the first kind.

First, we define generalized partially degenerate Gould-Hopper-Bernoulli polynomials of the first kind by the generating function

$$\sum_{j=0}^{\infty} {}_GB_j^{(r)}(w,\gamma,z|u,v|\lambda)\frac{z^j}{j!} = \left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r e^{wv+\zeta uz+\gamma u^p v^q}. \tag{4.2}$$

Note that for $r=0$, (4.2) reduces to

$$\sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma)\frac{v^m}{m!}e^{\zeta uz} = e^{wv+\zeta uz+\gamma u^p v^q},$$

where $G_m^{(q)}$ is defined by (1.8). The next generating function is a consequence of the above one (see [11]) and gives the closed expression of $R_\gamma^{p,q}(\zeta,w|u,v)$ in the form

$$R_\gamma^{p,q}(\zeta,w|u,v) = e^{\zeta u+wv+\gamma u^p v^q},$$

where

$$R_\gamma^{p,q}(z,w|u,v) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(\zeta,w|\gamma)\frac{u^j}{j!}\frac{v^m}{m!}. \tag{4.3}$$

Furthermore, the polynomials $H_{j,m}^{(p,q)}(\zeta,w|\gamma)$ are given by (1.8) and (1.9).

Theorem 4.3. *For every $u,v,w,\zeta,\in\mathbb{C}$ and $j,r\geqslant 0$, we have*

$$ {}_GB_j^{(r)}(w,\gamma,\zeta|u,v|\lambda) = \sum_{m=0}^{\infty}\sum_{s=0}^{j} H_{j-s,m}^{(p,q)}(\zeta,w|\gamma)B_s^{(r)}(\lambda)\frac{u^{j-s}v^m}{(j-s)!m!}.$$

Proof. Start with (1.8), replace γ by $u^p\gamma$ and multiply both sides by $e^{\zeta uz}$ to get

$$e^{wv+\gamma u^p v^q}e^{\zeta uz} = \sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma)\frac{v^m}{m!}e^{\zeta uz}.$$

Again, we multiply both sides by

$$\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r,$$

to get

$$e^{wv+\gamma u^p v^q} e^{\zeta uz}\left(\frac{\log(1+\lambda z)^{\frac{1}{\lambda}}}{e^z-1}\right)^r=\sum_{m=0}^{\infty} G_m^{(q)}(w|(uz)^p\gamma)\frac{v^m}{m!}e^{zut}\sum_{s=0}^{\infty} B_s^{(r)}(\lambda)\frac{z^s}{s!}.$$

Thus by using (4.2) and (4.3), we have

$$\sum_{s=0}^{\infty} {}_G B_s^{(r)}(w,\gamma,\zeta|u,v|\lambda)\frac{z^s}{s!}=\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} H_{j,m}^{(p,q)}(\zeta,w|\gamma)\frac{(uz)^j}{j!}\frac{v^m}{m!}\sum_{s=0}^{\infty} B_s^{(r)}(\lambda)\frac{z^s}{s!}.$$

Now replacing j with $j-s$ and comparing the coefficients of z^j, we get the required result. □

As an immediate consequence of the above theorem, we have the following.

Corollary 4.4. *For every* $u, v, w, \zeta, \in \mathbb{C}$ *and* $j \geqslant 0$, *we have*

$${}_G B_j^{(0)}(w,\gamma,\zeta|u,v|\lambda)=R_\gamma^{p,q}(\zeta,w|u,v)=e^{\zeta u+wv+\gamma u^p v^q}.$$

5. General identities

In our previous articles (Pathan and Khan [26, 31], Hiba and Khan [12], and Khan et al. [16, 19, 20]), it was shown that symmetric identities for Hermite-based generalized polynomials unify many properties and identities of Hermite-Bernoulli and Hermite-Euler polynomials. In this section, we give general symmetric identities for partially degenerate Hermite-Bernoulli polynomials of the first kind ${}_H B_j(\xi,\eta|\lambda)$ by applying the generating functions (1.3) and (3.1).

Theorem 5.1. *Let* $j \geqslant 0$. *Then*

$$\sum_{l=0}^{j}\binom{j}{l}a^{j-l}b^l {}_H B_{j-l}^{(r)}(b\xi,b^2\eta|\lambda){}_H B_l^{(r)}(a\xi,a^2\eta|\lambda)$$
$$=\sum_{l=0}^{j}\binom{j}{l}b^{j-l}a^l {}_H B_{j-l}^{(r)}(a\xi,a^2\eta|\lambda){}_H B_l^{(r)}(b\xi,b^2\eta|\lambda).$$

Proof. Suppose

$$\begin{aligned}A(z)&=\left(\frac{(\log(1+\lambda az)^{\frac{1}{\lambda}})(\log(1+\lambda bz)^{\frac{1}{\lambda}})}{(e^{az}-1)(e^{bz}-1)}\right)^r e^{ab\xi z+a^2b^2\eta z^2},\\ A(z)&=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}a^{j-l}b^l {}_H B_{j-l}^{(r)}(b\xi,b^2\eta|\lambda){}_H B_l^{(r)}(a\xi,a^2\eta|\lambda)\right)\frac{z^j}{j!}.\end{aligned}\tag{5.1}$$

On similar lines, we can show that

$$A(z)=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\binom{j}{l}b^{j-l}a^l {}_H B_{j-l}^{(r)}(a\xi,a^2\eta|\lambda){}_H B_l^{(r)}(b\xi,b^2\eta|\lambda)\right)\frac{z^j}{j!}.\tag{5.2}$$

Given (5.1) and (5.2), we arrive at the desired result. □

Corollary 5.2. *By setting* $b=1$ *in Theorem 5.1, we get*

$$\sum_{l=0}^{j}\binom{j}{l}a^{j-l}{}_H B_{j-l}^{(r)}(\xi,\eta|\lambda){}_H B_l^{(r)}(a\xi,a^2\eta|\lambda)=\sum_{l=0}^{j}\binom{j}{l}a^l {}_H B_{j-l}^{(r)}(a\xi,a^2\eta){}_H B_l^{(r)}(\xi,\eta).$$

Theorem 5.3. *Let $j \geqslant 0$. Then*

$$\begin{aligned}\sum_{l=0}^{j}\sum_{i=0}^{a-1}\sum_{p=0}^{b-1}\binom{j}{l}a^{j-l}b^{l}{}_{H}B_{j-l}^{(r)}\left(b\eta+\frac{b}{a}i+p,b^{2}\zeta|\lambda\right)B_{l}^{(r)}(a\eta|\lambda)\\=\sum_{l=0}^{j}\sum_{p=0}^{a-1}\sum_{i=0}^{b-1}\binom{j}{l}b^{j-l}a^{l}{}_{H}B_{j-l}^{(r)}\left(a\eta+\frac{a}{b}i+p,a^{2}\zeta|\lambda\right)B_{l}^{(r)}(b\eta|\lambda).\end{aligned}\tag{5.3}$$

Proof. Let

$$\begin{aligned}B(z)&=\left(\frac{(\log(1+\lambda az)^{\frac{1}{\lambda}})(\log(1+\lambda bz)^{\frac{1}{\lambda}})(e^{abz}-1)}{(e^{az}-1)^{2}(e^{bz}-1)^{2}}\right)^{r}e^{ab(\xi+\eta)z+a^{2}b^{2}\zeta z^{2}}\\&=\left(\frac{\log(1+\lambda az)^{\frac{1}{\lambda}}}{e^{az}-1}\right)^{r}e^{ab\xi z+a^{2}b^{2}\zeta z^{2}}\sum_{i=0}^{a-1}e^{bzi}\left(\frac{\log(1+\lambda bz)^{\frac{1}{\lambda}}}{e^{bz}-1}\right)^{r}e^{ab\eta z}\sum_{p=0}^{b-1}e^{azp},\end{aligned}\tag{5.4}$$

$$B(z)=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\sum_{i=0}^{a-1}\sum_{p=0}^{b-1}\binom{j}{l}a^{j-l}b^{l}{}_{H}B_{j-l}^{(r)}\left(b\xi+\frac{b}{a}i+p,b^{2}\zeta|\lambda\right)B_{l}^{(r)}(a\eta|\lambda)\right)\frac{z^{j}}{j!}.\tag{5.5}$$

On the other hand, we have

$$B(z)=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\sum_{p=0}^{a-1}\sum_{i=0}^{b-1}\binom{j}{l}b^{j-l}a^{l}{}_{H}B_{j-l}^{(r)}\left(a\xi+\frac{a}{b}i+p,a^{2}\zeta|\lambda\right)B_{l}^{(r)}(b\eta|\lambda)\right)\frac{z^{j}}{j!}.\tag{5.6}$$

Therefore, by (5.5) and (5.6), we get (5.3). □

Theorem 5.4. *Let $j \geqslant 0$. Then*

$$\begin{aligned}\sum_{l=0}^{j}\sum_{i=0}^{a-1}\sum_{p=0}^{b-1}\binom{j}{l}a^{j-l}b^{l}{}_{H}B_{j-l}^{(r)}\left(b\eta+\frac{b}{a}i,b^{2}\zeta|\lambda\right)B_{m}^{(r)}\left(a\eta+\frac{a}{b}p\right)\\=\sum_{l=0}^{j}\sum_{p=0}^{a-1}\sum_{i=0}^{b-1}\binom{j}{l}b^{j-l}a^{l}{}_{H}B_{j-l}^{(r)}\left(a\eta+\frac{a}{b}i,a^{2}\zeta|\lambda\right)B_{l}^{(r)}\left(b\eta+\frac{b}{a}p|\lambda\right).\end{aligned}\tag{5.7}$$

Proof. The proof is analogous to Theorem 5.3, but we need to write (5.4) in the form

$$B(z)=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\sum_{i=0}^{a-1}\sum_{p=0}^{b-1}\binom{j}{l}a^{j-l}b^{l}{}_{H}B_{j-l}^{(r)}\left(b\eta+\frac{b}{a}i,b^{2}\zeta|\lambda\right)B_{l}^{(r)}\left(a\eta+\frac{a}{b}p|\lambda\right)\right)\frac{z^{j}}{j!}.\tag{5.8}$$

On the other hand (5.4) can be shown equal to

$$B(z)=\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j}\sum_{p=0}^{a-1}\sum_{i=0}^{b-1}\binom{j}{l}b^{j-l}a^{l}{}_{H}B_{j-l}^{(r)}\left(a\eta+\frac{a}{b}i,a^{2}\zeta|\lambda\right)B_{l}^{(r)}\left(b\eta+\frac{b}{a}p|\lambda\right)\right)\frac{z^{j}}{j!}.\tag{5.9}$$

By (5.8) and (5.9), we get (5.7). □

Now, we prove the following symmetric identity involving a sum of integer powers $S_k(j)$ given by equation (1.16) and partially degenerate Hermite-Bernoulli polynomials of the first kind ${}_{H}B_{j}^{(r)}(\xi,\eta|\lambda)$.

Theorem 5.5. *Let* $j \geqslant 0$. *Then*

$$\begin{aligned}\sum_{k=0}^{j}\binom{j}{k}a^{j-k}b^{k}{}_{H}B_{j-k}^{(r)}(b\xi,b^{2}\zeta|\lambda)\sum_{i=0}^{k}\binom{k}{i}S_{i}(b-1)B_{k-i}^{(r)}(a\eta|\lambda)\\=\sum_{k=0}^{j}\binom{j}{k}b^{j-k}a^{k}{}_{H}B_{j-k}^{(r)}(a\xi,a^{2}\zeta|\lambda)\sum_{i=0}^{k}\binom{k}{i}S_{i}(a-1)B_{k-i}^{(r)}(b\eta|\lambda).\end{aligned}\tag{5.10}$$

Proof. Consider

$$\begin{aligned}C(z)&=\left(\frac{(\log(1+\lambda az)^{\frac{1}{\lambda}})(\log(1+\lambda bz)^{\frac{1}{\lambda}})(e^{abz}-1)}{(e^{az}-1)^{2}(e^{bz}-1)^{2}}\right)^{r}e^{ab(\xi+\eta)z+a^{2}b^{2}\zeta z^{2}}\\&=\sum_{j=0}^{\infty}{}_{H}B_{j}^{(r)}(b\xi,b^{2}\zeta|\lambda)\frac{(az)^{j}}{j!}\sum_{i=0}^{\infty}S_{i}(b-1)\sum_{k=0}^{\infty}B_{k}^{(r)}(a\eta|\lambda)\frac{(bz)^{k}}{k!},\\C(z)&=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}a^{j-k}b^{k}{}_{H}B_{j-k}^{(r)}(b\xi,b^{2}\zeta|\lambda)\sum_{i=0}^{k}\binom{k}{i}S_{i}(b-1)B_{k-i}^{(r)}(a\eta|\lambda)\right)\frac{z^{j}}{j!}.\end{aligned}$$

On the other hand, we have

$$C(z)=\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}b^{j-k}a^{k}{}_{H}B_{j-k}^{(r)}(a\xi,a^{2}\zeta|\lambda)\sum_{i=0}^{k}\binom{k}{i}S_{i}(a-1)B_{k-i}^{(r)}(b\eta|\lambda)\right)\frac{z^{j}}{j!}.$$

By comparing the coefficients of z^j on the right-hand sides of the last two equations, we obtain the result (5.10). □

6. Concluding remarks

In this paper, we have presented the generalized partially degenerate Hermite-Bernoulli polynomials of the first kind and discussed, in particular, some interesting series representations. We have deduced some relevant properties by using the structure and the relations satisfied by the recently generalized Hermite polynomials that incorporate the definition of partially degenerate Hermite-Bernoulli polynomials of the first kind and a preliminary study of these polynomials. We derived some theorems on implicit summation formulae for partially degenerate Hermite-Bernoulli polynomials of the first kind ${}_{H}B_{j}^{(r)}(\xi,\eta|\lambda)$ and their special cases are given. We derived the consequence of the definition of the two-index real Hermite-partially degenerate polynomials of the first kind and generalized Gould-Hopper-partially degenerate Bernoulli polynomials of the first kind combined with their properties and special cases. Finally, we derived symmetry identities for type partially degenerate Hermite-Bernoulli polynomials of the first kind.

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