On a minimal set of generators for the algebra $H^*(BE_d; F_2)$ and its applications

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Abstract
We investigate the Peterson hit problem for the polynomial algebra $P_d$, viewed as a graded left module over the mod-2 Steenrod algebra, $A$. For $d > 4$, this problem is still unsolved, even in the case of $d = 5$ with the help of computers. In this article, we study the hit problem for the case $d = 6$ in the generic degree $6(2^r - 1) + 6.2^r$, with $r$ an arbitrary non-negative integer. Furthermore, the behavior of the sixth Singer algebraic transfer in degree $6(2^r - 1) + 6.2^r$ is also discussed at the end of this paper.

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1. Introduction
Let $X$ be a topological space. Cohomology operations are generated by the natural transformations of degree $i$ which are so-called Steenrod squares

$$ Sq^i : H^*(X, F_2) \rightarrow H^{*+i}(X, F_2), $$

where $H^*(X, F_2)$ is the singular cohomology of $X$ with coefficients in the two-element field $F_2$, and $i$ is arbitrary non-negative integers. In 1952, Serre [13] proved that the Steenrod squares generate all stable cohomology operations with the usual addition and the composition of maps. The algebra of stable cohomology operations with coefficients in $F_2$ is known as the modulo 2 Steenrod algebra, $A$. Then, for each topological space $X$, $H^*(X, F_2)$ is an $A$-module.

Hence, the Steenrod algebra is able to be defined algebraically as a quotient algebra of $F_2$-free graded associative algebra generated by the symbols $Sq^i$ of degree $i$ where $i$ is a non-negative integer, by the two-sided ideal generated by the relation $Sq^0 = 1$ and the Adem’s relations

$$ Sq^aSq^b = \sum_{j=0}^{[a/2]} \binom{b - 1 - j}{a - 2j} Sq^{a+b-j}Sq^j, \ 0 < a < 2b. $$

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Let $E_d$ be an elementary abelian 2-group of rank $d$. Let us denote by $BE_d$ the classifying space of $E_d$. It may be thought of as the product of $d$ copies of real project space $\mathbb{R}P^\infty$. Then, using the Künneth formula for cohomology, one has an isomorphism of $F_2$-algebras

$$\mathcal{P}_d := H^*(BE_d; F_2) \cong F_2[x_1] \otimes F_2 \cdots \otimes F_2 F_2[x_d] \cong F_2[x_1, x_2, \ldots, x_d],$$

where $x_i \in H^1(BE_d; F_2)$ for every $i$.

As is well-known, $\mathcal{P}_d$ is a module over the mod-2 Steenrod algebra $A$. The action of $A$ on $\mathcal{P}_d$ is determined by the formula

$$Sq^k(x_j) = \begin{cases} x_j, & k = 0, \\ x_j^2, & k = 1, \\ 0, & k > 1, \end{cases}$$

and the Cartan formula $Sq^k(uv) = \sum_{i=0}^k Sq^i(u)Sq^{k-i}(v)$, where $u, v \in \mathcal{P}_d$ (see Steenrod and Epstein [16]).

The Peterson hit problem is to find a minimal generating set for $\mathcal{P}_d$ regarded as a module over the mod-2 Steenrod algebra. If we treat $F_2$ as a trivial $A$-module, the hit problem is analogous to the problem of finding a basis for the $F_2$-graded vector space $F_2 \otimes_A \mathcal{P}_d$.

This issue has first been studied by Peterson [7], Singer [14], Wood [29], Priddy [11], who show its relationship to several classical problems in cobordism theory, modular representation theory, Adams spectral sequence for the stable homotopy of spheres, stable homotopy type of the classifying space of finite groups.

Let $\alpha(n)$ be the number of digits 1 in the binary expansion of a natural $n$. The function $\mu : N \rightarrow N$ is defined as follows:

$$\mu(0) = 0, \text{ and } \mu(n) = \min(m \in N : \ n = \sum_{i=1}^{m} (2^n - 1), n_i > 0) = \min(m \in N : \ \alpha(n + m) \leq m).$$

Peterson [7] hypothesized that as a module over the Steenrod algebra $A$, $\mathcal{P}_d$ is generated by monomials of degree $m$ obeying the inequality $\alpha(m + d) \leq d$, and proved it for $d \leq 2$. After then, Wood [29] proved this in general. This is a fantastic tool for figuring out $A$-generators for $\mathcal{P}_d$.

The squaring operation of Kameko is one of the most essential tools in the study of the hit problem

$$\overline{Sq}_d^0 := \overline{S}_{d+2m}^d : (F_2 \otimes_A \mathcal{P}_d)_{2m+d} \rightarrow (F_2 \otimes_A \mathcal{P}_d)_m,$$

which is induced by an $F_2$-linear map $S_d : \mathcal{P}_d \rightarrow \mathcal{P}_d$, given by

$$S_d(x) = \begin{cases} y_r, & \text{if } x = x_1 x_2 \cdots x_k y_r^2, \\ 0, & \text{otherwise}, \end{cases}$$

for any monomial $x \in \mathcal{P}_d$. Clearly, $\overline{S}_{d+2m}^d$ is an $F_2$-epimorphism.

From the results of Wood [29], Kameko [4], and Sum [18], the hit problem is reduced to the case of degree $n$ of the form $n = r(2^t - 1) + 2^m t$, where $r, m, t$ are non-negative integers such that $0 \leq \mu(m) < r \leq d$.

Recently, the hit problem and its applications have been interested and studied by many authors (see Silverman [15], Repka-Selick [12], Janfada-Wood [2, 3], Nam [6], Sum [17, 18], Mothebe-Kaelo-Ramatapole [5], Phuc-Sum [8], Sum-Tin [20], Tin-Sum [22], Tin [23–26] and others).

The $F_2$-vector space $F_2 \otimes_A \mathcal{P}_d$ was entirely calculated for $d \leq 4$ (see Peterson [7] for $d = 1, 2$, Kameko [4] for $d = 3$, Sum [18] for $d = 4$), but it remains unresolved for $d \geq 5$, even with the aid of computers in the case of $d = 5$.

In this paper, we study the hit problem for the case $d = 6$ in the generic degree $6(2^r - 1) + 6.2^t$, with $r$ an arbitrary non-negative integer. The main goal of the current paper is to explicitly determine an admissible monomial basis of the $F_2$-vector space $F_2 \otimes_A \mathcal{P}_6$ in these degrees.
One of the primary applications of the hit problem is in surveying a homomorphism proposed by Singer [14], which is a homomorphism from the homology of the Steenrod algebra to the subspace of $F_2 \otimes_A \mathcal{P}_d$ consisting of all the $GL(d; F_2)$-invariant classes.

Noting that the general linear group $GL(d; F_2)$ acts naturally on $\mathcal{P}_d$ by matrix substitution. Due to the fact that the two actions of $\mathcal{A}$ and $GL(d; F_2)$ upon $\mathcal{P}_d$ commute with each other, there is an inherited action of $GL(d; F_2)$ on $F_2 \otimes_A \mathcal{P}_d$.

Recall that $\mathcal{P}_1$ is the submodule of $F_2[x_1,x_1^{-1}]$ spanned by all powers $x_1^i$ with $i \geq -1$. The usual $\mathcal{A}$-action on $\mathcal{P}_1 = F_2[x_1]$ is canonically extended to an $\mathcal{A}$-action on $F_2[x_1,x_1^{-1}]$. Hence, $\mathcal{P}_1$ is an $\mathcal{A}$-submodule of $F_2[x_1,x_1^{-1}]$. The inclusion $\mathcal{P}_1 \subset \mathcal{P}_1$ gives rise to a short exact sequence of $\mathcal{A}$-modules:

$$0 \to \mathcal{P}_1 \to \mathcal{P}_1 \to \sum_{-1}^{-1} F_2 \to 0.$$ 

Let $e_1$ be the corresponding element in $\text{Ext}^1_{\mathcal{A}}(\sum_{-1}^{d} F_2, \mathcal{P}_1)$. Using the cross and Yoneda products, Singer set

$$e_d := (e_1 \times \mathcal{P}_{d-1}) \circ (e_1 \times \mathcal{P}_{d-2}) \circ \ldots \circ (e_1 \times \mathcal{P}_1) \circ e_1 \in \text{Ext}^d_{\mathcal{A}}(\sum_{-d}^{-d} F_2, \mathcal{P}_d).$$

Then, he defined

$$\bar{\varphi}_d : \text{Tor}^d_{\mathcal{A}}(F_2, \sum_{-1}^{-1} F_2) \to \text{Tor}^d_{\mathcal{A}}(F_2, \mathcal{P}_d) = F_2 \otimes_A \mathcal{P}_d,$$

$$z \mapsto e_d \cap z.$$

Remarkably, $\text{Im} \bar{\varphi}_d$ is a submodule of $(F_2 \otimes_A \mathcal{P}_d)^{GL(d; F_2)}$. So, $\bar{\varphi}_d$ induces the homomorphism

$$\varphi_d : \text{Tor}^d_{\mathcal{A}}(F_2, \sum_{-1}^{-1} F_2) \to (F_2 \otimes_A \mathcal{P}_d)^{GL(d; F_2)}.$$ 

Let $F_2 \otimes_{GL(d; F_2)} \text{PH}_m((R^p)\mathcal{P}_d)$ be dual to $(F_2 \otimes_A \mathcal{P}_d)^{GL(d; F_2)}$. By passing to the dual, we have an algebraic homomorphism called Singer’s algebraic transfer

$$\psi_d : F_2 \otimes_{GL(d; F_2)} \text{PH}_m((R^p)\mathcal{P}_d) \to \text{Ext}^d_{\mathcal{A}}(F_2, F_2).$$ 

This is a useful tool in describing the cohomology groups of the Steenrod algebra, $\text{Ext}^d_{\mathcal{A}}(F_2, F_2)$. At the conclusion of this article, the behavior of the sixth Singer algebraic transfer in degree $6(2^r - 1) + 6.2^r$ is also discussed.

Next, in Section 2, we recall some needed information on admissible monomials in $\mathcal{P}_d$. The main results are presented in Section 3.

2. Preliminaries

We will review some key facts from Sum [18], Kameko [4], and Singer [14] in this section, which will be used in the next section. Let us denote by $N_d = \{1, 2, \ldots, d\}$ and

$$X_j = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j \in N_d \setminus J} x_j, \quad J = \{j_1, j_2, \ldots, j_s\} \subset N_d.$$ 

In particular, $X_{N_d} = 1$, $X_{\emptyset} = x_1 x_2 \ldots x_d$, $X_j = x_{1} \ldots \hat{x}_i \ldots x_d$, $1 \leq j \leq d$, and $X := X_d \in \mathcal{P}_{d-1}$.

Let $\alpha_t(n)$ be the $t$-th coefficient in dyadic expansion of $n$. Then, $n = \sum_{t=0}^{\infty} \alpha_t(n)2^t$ where $\alpha_t(n) \in \{0, 1\}$. Let $x = x_1^{a_1} x_2^{a_2} \ldots x_d^{a_d} \in \mathcal{P}_d$. Denote $\nu_j(x) = a_j, 1 \leq j \leq d$. Set

$$J_t(x) = \{j \in N_d : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geq 0$. Then, we have $x = \prod_{t \geq 0} X_{J_t(x)}^{2^t}$. 

Definition 2.1. For a monomial $x$ belongs to $\mathcal{P}_d$, define two sequences associated with $x$ by
\[
\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \sigma(x) = (v_1(x), v_2(x), \ldots, v_d(x)),
\]
where $\omega_i(x) = \sum_{1 \leq j \leq d} \alpha_{i-1}(v_j(x)) = \deg X_{f_{i-1}(x)}$, $i \geq 1$. The sequences $\omega(x)$ and $\sigma(x)$ are, respectively, called the weight vector and the exponent vector of $x$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence $\omega$ is called the weight vector if $\omega_i = 0$ for $i \gg 0$. Then, we define $\deg \omega = \sum_{i=0}^{\infty} 2^{i-1} \omega_i$. Denote by $\mathcal{P}_d(\omega)$ the subspace of $\mathcal{P}_d$ spanned by all monomials $y$ such that $\deg y = \deg \omega$, $\omega(y) \leq \omega$, and by $\mathcal{P}_d^-(\omega)$ the subspace of $\mathcal{P}_d$ spanned by all monomials $y \in \mathcal{P}_d(\omega)$ such that $\omega(y) < \omega$.

Definition 2.2. Let $A^+$ be an ideal of $A$ generated by all Steenrod squares of positive degrees, and $u, v$ two polynomials of the same degree in $\mathcal{P}_d$. We define the equivalence relations "\( \equiv \)" and "\( \equiv_\omega \)" on $\mathcal{P}_d$ by stating that
\begin{enumerate}
  \item[(i)] $u \equiv v$ if and only if $u - v \in A^+ \mathcal{P}_d$;
  \item[(ii)] $u \equiv_\omega v$ if and only if $u, v \in \mathcal{P}_d(\omega)$ and $u - v \in (A^+ \mathcal{P}_d \cap \mathcal{P}_d(\omega) + \mathcal{P}_d^-(\omega))$.
\end{enumerate}

Then, we have an $F_2$-quotient space of $\mathcal{P}_d$ by the equivalence relation "\( \equiv_\omega \)" as follows:
\[
A^+ \mathcal{P}_d(\omega) = \mathcal{P}_d(\omega)/((A^+ \mathcal{P}_d \cap \mathcal{P}_d(\omega)) + \mathcal{P}_d^-(\omega)).
\]

If a polynomial $u$ in $\mathcal{P}_d$ can be expressed as a finite sum $u = \sum_{l \geq 0} Sq^l(f_l)$ for suitable polynomials $f_l \in \mathcal{P}_d$, it is called a hit. That means $u$ belongs to $A^+ \mathcal{P}_d$.

Definition 2.3. Let $u, v$ be monomials of the same degree in $\mathcal{P}_d$. We say that $u < v$ if one of the following holds:
\begin{enumerate}
  \item[(i)] $\omega(u) < \omega(v)$;
  \item[(ii)] $\omega(u) = \omega(v)$, and $\sigma(u) < \sigma(v)$.
\end{enumerate}

Definition 2.4. Let $u$ be a monomial in $\mathcal{P}_d$. The monomial $u$ is said to be inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_i < u$ for $i = 1, 2, \ldots, m$ and $u - \sum_{i=1}^{m} v_i \in A^+ \mathcal{P}_d$. If $u$ is not inadmissible, we say it is admissible.

It is crucial to note that the set of all admissible monomials of degree $n$ in $\mathcal{P}_d$ is a minimal set of $A$-generators for $\mathcal{P}_d$ in degree $n$. And therefore, $(F_2 \otimes_A \mathcal{P}_d)_n$ is an $F_2$-vector space with a basis consisting of all the classes represent by the elements in $(\mathcal{P}_d)_n$.

Definition 2.5. Let $u$ be a monomial in $\mathcal{P}_d$. We say $u$ is strictly inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_j < u$, for $j = 1, 2, \ldots, m$ and $u = \sum_{j=1}^{m} v_j + \sum_{i=1}^{2^s-1} Sq^i(f_i)$ with $s = \max(k : \omega_k(u) > 0)$, $f_i \in \mathcal{P}_d$.

Observe that if $u$ is strictly inadmissible monomial, then it is inadmissible monomial, as defined by the Definitions 2.4 and 2.5. In general, the inverse is not true.

Theorem 2.6 (Kameko [4], Sum [18]). Let $u, v, w$ be monomials in $\mathcal{P}_d$ such that $\omega_t(u) = 0$ for $t > k > 0$, $\omega_t(v) \neq 0$ and $\omega_t(w) = 0$ for $t > r > 0$. Then,
\begin{enumerate}
  \item[(i)] $uw2^k$ is inadmissible if $w$ is inadmissible;
  \item[(ii)] $vw2^r$ is strictly inadmissible if $w$ is strictly inadmissible.
\end{enumerate}

Definition 2.7. Let $z = x_1^{a_1}x_2^{a_2}\ldots x_d^{a_d}$ in $\mathcal{P}_d$. The monomial $z$ is called a spike if $a_j = 2^t_j - 1$ for $t_j$ a non-negative integer and $j = 1, 2, \ldots, d$. Moreover, $z$ is called the minimal spike, if it is a spike such that $t_1 > t_2 > \ldots > t_{r-1} \geq t_r > 0$ and $t_j = 0$ for $j > r$. 

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The following is a Singer’s criterion on the hit monomials in $\mathcal{P}_d$.

**Theorem 2.8** (Singer [14]). Assume that $z$ is the minimal spike of degree $n$ in $\mathcal{P}_d$, and $u \in (\mathcal{P}_d)_n$ satisfying the condition $\mu(n) \leq d$. If $\omega(u) < \omega(z)$, then $u$ is hit.

The $A$-submodules of $\mathcal{P}_d$ that spanned all the monomials $x_1^{s_1}x_2^{s_2}\ldots x_d^{s_d}$ such that $s_1\ldots s_d = 0$, and $s_1\ldots s_d > 0$, respectively, will be denoted by $\mathcal{P}_d^0$ and $\mathcal{P}_d^+$. It is easy to check that $\mathcal{P}_d^0$ and $\mathcal{P}_d^+$ are the $A$-submodules of $\mathcal{P}_d$. Then, we have a direct summand decomposition of the $F_2$-vector spaces:

$$F_2 \otimes_A \mathcal{P}_d = (F_2 \otimes_A \mathcal{P}_d^0) \oplus (F_2 \otimes_A \mathcal{P}_d^+).$$

From now on, we set $A\mathcal{P}_d := F_2 \otimes_A \mathcal{P}_d$, $A\mathcal{P}_d^0 := F_2 \otimes_A \mathcal{P}_d^0$, $A\mathcal{P}_d^+ := F_2 \otimes_A \mathcal{P}_d^+$. Let us denote by $\mathcal{D}_d^\otimes(m)$ the set of all admissible monomials of degree $m$ in $\mathcal{P}_d$. For $f$ an element of $\mathcal{P}_d$, we denote by $[f]$ the class in $A\mathcal{P}_d$ represented by $f$. The cardinality of a set $\mathcal{U}$ is denoted by $|\mathcal{U}|$.

3. The main results

First, we study the hit problem for the polynomial algebra of six variables in the generic degree $m_r := 6(2^r - 1) + 6.2^r$, with $r$ an arbitrary non-negative integer. For $r = 0$, and $1 \leq i, j, k, \ell, t, m_s, r \leq 6$, we set

$$\mathcal{F} = \left\{ \prod_{l=1}^6 x_l; x_l^3; x_l x_l x_{k}^3; x_l x_k x_l x_{1m}^2; x_l x_m x_{2}^2 x_{7}^2 : t < m_s, s < r \right\}.$$

An easy computation shows that the following proposition, which is an immediate consequence of the result in [18].

**Proposition 3.1.** The set $\mathcal{M} = \{[a_i] : a_i \in \mathcal{F}, 1 \leq i \leq 190\}$ is a basis of $F_2$-vector space $(A\mathcal{P}_6)_6(2^0 - 1) + 6.2^0$. Consequently, $|\mathcal{M}| = 190$.

For $r = 1$, then $m_1 = 6(2^1 - 1) + 6.2^1$. Consider the homomorphism $\mathcal{T}_j : \mathcal{P}_3 \to \mathcal{P}_6$, for $1 \leq j \leq 6$ by substituting:

$$\mathcal{T}_j(x_l) = \begin{cases} x_{4j}, & \text{if } 1 \leq i < j - 1, \\ x_{4j + 1}, & \text{if } j \leq i < 6. \end{cases}$$

Then, the $F_2$-vector space $(A\mathcal{P}_6)_6(2^1 - 1) + 6.2^1$ is explicitly determined by the following theorem.

**Theorem 3.2.** Let $\bar{\omega}_1 := (2, 2, 1, 1), \bar{\omega}_2 := (2, 2, 3), \bar{\omega}_3 := (2, 4, 2), \bar{\omega}_4 := (4, 1, 1, 1), \bar{\omega}_5 := (4, 1, 3), \bar{\omega}_6 := (4, 5, 1), \bar{\omega}_7 := (4, 3, 2)$. Then:

(i) $\operatorname{Im}\mathcal{S}_{m_1}^\wedge$ is isomorphic to a subspace of $(A\mathcal{P}_6)_m$ generated by all the classes represented by the admissible monomials of the form $\prod_{l=1}^6 x_l a_l^2$, for all $1 \leq j \leq 190$. Consequently, $\dim \operatorname{Im}\mathcal{S}_{m_1}^\wedge = 190$.

(ii) The set $\{[b_i] : b_i \in \bigcup_{j=1}^6 \mathcal{T}_j(\mathcal{C}_5^\otimes(18)), 1 \leq i \leq 2865\}$ is a basis of the $F_2$-vector space $(A\mathcal{P}_6)_m$. This implies that $(A\mathcal{P}_6)_m$, has dimension 2865.

(iii) We have $(\operatorname{Ker}\mathcal{S}_{m_1}^\wedge \cap (A\mathcal{P}_6^+)_m) \cong \bigoplus_{L=1}^7 \mathcal{P}_6^+(\bar{\omega}_i)$. Moreover, the space $(\operatorname{Ker}\mathcal{S}_{m_1}^\wedge \cap (A\mathcal{P}_6^+)_m)$ is 886-dimensional.

**Proof.** Since $\mathcal{P}_d = \oplus_{m \geq 0}(\mathcal{P}_d)_m$ is the graded polynomial algebra, and the homomorphism $\mathcal{S}_{m_1}^\wedge$ is an $F_2$-epimorphism, it follows that

$$(A\mathcal{P}_6)_m \cong (A\mathcal{P}_6)_m \bigoplus (\operatorname{Ker}\mathcal{S}_{m_1}^\wedge \cap (A\mathcal{P}_6^+)_m) \bigoplus \operatorname{Im}\mathcal{S}_{m_1}^\wedge.$$
Recall that, Phuc [9] demonstrated that the space $(\mathcal{A}\mathcal{P}_5)_{6(2^1−1)+6.2^1}$ is an $F_2$-vector space of dimension 730 with a basis consisting of all the classes represented by the monomials $a_k$, $1 \leq k \leq 730$. Consequently, $|D_6^\otimes(6(2^1−1)+6.2^1)| = 730$. An easy computation shows that

$$\left| \bigcup_{j=1}^{6} T_j(D_6^\otimes(6(2^1−1)+6.2^1)) \right| = 2865,$$

and the set

$$\{b_t : b_t \in \bigcup_{j=1}^{6} T_j(a_t), 1 \leq t \leq 730, 1 \leq i \leq 2865\}$$

is a minimal set of generators for $\mathcal{A}$-modules $\mathcal{P}_6^0$ in degree $6(2^1−1)+6.2^1$. This implies $(\mathcal{A}\mathcal{P}_5)_{6(2^1−1)+6.2^1}$ has dimension 2865. Part (ii) is proved.

**Remark 3.3.** We set $\mathcal{X}_{(6,t)} = \{1 = (i_1, i_2, \ldots, i_t) : 1 \leq i_1 < \ldots < i_t \leq 6\}$, with $1 \leq t \leq 6$. For $H \in \mathcal{X}_{(6,t)}$, consider the homomorphism $\varphi_H : \mathcal{P}_6 \rightarrow \mathcal{P}_6$ of algebras by substituting $\varphi_H(x_t) = x_{i_t}$ with $1 \leq i_t \leq t$. Then, $\varphi_H$ is an $\mathcal{A}$-modules monomorphism. From the result in [5], one has

$$\mathcal{A}\mathcal{P}_6^0 = \bigoplus_{1 \leq t \leq 5} \bigoplus_{H \in \mathcal{X}_{(6,t)}} (Q\varphi_H(\mathcal{P}_6^+)),$$

where $Q\varphi_H(\mathcal{P}_6^+) = F_2 \otimes \mathcal{A} \varphi_H(\mathcal{P}_6^+)$. Then, $\dim(Q\varphi_H(\mathcal{P}_6^+)) = \dim(\mathcal{A}\mathcal{P}_6^+) = \dim(\mathcal{A}\mathcal{P}_6^+)$, and $|\mathcal{X}_{(6,t)}| = \binom{6}{t}$. Combining with the results in Wood [29], one gets

$$\dim(\mathcal{A}\mathcal{P}_6^+)_n = \sum_{\mu(n) \leq t \leq 6} \binom{6}{t} \dim(\mathcal{A}\mathcal{P}_6^+)_n.$$

Since $\mu(6(2^1−1)+6.2^1) = 2$, the vector space $(\mathcal{A}\mathcal{P}_6)_{6(2^1−1)+6.2^1}$ is trivial. Using the results in Peterson [7], Kameko [4], Sum [18], and Phuc [9], we have

$$\dim(\mathcal{A}\mathcal{P}_6^+)_{6(2^1−1)+6.2^1} = \begin{cases} 3, & \text{if } t = 2, \\ 12, & \text{if } t = 3, \\ 60, & \text{if } t = 4, \\ 280, & \text{if } t = 5. \end{cases}$$

From the above results, we get

$$\dim(\mathcal{A}\mathcal{P}_6^+)_{6(2^1−1)+6.2^1} = \left(\frac{6}{2}\right).3 + \left(\frac{6}{3}\right).12 + \left(\frac{6}{4}\right).60 + \left(\frac{6}{5}\right).280 = 2865.$$

Next, we prove Part (iii) of the theorem by explicitly determining the admissible monomial basis of the $F_2$-vector space $(\text{Ker}S_6^\otimes \cap (\mathcal{A}\mathcal{P}_6^+)_{m_1})$.

Denote $D_6^\otimes(\omega) := D_6^\otimes(m) \cap \mathcal{P}_6(\omega)$. It is easy to see that $D_6^\otimes(m) = \bigcup_{\deg \omega = m} D_6^\otimes(\omega)$. Put

$$Q\mathcal{P}_6^\omega := \{(x) \in \mathcal{A}\mathcal{P}_6 : \omega(x) = \omega, \text{ and } x \in D_6^\otimes(\omega)\}.$$

It is simple to verify that the map $\mathcal{A}\mathcal{P}_6(\omega) \rightarrow Q\mathcal{P}_6^\omega$, $[x]_\omega \rightarrow [x]$ is an isomorphism of $F_2$-vector spaces. Hence, $Q\mathcal{P}_6^\omega \subset \mathcal{A}\mathcal{P}_6$ can be used to identify the vector space $\mathcal{A}\mathcal{P}_6(\omega)$. As a result of this, one obtains
(\mathcal{AP}_6)_m = \bigoplus_{\deg \omega = m} Q^0_{\mathcal{P}_6}(\omega) \approx \bigoplus_{\deg \omega = m} \mathcal{AP}_6(\omega).

From this, it follows that \((\mathcal{AP}_6^+)_m = \bigoplus_{\deg \omega = m} \mathcal{AP}_6^+(\omega).\)

Assume that \(x\) belongs to \((\mathcal{D}_6^\omega(6(2^1 - 1) + 6.2^1) \cap \mathcal{P}_6^+))\) such that \([x]\) does not an element of \(\text{Im}\tilde{S}_{18}^\omega\). It is easy to check that \(y = x_1^6 x_2^3\) is the minimal spike of degree eighteen in \(\mathcal{P}_6\) and \(\omega(y) = (2, 2, 1, 1)\). Since \(x\) is an admissible monomial, by Theorem 2.8 it shows that \(\omega_1(x) \geq \omega_1(y)\). Moreover, \(\deg(x)\) is an even number, it implies \(\omega_1(x) = 2\), or \(\omega_1(x) = 4\), or \(\omega_1(x) = 6\).

If \(\omega_1(x) = 2\), then \(x = x_1 x_1 u^2\) with \(u\) a monomial of degree eight in \(\mathcal{P}_6\) and \(1 \leq i < j \leq 6\). By Theorem 2.6, \(u\) is an admissible monomial. Moreover, using Theorem 2.8, we also have \(\omega_1(u) \geq \omega_2(y) = 2\). Hence, \(\omega_1(u) = 6\), or \(\omega_1(u) = 4\), or \(\omega_1(u) = 2\).

If \(\omega_1(u) = 6\), then \(\omega_2(x) = 6\). Using the results in Sum [17], we see that \(x\) is strictly inadmissible. And therefore, \(x\) is inadmissible. This contradicts the fact that \(x\) belongs to \(\mathcal{D}_6^\omega(18)\). In case of \(\omega_1(u) = 4\), then \(u = x_m x_r x_\ell x_i v^2\) with \(1 \leq m < r < s < t \leq 6\), where \(v \in \mathcal{D}_6^\omega(2)\), and \(\omega(v) = (2, 0)\). From this, we obtain \(\omega(x) = \tilde{\omega}_3\).

If \(\omega_1(u) = 2\), then \(u = x_m x_m w^2\) with \(1 \leq n < m \leq 6\), where \(w \in \mathcal{D}_6^\omega(3)\). Since \(w \in \mathcal{D}_6^\omega(3)\), yields \(\omega(w) = (3, 0)\) or \(\omega(w) = (1, 1)\). So, either \(\omega(x) = \tilde{\omega}_1\), or \(\omega(x) = \tilde{\omega}_2\).

If \(\omega_1(x) = 4\), then \(x = x_i x_k x_\ell x_i f^2\), where \(f\) is an admissible monomial of degree seven in \(\mathcal{P}_6\) and \(1 \leq i < j < k \leq 6\). An easy computation, using the result in [21], we obtain the set

\[
\{[x_i x_k x_\ell x_i^2 x_m^2] : 1 \leq i, j, k, \ell, m \leq 6, t < m\} \cup \{[q] : q \in \bigcup_{m=1}^6 \mathcal{F}_m(\mathcal{D}_6^\omega(7))\}
\]

is a basis of \(F_2\)-vector space \((\mathcal{AP}_6)_7\). Since \(f \in \mathcal{D}_6^\omega(7)\), it yields that \(\omega(f) = (5, 1)\) or \(\omega(f) = (3, 2)\), or \(\omega(f) = (1, 3)\), or \(\omega(f) = (1, 1, 1)\). So, \(\omega(x) = \tilde{\omega}_4\) for \(4 \leq i \leq 7\).

If \(\omega_1(x) = 6\), then \(x = \prod_{i=1}^9 x_i g^2\) with \(g\) an admissible monomial of degree six in \(\mathcal{P}_6\). By Theorem 2.6, \(g\) is an admissible monomial, and therefore \([g] \neq 0\). Thus, we have \([g] = \tilde{S}_{18}^\omega([x]) \neq 0\). This contradicts the fact that \([x]\) belongs to \(\text{Ker}\tilde{S}_{18}^\omega\).

From the above results, we get \(\omega(x) = \tilde{\omega}_4\), for all \(1 \leq i \leq 7\). Furthermore, one gets

\[
\text{Ker}\tilde{S}_{18}^\omega \cap (\mathcal{AP}_6^+)_7 \approx \bigoplus_{i=1}^7 \mathcal{AP}_6^+(\tilde{\omega}_i).
\]

We will denote by \(\mathcal{D}_d^\omega(\omega)\) the set of all admissible monomials in \(\mathcal{P}_d^+\). In order to explicitly determine the space \(\text{Ker}\tilde{S}_{18}^\omega \cap (\mathcal{AP}_6^+)_7\), we show all admissible monomials in \(\mathcal{P}_6^+(\tilde{\omega}_1)\), for all \(1 \leq i \leq 7\). The proof is divided into the following steps.

Step 1. Consider the weight vector \(\omega = \tilde{\omega}_1\). Assume that \(x\) is an admissible monomial in \(\mathcal{P}_6\) such that \(\omega(x) = \tilde{\omega}_1\), then \(x = x_i x_j y^2\), where \(y \in \mathcal{D}_6^\omega(1, 1, 1)\), and \(1 \leq i < j \leq 6\). We set

\[
M_{ij}^1 := \{x_i x_j y^2 : \omega(y) = (1, 1, 1), 1 \leq i < j \leq 6\} \cap \mathcal{P}_6^+.
\]

It is easy to see that \(\text{Span}(M_{ij}^1) = \mathcal{P}_6^+(\tilde{\omega}_1)\), and if \(u\) is an element in \(M_{ij}^1\), then \(u\) has the form \(x_i x_j x_k x_\ell x_m\) with \(k < \ell < m\), where \((i, j, k, \ell, m, n)\) is a permutation of \((1, 2, 3, 4, 5, 6)\).

Clearly, the monomials \(x_i^2 x_j x_k x_\ell x_m\) are inadmissible (more precisely by \(S_1^d\)), where \((i, j, k, \ell, m)\) is an arbitrary permutation of \((2, 3, 4, 5, 6)\). Furthermore, for \(1 < i < j < \ell\), one has

\[
x_i^2 x_j x_k x_\ell x_m = S_1^d(x_i x_j x_k x_\ell x_m) + S_1^d(x_i^2 x_j x_k x_\ell x_m) + \text{smaller than}.
\]

From this, the monomials \(x_i^2 x_j x_k x_\ell x_m\) are inadmissible.
As may be seen from the preceding findings, \( \mathcal{P}_6^+ (\bar{\omega}_1) \) is generated by 9 elements \( c_{i,1} \), with \( 1 \leq i \leq 9 \), as follows:

1. \( x_1^2 x_2 x_3 x_4 x_5 x_6 \)
2. \( x_1^2 x_2 x_4 x_5 x_6 \)
3. \( x_1^2 x_2 x_3 x_4^2 x_6 \)
4. \( x_1^2 x_2 x_3 x_4^3 x_6 \)
5. \( x_1^2 x_2 x_3 x_4^4 x_6 \)
6. \( x_1^2 x_2 x_3 x_4^5 x_6 \)
7. \( x_1^2 x_2 x_3 x_4^6 x_6 \)
8. \( x_1^2 x_2 x_3 x_4^7 x_6 \)
9. \( x_1^2 x_2 x_3 x_4^8 x_6 \)

We then prove the set \( \{ c_{i,1} : 1 \leq i \leq 9 \} \) is linearly independent in \( \mathcal{A}\mathcal{P}_6 (\bar{\omega}_1) \). Denote \( \mathcal{N}_d = \{ (i;1) : 1 = (i_1, i_2, \ldots, i_d), 1 \leq i < i_1 < \ldots < i_t \leq d, 0 \leq t < d \} \), where by convention \( I = 0 \) if \( t = 0 \). Write \( t = \ell (1) \) for the length of \( I \).

For each \( (i;1) \in \mathcal{N}_6 \), consider the homomorphism \( \Omega_{(i,1)} : \mathcal{P}_6 \to \mathcal{P}_5 \) which is defined as:

\[
\Omega_{(i,1)} (x_k) = \begin{cases} 
\sum_{s \in I} x_{s-1}, & \text{if } k = i, \\
0, & \text{if } 1 \leq k \leq i - 1, \\
x_k, & \text{if } i < k \leq 6.
\end{cases}
\]

We use them to prove that a given set of monomials is the set of admissible monomials in \( \mathcal{P}_6 \) by showing that they are linearly independent in \( \mathcal{A}\mathcal{P}_6 \).

Assume that there is a linear relation

\[
S_1 = \sum_{1 \leq i \leq 9} \gamma_i c_{i,1} = 0, \quad \text{where } \gamma_i \in \mathcal{F}_2.
\]

From a result in [9], one has \( \dim \mathcal{A}\mathcal{P}_6^+ (\bar{\omega}_1) = 25 \), with a basis consisting of all the classes represented by the monomials \( a_{k,1} \), \( 1 \leq k \leq 25 \), which is determined as follows:

1. \( x_1^2 x_2 x_3 x_4 x_5^2 \)
2. \( x_1^2 x_2 x_3 x_4^2 x_5 \)
3. \( x_1^2 x_2 x_3^2 x_4 x_5 \)
4. \( x_1^2 x_2 x_3 x_4^3 x_5 \)
5. \( x_1^2 x_2 x_3 x_4^4 x_5 \)
6. \( x_1^2 x_2 x_3 x_4^5 x_5 \)
7. \( x_1^2 x_2 x_3 x_4^6 x_5 \)
8. \( x_1^2 x_2 x_3 x_4^7 x_5 \)
9. \( x_1^2 x_2 x_3 x_4^8 x_5 \)
10. \( x_1^2 x_2 x_3 x_4^9 x_5 \)
11. \( x_1^2 x_2 x_3 x_4^{10} x_5 \)
12. \( x_1^2 x_2 x_3 x_4^{11} x_5 \)
13. \( x_1^2 x_2 x_3 x_4^{12} x_5 \)
14. \( x_1^2 x_2 x_3 x_4^{13} x_5 \)
15. \( x_1^2 x_2 x_3 x_4^{14} x_5 \)
16. \( x_1^2 x_2 x_3 x_4^{15} x_5 \)
17. \( x_1^2 x_2 x_3 x_4^{16} x_5 \)
18. \( x_1^2 x_2 x_3 x_4^{17} x_5 \)
19. \( x_1^2 x_2 x_3 x_4^{18} x_5 \)
20. \( x_1^2 x_2 x_3 x_4^{19} x_5 \)
21. \( x_1^2 x_2 x_3 x_4^{20} x_5 \)
22. \( x_1^2 x_2 x_3 x_4^{21} x_5 \)
23. \( x_1^2 x_2 x_3 x_4^{22} x_5 \)
24. \( x_1^2 x_2 x_3 x_4^{23} x_5 \)
25. \( x_1^2 x_2 x_3 x_4^{24} x_5 \).

Acting the homomorphism \( \Omega_{(5,6)} \) on both sides of (3.1), and explicitly computing \( \Omega_{(5,6)} (S_1) \) in terms of \( a_{k,1} \), \( 1 \leq k \leq 25 \) in \( \mathcal{P}_5 (\mod (\mathcal{A}^+ \mathcal{P}_5)) \), we obtain

\[
\Omega_{(5,6)} (S_1) = \bar{\omega}_1 \gamma_1 a_1 + (\gamma_2 + \gamma_3) a_6 + \gamma_4 a_{3} + (\gamma_5 + \gamma_6) a_7 + (\gamma_7 + \gamma_8) a_8 + \gamma_9 a_{23} = \bar{\omega}_1 0.
\]

From the above equation, we can derive that \( \gamma_1 = \gamma_4 = \gamma_9 = 0 \).

Similarly, the homomorphism \( \Omega_{(5,4)} \) sends the relation (3.1) to the following relation in \( \mathcal{P}_5 (\mod (\mathcal{A}^+ \mathcal{P}_5)) \)

\[
\Omega_{(5,4)} (S_1) = \bar{\omega}_1 \gamma_2 a_9 + \gamma_3 a_{2} + \gamma_5 a_{11} + \gamma_6 a_{4} + \gamma_7 a_{22} + \gamma_8 a_{25} = \bar{\omega}_1 0.
\]

From the above results, one gets \( \gamma_i = 0 \), for all \( 1 \leq i \leq 9 \).

In summary, the set \( \{ c_{i,1} : 1 \leq i \leq 9 \} \) is a basis of the \( \mathcal{F}_2 \)-vector space \( \mathcal{A}\mathcal{P}_6^+ (\bar{\omega}_1) \). Consequently, \( |\mathcal{D}_6^+ (\bar{\omega}_1)| = 9 \).

Step 2. Consider the weight vector \( \omega = (4,1,1,1) \). Let us denote by

\[
M_5^2 := \{ x_1 x_2 x_3 x_4 x_5 : \omega(z) = (1,1,1), \ 1 \leq i < j < k < \ell \leq 6 \} \subset \mathcal{P}_6^+.
\]

It is easy to see that \( \mathcal{P}_6^+ (\bar{\omega}_4) = \text{Span} \{ M_5^2 \} \), and if \( v \) is an element in \( M_5^2 \), then \( v \) has the form:

\[
x_1 x_2 x_3 x_4 x_5 x_6, \ x_1 x_3 x_2 x_4 x_5 x_6, \ x_1 x_4 x_2 x_3 x_5 x_6, \ x_1 x_5 x_2 x_3 x_4 x_6, \ x_1 x_6 x_2 x_3 x_4 x_5.
\]

where \( (i,j,k,\ell,m,n) \) is a permutation of \( (1,2,3,4,5,6) \).
By direct calculations, using Theorem 2.6, we remove the inadmissible monomials in \( \mathcal{N}_6^0 \), and we see that \( \mathcal{P}_6^+ (\tilde{\omega}_4) \) is generated by 50 elements \( c_{i \mu i} \), \( 1 \leq i \leq 50 \), as follows:

1. \( x_1^2 x_3 x_4 x_6^2 \), 2. \( x_1^3 x_2 x_4 x_5 x_6^2 \), 3. \( x_1^2 x_3 x_4 x_5^2 x_6^6 \), 4. \( x_1 x_2 x_3 x_4 x_6^4 x_8 \), 5. \( x_1 x_2 x_3 x_4 x_5 x_6^9 \), 6. \( x_1 x_3 x_4 x_5 x_6^2 \), 7. \( x_1 x_3 x_4 x_5^2 x_6^8 \), 8. \( x_1 x_3 x_4 x_6^4 x_8 \), 9. \( x_1 x_3 x_4 x_5^2 x_6^8 \), 10. \( x_1 x_3 x_4 x_5 x_6^2 \), 11. \( x_1 x_3 x_4 x_5^2 x_6^8 \), 12. \( x_1 x_3 x_4 x_5^2 x_6^8 \), 13. \( x_1 x_3 x_4 x_5^2 x_6^8 \), 14. \( x_1 x_3 x_4 x_5^2 x_6^8 \), 15. \( x_1 x_3 x_4 x_5^2 x_6^8 \),

We now show that the set \( \{ c_{i \mu i} : 1 \leq i \leq 50 \} \) is linearly independent in \( A \mathcal{P}_6^+ (\tilde{\omega}_4) \). Assume that there is a linear relation

\[
S_2 = \sum_{1 \leq i \leq 50} \gamma_i c_{i \mu i} = 0, \quad \text{where} \quad \gamma_i \in F_2, i \in N_{50}.
\]

Recall that \( \dim A \mathcal{P}_5^+ (\tilde{\omega}_4) = 40 \), with a basis consisting of all the classes represented by the monomials \( \alpha_k, 26 \leq k \leq 65 \), which are determined as follows:

26. \( x_1 x_2 x_3 x_4 x_5^{14} \), 27. \( x_1 x_2 x_3 x_4 x_5^{14} x_6 \), 28. \( x_1 x_2 x_3 x_4 x_5^{14} x_6^3 \), 29. \( x_1 x_2 x_3 x_4 x_5^{14} x_6^5 \), 30. \( x_1 x_2 x_3 x_4 x_5^{14} x_6^9 \),

Acting the homomorphism \( \Omega_{(4,5)} \) on both sides of (3.2), and explicitly computing \( \Omega_{(4,5)} (S_2) \) in terms of \( \alpha_k, 26 \leq k \leq 65 \) in \( \mathcal{P}_5 (\text{mod} \ A^+ \mathcal{P}_5) \), we obtain \( \gamma_i = 0 \), for all \( i \in L = \{11, 12, 13, 14\} \). Therefore, the relation (3.2) becomes

\[
\mathcal{U} = \sum_{i \in N_{50} \setminus L} \gamma_i c_{i \mu i} \equiv 0, \quad (3.3)
\]

By the same calculation as above, we explicitly compute \( \Omega_{(j,1)} (S_j), (j,1) \in N_{60} \), in terms of \( \alpha_k, 26 \leq k \leq 65 \) in \( \mathcal{P}_5 (\text{mod} \ A^+ \mathcal{P}_5) \), and from the relation \( \varphi_{(j,1)} (\mathcal{U}) \equiv 0 \), with \( \ell (1) = 1 \), we get \( \gamma_i = 0 \) for all \( i \in N_{50} \setminus L \). That means, the set \( \{ c_{i \mu i} : 1 \leq i \leq 50 \} \) is a basis of the \( F_2 \)-vector space \( A \mathcal{P}_6^+ (\tilde{\omega}_4) \). Consequently, \( |\mathcal{D}_6^+ (\tilde{\omega}_4)| = 50 \).

Step 3. Consider the weight vector \( \omega = \tilde{\omega}_1 \), with \( i \in J = \{2, 3, 5, 6, 7\} \). Let us denote by \( \mathcal{D}_6^+ (\omega) := \bigcup_{i \in J} \mathcal{D}_6^+ (\tilde{\omega}_i) \). For each \( i \in J \), by the same method as in the previous section, we explicitly determine the \( F_2 \)-vector spaces \( A \mathcal{P}_6^+ (\tilde{\omega}_i) \). By direct calculations, using Theorem 2.6, one gets \( |\mathcal{D}_6^+ (\omega)| = \sum_{i \in J} \dim A \mathcal{P}_6^+ (\tilde{\omega}_i) = 827 \). Hence, one gets \( \dim (\text{Ker} \mathcal{S}_{m_1} (A \mathcal{P}_6^+ (\tilde{\omega}_1))) = 886 \). Part (iii) has been established. So, the theorem is proved.
From the above results, we obtain the following corollary.

**Corollary 3.4.** There exist exactly 3941 admissible monomials in \(P_6\) of degree \(6(2^1 - 1) + 6.2^r\). Consequently, \(|D_6^5(6(2^1 - 1) + 6.2^r)| = 3941.\)

Next, we consider the degree \(m_r := 6(2^r - 1) + 6.2^r\), for any \(r \geq 2\). Since the homomorphism \(S^6_m\) : \((A^6)^m \rightarrow (A^6)^m\) is an \(F_2\)-epimorphism, it shows that

\[
\text{dim \(\operatorname{Im} S^6_{m_2}\)} = \left\{ \prod_{i=1}^{6} x_i x^2 : x \in D_6^5(18) \right\} = 3941.
\]

Consider the homomorphism \(\Gamma : P_6 \rightarrow P_6\) is an \(F_2\)-homomorphism determined by \(\Gamma(x) = \prod_{i=1}^{6} x_i x^2\), for \(x \in P_6\). Thus, we have the following theorem.

**Theorem 3.5.** The following statements are true.

(i) \(\text{Im} S^6_{m_2}\) is isomorphic to a subspace of \((A^6)^m\) generated by all the classes represented by the admissible monomials of the form \(\Gamma(u)\) for every \(u \in D_6^5(18)\). Consequently, \(\text{dim} \text{Im} S^6_{m_2} = 3941\).

(ii) The set \(\{ d_i : d_i \in \bigcup_{i=1}^{r} s_j(D_5^6(42)), 1 \leq i \leq 13020 \}\) is a basis of the \(F_2\)-vector space \((A^6)^m\). This implies that \((A^6)^m\) has dimension 13020.

**Proof.** The proof of Part (i) of the above theorem is straightforward. It occurs as a direct result of Corollary 3.4. Observe, from the result in Corollary 3.4, it shows that

\[
\text{dim \(\text{Im} S^6_{m_2}\)} = \left\{ \prod_{i=1}^{6} x_i x^2 : x \in D_6^5(18) \right\} = 3941.
\]

Consider the degree \(m_r := 6(2^r - 1) + 6.2^r\), for \(r = 2\). By using the MAGMA computer algebra system, Phuc showed in [10] that the \(F_2\)-vector space \((A^6)^m\) has 2520-dimensional (see [10], pp.4), where \(\text{dim}(A^6)^m = 700\), and \(\text{dim}(A^6)^m = 1820\). Assume that the set \(\{ e_i \in (A^6)^m : 1 \leq i \leq 2520 \}\) is a minimal set of generators for \(A\)-modules \(P_6\) in degree forty-two.

That means, \(D_6^5(42) = \{ e_i \in (A^6)^m : 1 \leq i \leq 2520 \}\). It is easy to check that \(\bigcup_{i=1}^{6} s_j(D_5^6(42))\) is a minimal set of generators for \(A\)-module \(P_{6}\) in degree forty-two. This implies that \((A^6)^m\) has dimension 13020. The second part has been established.

**Remark 3.6.** By the same argument as the previous part, we set

\[\mathcal{H}_{(d,1)} = \{ I = (i_1, i_2, \ldots, i_t) : 1 \leq i_1 < \ldots < i_t \leq d \}, \quad \text{with} \quad 1 \leq t < d.\]

For each \(H \in \mathcal{H}_{(d,1)}\), consider the homomorphism \(F_H : P_t \rightarrow P_{d}\) of algebras by substituting \(F_H(x_t) = x_t\), with \(1 \leq t \leq t\). Then, \(F_H\) is an \(A\)-modules monomorphism. From the result in [3], we have a direct summand decomposition of the \(F_2\)-vector subspaces:

\[
A^6 = \bigoplus_{1 \leq t \leq d \leq \mathcal{H}_{(d,1)} \cap \mathcal{H}_{(d,1)}} (Q F_H(P_t^+) ),
\]

where \(Q F_H(P_t^+) = F_2 \otimes_A F_H(P_t^+)\). Hence, \(\text{dim}(Q F_H(P_t^+)) = \text{dim}(A^6)\), and \(|\mathcal{H}_{(d,1)}| = \binom{d}{t}\). Combining with the results in Wood [29], one gets
\[
\dim(\mathcal{A}\mathcal{P}_d^0)_m = \sum_{\mu(m) \leq t \leq d} \binom{d}{t} \dim(\mathcal{A}\mathcal{P}_d^+)_m.
\]

Since \(\mu(6^2 - 1 + 6^2) = 4\), then for \(t < 4\) the vector space \([\mathcal{A}\mathcal{P}_4]_{6^2 - 1 + 6^2}\) is trivial. On the other hand, using the result in Sum [18] we have \(\dim(\mathcal{A}\mathcal{P}_4^+)_42 = 140\).

From the above results, one obtains
\[
\dim(\mathcal{A}\mathcal{P}_6^0)_42 = \binom{6}{4}. \dim(\mathcal{A}\mathcal{P}_4^+)_42 + \binom{6}{5}. \dim(\mathcal{A}\mathcal{P}_5^+)_42 = 13020.
\]

The theorem has been established.

Theorem 3.7. Suppose that \(u\) belongs to \((\mathcal{D}^\circ_6(42) \cap \mathcal{P}_6^+\) such that \(\tilde{S}_{42}^\circ([u])\) is not an element of \(\text{Im}\tilde{S}_{42}^\circ\). Then \(\omega(u) = \omega_{[i]}\) for \(1 \leq i \leq 7\). Furthermore, we have an isomorphism of the \(F_2\)-vector spaces:

\[
(Ker\tilde{S}_{42}^\circ \cap (\mathcal{A}\mathcal{P}_6^+)_42) \cong \bigoplus_{i=1}^{7} \mathcal{A}\mathcal{P}_6^+\left(\tilde{\omega}_{[i]}\right).
\]

Proof. Let \(\omega\) be a weight vector of degree \(m\), we set
\[
Q^\omega_d := \text{Span}\{[u] \in \mathcal{A}\mathcal{P}_d : \omega(u) = \omega, \text{ and } u \in \mathcal{D}^\circ_6(\omega)\}.
\]

By the same arguments as in the proof of the previous theorem, it is easy to check that the map \(\mathcal{A}\mathcal{P}_d(\omega) \to Q^\omega_d, [u] \omega \to [u]\) is an isomorphism of \(F_2\)-vector spaces. Thus, \(Q^\omega_d \subset \mathcal{A}\mathcal{P}_d\) can be used to identify the vector space \(\mathcal{A}\mathcal{P}_d(\omega)\). As a result of this, one gets
\[
(\mathcal{A}\mathcal{P}_d)_m = \bigoplus_{\text{deg }\omega = m} Q^\omega_d \cong \bigoplus_{\text{deg }\omega = m} \mathcal{A}\mathcal{P}_d(\omega).
\]

Hence, it follows that \((\mathcal{A}\mathcal{P}_d^+)_42 = \bigoplus_{\text{deg }\omega = 42} \mathcal{A}\mathcal{P}_d^+\omega\).

Assume that \(u\) is an admissible monomial of degree forty-two in \(\mathcal{P}_6\) such that \([u]\) belongs to \(\text{Ker}\tilde{S}_{42}^\circ\). Observe that \(v = x_1^3x_2^3x_3^3x_4^3\) is the minimal spike of degree forty-two in \(\mathcal{P}_6\), and \(\omega(v) = \tilde{\omega}_{[1]}\). Using Theorem 2.8, one obtains \(\omega_1(u) \geq \omega_1(v) = 4\). Since the degree of \(u\) is even, one gets either \(\omega_1(u) = 4\), or \(\omega_1(u) = 6\).

If \(\omega_1(u) = 4\) then \(u = X_{[i,j]}w^2\), with \(w\) a monomial of degree nineteen in \(\mathcal{P}_6\), and \(1 \leq i < j \leq 6\). By Theorem 2.6, it shows that \(w\) is admissible. Clearly, \(v = x_1^3x_2^2x_3^3x_4^3\) is the minimal spike of degree nineteen in \(\mathcal{P}_6\), and \(\omega(y) = (3, 2, 1, 1)\). Using Theorem 2.8, we have \(\omega_1(w) \geq 3\). Since the degree of \(w\) is even, one gets either \(\omega_1(w) = 3\), or \(\omega_1(w) = 5\).

Case 1. If \(\omega_1(w) = 3\) then \(w = x_1x_2x_3f^2\), where \(f\) is an admissible monomial of degree eight in \(\mathcal{P}_6\), and \(1 \leq k < \ell < t \leq 6\). Since \(f \in \mathcal{D}^\circ_6(8)\), and using the result in [5], one has \(\omega(f)\) belongs to \{\((2, 1, 1), (2, 3), (4, 2), (6, 1)\)\).

Remarkably, if \(w\) is a monomial in \(\mathcal{P}_6\) such that \(\omega(w) = (3, 6, 1)\) then \(w\) is strictly inadmissible (see Sum [17], Prop. 4.3). Hence, \(w\) is inadmissible. Thus, \(\omega(w)\) belongs to \{\((3, 2, 1, 1), (3, 2, 3), (3, 4, 2)\)\}. So \(\omega(u) = \tilde{\omega}_{[i]}\) for \(i = 1, 2, 3\).
Case 2. If \( \omega_1(w) = 5 \) then \( w = X_{(k)}^2 \), with \( g \) a monomial of degree seven in \( \mathcal{P}_6 \), \( 1 \leq k \leq 6 \). Using the results in the previous section, we see that if \( g \) belongs to \( \mathcal{D}_6^{\otimes}(7) \), one has \( \omega(g) \) belongs to \( \{(5, 1), (3, 2), (1, 3), (1, 1, 1)\} \). Then, \( \omega(w) \) belongs to \( \{(5, 5, 1), (3, 5, 2), (5, 1, 3), (5, 1, 1, 1)\} \). So \( \omega(u) = \delta(i) \) for \( i = 4, 5, 6, 7 \).

If \( \omega_1(u) = 6 \) then \( x = X_9h^2 \), with \( h \) a monomial of degree eighteen in \( \mathcal{P}_6 \). Since \( u \) is admissible, by Theorem 2.6, it shows that \( h \) is also admissible, and \( [h] \neq 0 \). From this, it implies \( [h] = \text{Ker}^{\otimes} _{(2^r)}(u) \neq 0 \). This contradicts the fact that \( [x] \in \text{Ker}^{\otimes} _{(2^r)} \).

In summary, \( \omega(u) = \delta(i) \) for all \( 1 \leq i \leq 7 \). From the above results, one obtains

\[
\text{Ker}^{\otimes} _{(2^r)} \cap (\mathcal{A} \mathcal{P} _{6}^+)_{42} \approx \bigoplus_{i=1}^{7} \mathcal{A} \mathcal{P} _{6}^+(\delta(i)).
\]

The theorem has been established.

For each integer \( r > 2 \), we consider the degree \( m_r = 6(2^r - 1) + 6.2^r \). Let \( m \) be an arbitrary non-negative integer, and let \( \xi(m) \) be the greatest integer \( v \) such that \( m \) is divisible by \( 2^v \). That means \( m = 2^v k \), with \( k \) an odd integer. Put

\[
\lambda(d, m) = \max(0, d - \alpha(d + m) - \xi(d + m)).
\]

Then, the map

\[
(Sq_s^0)^{s-t} : (\mathcal{A} \mathcal{P} _{d})_{d(2^r - 1) + 2^r m} \rightarrow (\mathcal{A} \mathcal{P} _{d})_{d(2^r - 1) + 2^r m}
\]

is an isomorphism of \( \text{GL}(d; F_2) \)-modules for every \( s \geq t \) if and only if \( t \geq \lambda(d, m) \) (see Tin-Sum [22]).

For \( d = m = 6 \), \( m_r = 6(2^r - 1) + 6.2^r \), then \( \alpha(d + m) = \alpha(12) = 2 \), and \( \xi(d + m) = \xi(2^2.3) = 2 \). Then \( \lambda(n, d) = 2 \). Using the above result, we have an isomorphism of \( F_2 \)-vector space

\[
(\mathcal{A} \mathcal{P} _{6})_{6(2^r - 1) + 2^r 6} \cong (\mathcal{A} \mathcal{P} _{6})_{m_2}
\]

for all \( r \geq 2 \). Hence, the set \( \{ [x] : x \in \Gamma_{r}^{-2}(\mathcal{D}^{\otimes}_6(m_2)) \} \) is a basis of the \( F_2 \)-vector space \( \mathcal{A} \mathcal{P} _{6} \) in degree \( 6(2^r - 1) + 6.2^r \) for any integer \( r \geq 2 \). So, we obtain the following theorem.

**Theorem 3.8.** The set \( \{ [x] : x \in \Gamma_{r}^{-2}(\mathcal{D}^{\otimes}_6(m_2)) \} \) is a basis of the \( F_2 \)-vector space \( \mathcal{A} \mathcal{P} _{6} \) in degree \( 6(2^r - 1) + 6.2^r \), for any \( r \geq 2 \).

**Remark.** It could be seen from the work of Singer the meaning and necessity of the hit problem. In [14], Singer defined the algebraic transfer, which is a homomorphism

\[
\psi_d : F_2^{\otimes} \text{GL}(d; F_2) \text{PH}_s((R^{p^{\infty}})^d) \rightarrow \text{Ext}^{d,d+s}_d(F_2, F_2),
\]

where \( F_2^{\otimes} \text{GL}(d; F_2) \text{PH}_m((R^{p^{\infty}})^d) \) is dual to \( (\mathcal{A} \mathcal{P} _{d})^{\otimes} \text{GL}(d; F_2) \), and \( \text{Ext}^{d,d+s}_d(F_2, F_2) \) is the cohomology groups of the Steenrod algebra.

Singer has indicated the importance of the algebraic transfer by showing that \( \psi_d \) is an isomorphism with \( d = 1, 2 \) and at some other degrees with \( d = 3, 4 \), but he also disproved this for \( \psi_5 \) at degree 9, and then gave the following conjecture.

**Conjecture 3.9.** The algebraic transfer \( \psi_d \) is a monomorphism for any \( d \geq 0 \).

Boardman [1] then corroborated this by demonstrating that \( \psi_3 \) is likewise an isomorphism using the modular representation theory of linear groups. Singer’s conjecture, however, remains open for \( d \geq 4 \).

In [19] and [24], we based on the results for the hit problem to verify Singer’s conjecture is true for \( n = 5 \) and the generic degrees \( d_n = 5(2^4 - 1) + 2^4 m \), where \( m \in \{1, 2, 3\} \). Continuing this work, using the results of the hit problem, we will investigate and validate Singer’s conjecture for the sixth algebraic...
transfer in the aforementioned degrees by combining the computations of the cohomology groups of the Steenrod algebra $\text{Ext}^i_A(\mathbb{F}_2, \mathbb{F}_2)$. Remarkably, by using the result in Tin-Sum [22] (see Theorem 3, pp. 2), we also obtain an isomorphism of $\text{GL}(6; \mathbb{F}_2)$-modules

$$ (A^p_6)^{\text{GL}(6; \mathbb{F}_2)} \cong (A^p_6)^{\text{GL}(6; \mathbb{F}_2)}_{6(2^r-1)+6.2^r} \text{ for all } r \geq 2. $$

Hence, one obtains

$$ F_2 \otimes_{\text{GL}(6; \mathbb{F}_2)} \mathbb{P}H_6(2^r-1)+6.2^r \langle (\mathbb{R}P^\infty)^6 \rangle \cong (F_2 \otimes_{\text{GL}(6; \mathbb{F}_2)} \mathbb{P}H_6(2^r-1)+6.2^r \langle (\mathbb{R}P^\infty)^6 \rangle), $$

for all $r \geq 2$.

And therefore, we need only to compute the dimension of spaces $F_2 \otimes_{\text{GL}(6; \mathbb{F}_2)} \mathbb{P}H_6(2^r-1)+6.2^r \langle (\mathbb{R}P^\infty)^6 \rangle$ for $r \leq 2$. This is an open problem.

Furthermore, Walker and Wood have recently published volumes on the hit problem and its applications to representations of general linear groups in the books [27] and [28]. This is yet another application of the hit problem that has to be investigated further in the future.

4. Conclusion

In the article, we study the hit problem for the polynomial algebra of six variables, viewed as a module over the Steenrod algebra in the generic degree $6(2^r-1)+6.2^r$ with $r$ an arbitrary positive integer, and its application to the sixth algebraic transfer of Singer. In the future, we will verify the Singer conjecture for the sixth algebraic transfer in degree $6(2^r-1)+6.2^r$, with $r$ an arbitrary positive integer, by combining the computations of the cohomology groups of the Steenrod algebra in these cases.

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