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Multiplication mappings on a new stochastic space of a sequence of fuzzy functions

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Abstract

A number of topological and geometrical properties of the weighted Gamma matrix of order r in Nakano sequence space for fuzzy functions equipped with definite pre-modular functions are defined and investigated in this paper. We begin by defining the necessary conditions for the formation of pre-modular Banach in this space. Second, we specify the conditions under which the multiplication operator defined on this pre-modular space is bounded, approximable, invertible, Fredholm, and closed on the basis of this space.

Keywords: Gamma matrix, Nakano sequence space, Fredholm mapping, multiplication mapping, approximable mapping.

2020 MSC: 46B10, 46C05, 46E30.

1. Introduction

In the study of uncertainty, probability theory, fuzzy set theory, soft sets, and rough sets have all had a significant impact. However, there are certain downsides to these hypotheses. Please see [1, 2, 7, 9, 11, 17, 19, 21, 31, 32] for additional details and real-world examples. Suppose that $\mathcal{R}$ is the set of real numbers and $\mathbb{N}_0$ is the set of nonnegative integers. Given that the proof of many fixed point theorems in a given space requires either growing the space itself or expanding the self-operator that acts on it, both of these options are viable, we have constructed the space, $\left(\Gamma_p^F(x,y)\right)_{\varpi}$, which is the domain of weighted Gamma matrix of order p in Nakano fuzzy sequence space since it is constructed by the domain of weighted Gamma matrix of order p defined in $\ell^F_{((y_m))}$, where the weighted Gamma matrix of order p, $W_p = (\gamma^p_{b\,a}(x))$ is defined as:

$$\gamma^p_{b\,a}(x) = \begin{cases} \frac{\left[{p+a-1 \atop a}\right] x_a}{\left[{p+b \atop b}\right]}, & 0 \leqslant a \leqslant b, \\ 0, & a > b, \end{cases}$$

where p is a positive integer, $x_a \in (0,\infty)$, for all $a \in \mathbb{N}_0$, and $\left[{p+a-1 \atop a}\right] = \frac{(p+a-1)!}{a!(p-1)!}$.

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In [30], Roopaei and Başar studied the Gamma spaces, including the spaces of absolutely p-summable, null, convergent, and bounded sequences. By c_0, ℓ_∞ and ℓ_q, we indicate the space of null, bounded and q-absolutely summable sequences of reals. We mark the space of all bounded, finite rank linear operators from an infinite-dimensional Banach space $\mathcal{O}$ into an infinite-dimensional Banach space $\mathcal{M}$ by $\mathcal{L}(\mathcal{O},\mathcal{M})$, and $\mathbb{I}(\mathcal{O},\mathcal{M})$ and when $\mathcal{O}=\mathcal{M}$, we put $\mathcal{L}(\mathcal{O})$ and $\mathbb{I}(\mathcal{O})$. The space of approximable and compact bounded linear operators from $\mathcal{O}$ into $\mathcal{M}$ will be represented by $\mathfrak{P}(\mathcal{O},\mathcal{M})$, and $\mathcal{C}(\mathcal{O},\mathcal{M})$, respectively. The ideal of bounded, approximable and compact operators between every two infinite-dimensional Banach spaces will be denoted by $\mathcal{L}$, $\mathfrak{P}$ and $\mathcal{C}$, respectively.

Definition 1.1 ([29])**.** An s-number is a function $s:\mathcal{L}(\mathcal{O},\mathcal{M})\to\mathcal{R}^{+\mathbb{N}_0}$ that sorts all $A\in\mathcal{L}(\mathcal{O},\mathcal{M})$ and $(s_q(A))_{q=0}^{\infty}$ satisfies the next setups:

(1) $\|A\|=s_0(A)\geqslant s_1(A)\geqslant s_2(A)\geqslant\cdots\geqslant 0$, for all $A\in\mathcal{L}(\mathcal{O},\mathcal{M})$;
(2) $s_q(ABG)\leqslant \|A\|s_q(B)\,\|G\|$, for every $G\in\mathcal{L}(\mathcal{O}_0,\mathcal{O})$, $B\in\mathcal{L}(\mathcal{O},\mathcal{M})$ and $A\in\mathcal{L}(\mathcal{M},\mathcal{M}_0)$, where $\mathcal{O}_0$ and $\mathcal{M}_0$ are arbitrary Banach spaces;
(3) $s_{k+m-1}(G_1+G_2)\leqslant s_k(G_1)+s_m(G_2)$, for every $G_1,G_2\in\mathcal{L}(\mathcal{O},\mathcal{M})$ and $k,m\in\mathbb{N}_0$;
(4) if $A\in\mathcal{L}(\mathcal{O},\mathcal{M})$ and $\zeta\in\mathcal{R}$, then $s_q(\zeta A)=|\zeta|s_q(A)$;
(5) suppose $\text{rank}(A)\leqslant q$, then $s_q(A)=0$, for all $A\in\mathcal{L}(\mathcal{O},\mathcal{M})$;
(6) $s_{p\geqslant q}(I_q)=0$ or $s_{p<q}(I_q)=1$, where I_q marks the unit operator on the q-dimensional Hilbert space ℓ_2^q.

Some examples of s-numbers are as

(a) the p-th approximation number defined as $\alpha_p(T)=\inf\{\|T-V\|:V\in\mathcal{L}(\mathcal{O},\mathcal{M})\text{ and rank}(V)\leqslant p\}$;
(b) the p-th Kolmogorov number defined as $d_p(T)=\inf_{\dim(B)\leqslant p}\sup_{\|i\|\leqslant 1}\inf_{k\in B}\|Ti-k\|$.

Notations 1.2 ([5])**.** Assume $\mathcal{E}$ is a linear space of sequences of real numbers.

$$\mathcal{L}^s_{\mathcal{E}}:=\left\{\mathcal{L}^s_{\mathcal{E}}(\mathcal{O},\mathcal{M})\right\},\text{ where }\mathcal{L}^s_{\mathcal{E}}(\mathcal{O},\mathcal{M}):=\left\{B\in\mathcal{L}(\mathcal{O},\mathcal{M}):((s_q(B))_{q=0}^{\infty}\in\mathcal{E}\right\},$$
$$\mathcal{L}^\alpha_{\mathcal{E}}:=\left\{\mathcal{L}^\alpha_{\mathcal{E}}(\mathcal{O},\mathcal{M})\right\},\text{ where }\mathcal{L}^\alpha_{\mathcal{E}}(\mathcal{O},\mathcal{M}):=\left\{B\in\mathcal{L}(\mathcal{O},\mathcal{M}):((\alpha_q(B))_{q=0}^{\infty}\in\mathcal{E}\right\},$$
$$\mathcal{L}^d_{\mathcal{E}}:=\left\{\mathcal{L}^d_{\mathcal{E}}(\mathcal{O},\mathcal{M})\right\},\text{ where }\mathcal{L}^d_{\mathcal{E}}(\mathcal{O},\mathcal{M}):=\left\{B\in\mathcal{L}(\mathcal{O},\mathcal{M}):((d_q(B))_{q=0}^{\infty}\in\mathcal{E}\right\}.$$

The multiplication operators have a wide field of mathematics in functional analysis, for instance, in eigenvalue distributions theorem, geometric structure of Banach spaces, theory of fixed point, and so forth. A few of operator ideals in the class of Hilbert spaces or Banach spaces are defined by distinct scalar sequence spaces. Such as the ideal of compact operators $\mathcal{C}$ formed by $(d_q(B)$ and c_0. Pietsch [28], studied the quasi-ideals $\mathcal{L}^\alpha_{\ell_q}$ for $q>0$, the ideals of Hilbert Schmidt operators between Hilbert spaces constructed by ℓ_2 and the ideals of nuclear operators generated by ℓ_1. He explained that the closure of $\mathbb{I}=\mathcal{L}^\alpha_{\ell_q}$ for $q\geqslant 1$, and the class $\mathcal{L}^\alpha_{\ell_q}$ became simple Banach and small [27]. The strictly inclusion $\mathcal{L}^\alpha_{\ell_p}(\mathcal{O},\mathcal{M})\subsetneqq\mathcal{L}^\alpha_{\ell_q}(\mathcal{O},\mathcal{M})\subsetneqq\mathcal{L}(\mathcal{O},\mathcal{M})$, if $q>p>0$, investigated through Makarov and Faried [18]. Faried and Bakery [10], gave a generalization of the class of quasi operator ideal which is the pre-quasi operator ideal, they examined several geometric and topological structure of $\mathcal{L}^s_{\ell_M}$ and $\mathcal{L}^s_{ces(r)}$. On sequence spaces, Mursaleen and Noman ([23, 24]) investigated the Compact operators on some difference sequence spaces. For more updates on sequence spaces and their applications see [13–15]. The multiplication operators on $(ces(r),\|.\|)$ with the Luxemburg norm $\|.\|$ elaborated by Komal et al. [16]. Bakery et al. [6] studied the multiplication Operators acting on weighted Nakano sequence space. The aim of this article to define and offer some geometric and topological structures of the weighted Gamma matrix of order r in Nakano sequence space of fuzzy functions, $\left(\Gamma^F_p(x,y)\right)_{\varpi}$, equipped with the pre-modular function. First, we give the sufficient conditions on this space to form pre-modular Banach. Second, we give the necessity and sufficient conditions on this pre-modular space such that the multiplication operator defined on it is bounded, approximable, invertible, Fredholm and closed range operator.

2. Preliminaries and definitions

Let Φ be the set of all closed and bounded intervals on $\mathcal{R}$. If $h = [h_1, h_2]$ and $j = [j_1, j_2]$ in Φ, assume

$$h \leqslant j \text{ if and only if } h_1 \leqslant j_1 \text{ and } h_2 \leqslant j_2.$$

Define a metric τ on Φ by

$$\tau(h, j) = \max\{|h_1 - j_1|, |h_2 - j_2|\}.$$

Matloka [20] showed that τ is a metric on Φ and (Φ, τ) is a complete metric space. The relation $\leqslant$ is a partial order on Φ.

Definition 2.1. A fuzzy number h is an operator $h : \mathcal{R} \to [0, 1]$ that satisfies the next setups:

(a) h is fuzzy convex, i.e., for $p, q \in \mathcal{R}$ and $\lambda \in [0, 1]$, $h(\lambda p + (1 - \lambda)q) \geqslant \min\{h(p), h(q)\}$;
(b) h is normal, i.e., there is $p_0 \in \mathcal{R}$ such that $h(p_0) = 1$;
(c) h is an upper-semi continuous, i.e., for all $\lambda > 0$, $h^{-1}([0, p + \lambda))$ for all $p \in [0, 1]$ is open in the usual topology of $\mathcal{R}$;
(d) the closure of $h^0 := \{p \in \mathcal{R} : \ h(p) > 0\}$ is compact.

The λ-level set of a fuzzy real number h, $\ 0 < \lambda < 1$, denoted by h^λ, is defined as

$$h^\lambda = \{p \in \mathcal{R} : \ h(p) \geqslant \lambda\}.$$

The set of all upper semi-continuous, normal, convex fuzzy number, and h^λ is compact, is denoted by $\mathcal{R}([0, 1])$. The set $\mathcal{R}$ can be embedded in $\mathcal{R}([0, 1])$, if we define $t \in \mathcal{R}([0, 1])$ by

$$\bar{t}(y) = \begin{cases} 1, & y = t, \\ 0, & y \neq t. \end{cases}$$

The additive identity and multiplicative identity in $\mathcal{R}[0, 1]$ are denoted by $\bar{0}$ and $\bar{1}$, respectively. Assume $h, j \in \mathcal{R}[0, 1]$ and the λ-level sets are $[h]^\lambda = [h_1^\lambda, h_2^\lambda]$, $[j]^\lambda = [j_1^\lambda, j_2^\lambda]$, $\lambda \in [0, 1]$. A partial ordering for any $h, j \in \mathcal{R}[0, 1]$ is as follows: $h \preceq j$ if and only if $h^\lambda \leqslant j^\lambda$, for all $\lambda \in [0, 1]$.

If $\overline{\tau} : \mathcal{R}[0, 1] \times \mathcal{R}[0, 1] \to \mathcal{R}^+ \cup \{0\}$ is defined by $\overline{\tau}(h, j) = \sup_{0 \leqslant \lambda \leqslant 1} \tau(h^\lambda, j^\lambda)$, then the following are verified:

1. $(\mathcal{R}[0, 1], \overline{\tau})$ is a complete metric space;
2. $\overline{\tau}(h + t, j + t) = \overline{\tau}(h, j)$ for all $h, j, t \in \mathcal{R}[0, 1]$;
3. $\overline{\tau}(h + t, j + m) \leqslant \overline{\tau}(h, j) + \overline{\tau}(t, m)$;
4. $\overline{\tau}(\zeta h, \zeta j) = |\zeta| \overline{\tau}(h, j)$, for all $\zeta \in \mathcal{R}$.

For more details on fuzzy functions and their properties, see [4, 8, 12, 25, 26].

Definition 2.2 ([28]). An operator $A \in \mathcal{L}(\mathcal{M})$ is said to be approximable if there are $D_r \in \mathbb{I}(\mathcal{M})$, for every $r \in \mathbb{N}$ and $\lim_{r \to \infty} \|A - D_r\| = 0$.

Theorem 2.3 ([28]). *If $\mathcal{M}$ is Banach space with $\dim(\mathcal{M}) = \infty$, then*

$$\mathbb{I}(\mathcal{M}) \subsetneqq \mathfrak{P}(\mathcal{M}) \subsetneqq \mathcal{C}(\mathcal{M}) \subsetneqq \mathcal{L}(\mathcal{M}).$$

Definition 2.4 ([22]). An operator $B \in \mathcal{L}(\mathcal{E})$ is called Fredholm if $\dim(\text{Range}(B))^c < \infty$, $\text{Range}(B)$ is closed and $\dim(\ker(B)) < \infty$.

3. Properties of $\left(\Gamma_p^F(x, y)\right)_\varpi$

We have discussed in this section some geometric and topological properties of the fuzzy functions space, $\left(\Gamma_p^F(x, y)\right)_\varpi$ equipped with the pre-modular function. Suppose ω^F is the space of all sequences of fuzzy reals.

Definition 3.1. If $(y_b) \in \mathcal{R}^{+\mathbb{N}_0}$, where $\mathcal{R}^{+\mathbb{N}_0}$ is the space of all sequences of positive reals, the sequence space $\left(\Gamma_p^F(x,y)\right)_\varpi$ equipped with the function ϖ is defined as:

$$\left(\Gamma_p^F(x,y)\right)_\varpi = \left\{\bar{j} = (\overline{j_b}) \in \omega^F : \varpi(\eta\bar{j}) < \infty, \text{ for some } \eta > 0\right\},$$

where

$$\varpi(\bar{j}) = \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \overline{j_a}, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b}.$$

Lemma 3.2 ([3]). *If $y_a > 0$ and $x_a, z_a \in \mathcal{R}$, for all $a \in \mathbb{N}_0$, and $\hbar = \max\{1, \sup_a y_a\}$, then*

$$|x_a + z_a|^{y_a} \leqslant 2^{\hbar-1}\left(|x_a|^{y_a} + |z_a|^{y_a}\right).$$

Theorem 3.3. *Suppose $(y_a) \in \ell_\infty \cap \mathcal{R}^{+\mathbb{N}_0}$, then*

$$\left(\Gamma_p^F(x,y)\right)_\varpi = \left\{\bar{j} = (\overline{j_a}) \in \omega^F : \varpi(\eta\bar{j}) < \infty, \textit{ for all } \eta > 0\right\}.$$

Proof. Obviously, as (y_a) is bounded. □

Theorem 3.4. *Suppose $(y_a) \in [1,\infty)^{\mathbb{N}_0} \cap \ell_\infty$, then the space $\left(\Gamma_p^F(x,y)\right)_\varpi$ is a non-absolute type.*

Proof. Clearly, since

$$\begin{aligned}\varpi\left(\bar{1}, -\bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots\right) &= (x_0)^{y_0} + \left(\frac{|x_0 - px_1|}{1+p}\right)^{y_1} + \left(\frac{|x_0 - px_1|}{\begin{bmatrix} p+2 \\ 2 \end{bmatrix}}\right)^{y_2} + \cdots \\ &\neq (x_0)^{y_0} + \left(\frac{x_0 + px_1}{1+p}\right)^{y_1} + \left(\frac{x_0 + px_1}{\begin{bmatrix} p+2 \\ 2 \end{bmatrix}}\right)^{y_2} + \cdots = \varpi\left(\bar{1}, \bar{1}, \bar{0}, \bar{0}, \bar{0}, \dots\right).\end{aligned}$$

□

Definition 3.5. Assume $(y_a) \in \mathcal{R}^{+\mathbb{N}_0}$ and $y_a \geqslant 1$, for all $a \in \mathbb{N}_0$.

$$\left(|\Gamma_p^F|(x,y)\right)_\wp := \left\{\bar{j} = (\overline{j_a}) \in \omega^F : \wp(\eta\bar{j}) < \infty, \text{ for some } \eta > 0\right\},$$

where

$$\wp(\bar{j}) = \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a |\overline{j_a}|, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b}.$$

Theorem 3.6. *If $(y_a) \in (1,\infty)^{\mathbb{N}_0} \cap \ell_\infty$ with $\left(\frac{a+1}{\begin{bmatrix} p+a \\ a \end{bmatrix}}\right) \notin \ell_{(y_a)}$, then $\left(|\Gamma_p^F|(x,y)\right)_\wp \subsetneqq \left(\Gamma_p^F(x,y)\right)_\varpi$.*

Proof. Assume $\bar{j} \in \left(|\Gamma_p^F|(x,y)\right)_\wp$, as

$$\sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \overline{j_a}, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \leqslant \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a |\overline{j_a}|, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} < \infty.$$

Therefore, $\bar{j} \in \left(\Gamma_p^F(x,y)\right)_\varpi$. Take $\bar{i} = \left(\frac{(-\bar{1})^a}{\begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a}\right)_{a \in \mathbb{N}_0}$, one has $\bar{i} \in \left(\Gamma_p^F(x,y)\right)_\varpi$ and $\bar{i} \notin \left(|\Gamma_p^F|(x,y)\right)_\wp$. □

Consider $\mathcal{E}^F$ is a linear space of sequences of fuzzy functions, and $[b]$ indicates an integral part of the real number b.

Definition 3.7. The space $\mathcal{E}^F$ is said to be a private sequence space of fuzzy functions ($\mathfrak{pssff}$), when the next setups are satisfied:

(a1) for all $d \in \mathbb{N}_0$, then $\overline{e_d} \in \mathcal{E}^F$, where $\overline{e_d} = (\overline{0}, \overline{0}, \dots, \overline{1}, \overline{0}, \overline{0}, \cdots)$, while $\overline{1}$ displays at the d^{th} place;
(a2) assume $\overline{i} = (\overline{i_a}) \in \omega^F$, $|\overline{j}| = (|\overline{j_a}|) \in \mathcal{E}^F$ and $|\overline{i_a}| \leqslant |\overline{j_a}|$, with $a \in \mathbb{N}_0$, then $|\overline{i}| \in \mathcal{E}^F$;
(a3) $\left(\left|\overline{k_{[\frac{a}{2}]}}\right|\right)_{a=0}^{\infty} \in \mathcal{E}^F$, if $\left(\left|\overline{k_a}\right|\right)_{a=0}^{\infty} \in \mathcal{E}^F$.

Assume $\overline{\theta} = (\overline{0}, \overline{0}, \overline{0}, \dots)$ and $\mathcal{F}$ is the space of finite sequences of fuzzy numbers.

Definition 3.8. A subspace of the $\mathfrak{pssff}$ is called a pre-modular $\mathfrak{pssff}$, if there is a function $\varpi : \mathcal{E}^F \to [0, \infty)$ that satisfies the next setups:

(i) if $\overline{i} \in \mathcal{E}^F$, $\overline{i} = \overline{\theta} \iff \varpi(|\overline{i}|) = 0$, and $\varpi(\overline{i}) \geqslant 0$;
(ii) assume $\overline{i} \in \mathcal{E}^F$ and $\sigma \in \mathcal{R}$, one has $E_0 \geqslant 1$ so that $\varpi(\sigma \overline{i}) \leqslant |\sigma| E_0 \varpi(\overline{i})$;
(iii) one has $G_0 \geqslant 1$ with $\varpi(\overline{i} + \overline{j}) \leqslant G_0(\varpi(\overline{i}) + \varpi(\overline{j}))$, for all $\overline{i}, \overline{j} \in \mathcal{E}^F$;
(iv) suppose $|\overline{i_q}| \leqslant |\overline{j_q}|$, for all $q \in \mathbb{N}_0$, then $\varpi(|\overline{i_q}|) \leqslant \varpi(|\overline{j_q}|)$;
(v) we have $D_0 \geqslant 1$ such that $\varpi(|\overline{i}|) \leqslant \varpi(|\overline{i_{[.]}}|) \leqslant D_0 \varpi(|\overline{i}|)$,
(vi) the closure of $\mathcal{F} = \mathcal{E}^F_{\varpi}$;
(vii) one has $\lambda > 0$ with $\varpi(\overline{\gamma}, \overline{0}, \overline{0}; \overline{0}, \dots) \geqslant \lambda |\gamma| \varpi(\overline{1}, \overline{0}, \overline{0}, \overline{0}, \dots)$.

The space $\mathcal{E}^F_{\varpi}$ is called a pre-modular Banach $\mathfrak{pssff}$, if $\mathcal{E}^F$ is complete under ϖ.

Definition 3.9. The $\mathfrak{pssff}$ $\mathcal{E}^F_{\varpi}$ is said to be a pre-quasi normed $\mathfrak{pssff}$, if ϖ verifies the setups (i)-(iii) of Definition 3.8.

Theorem 3.10. *The space $\mathcal{E}^F_{\varpi}$ is a pre-quasi normed $\mathfrak{pssff}$, whenever it is pre-modular $\mathfrak{pssff}$.*

By $\uparrow$ and $\downarrow$, we mark the space of all monotonic increasing and decreasing sequences of positive reals, respectively.

Theorem 3.11. *Suppose*

(f1) $(y_a) \in \uparrow \cap \ell_\infty$ *with* $y_0 > \frac{1}{p}$;
(f2) $\left(\begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a\right)_{a=0}^{\infty} \in \downarrow$ *or,* $\left(\begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a\right)_{a=0}^{\infty} \in \uparrow \cap \ell_\infty$ *and there exists* $C \geqslant 1$ *so that*

$$\begin{bmatrix} 2a+p \\ 2a+1 \end{bmatrix} x_{2a+1} \leqslant C \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a,$$

then $(\Gamma^F_p(x, y))_{\varpi}$ *is a pre-modular Banach* $\mathfrak{pssff}$.

Proof.

(i). Evidently, $\varpi(|\overline{i}|) = 0 \Leftrightarrow \overline{i} = \overline{\theta}$ and $\varpi(\overline{i}) \geqslant 0$.

(a1) and (iii). If $\overline{i}, \overline{j} \in (\Gamma^F_p(x, y))_{\varpi}$, then

$$\begin{aligned}
\varpi(\overline{i} + \overline{j}) &= \sum_{b=0}^{\infty} \left(\frac{\overline{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \left(\overline{i_a} + \overline{j_a}\right), \overline{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \\
&\leqslant 2^{\hbar-1} \left(\sum_{b=0}^{\infty} \left(\frac{\overline{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \overline{i_a}, \overline{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} + \sum_{b=0}^{\infty} \left(\frac{\overline{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \overline{j_a}, \overline{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \right) \\
&= 2^{\hbar-1}(\varpi(\overline{i}) + \varpi(\overline{j})) < \infty,
\end{aligned}$$

so $\overline{f} + \overline{g} \in (\Gamma^F_p(x, y))_{\varpi}$.

(ii). Suppose $\zeta \in \mathcal{R}$, $\bar{j} \in (\Gamma^F_p(x,y))_\varpi$ and as $(y_b) \in \uparrow \cap \ell_\infty$, we have

$$\varpi(\zeta\bar{j}) = |\sum_{m=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{m}\left[{a+p-1 \atop a}\right]x_a\overline{\zeta j_a},\bar{0}\right)}{\left[{p+m \atop m}\right]}\right)^{y_m} \leqslant \sup_m|\zeta|^{y_m}|\sum_{m=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{m}\left[{a+p-1 \atop a}\right]x_a\overline{j_a},\bar{0}\right)}{\left[{p+m \atop m}\right]}\right)^{y_m} \leqslant E_0|\zeta|\varpi(\bar{j}) < \infty,$$

where $E_0 = \max\left\{1, \sup_b|\zeta|^{y_b-1}\right\} \geqslant 1$. Hence $\zeta\bar{j} \in (\Gamma^F_p(x,y))_\varpi$. As $(y_b) \in \uparrow \cap \ell_\infty$ and $y_0 > \frac{1}{p}$, one obtains

$$\begin{aligned} |\sum_{m=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{m}\left[{a+p-1 \atop a}\right]x_a\overline{(e_b)_a},\bar{0}\right)}{\left[{p+m \atop m}\right]}\right)^{y_m} &= |\sum_{m=b}^{\infty}\left(\frac{\left[{b+p-1 \atop b}\right]x_b}{\left[{p+m \atop m}\right]}\right)^{y_m} \\ &\leqslant |\sup_{m=b}^{\infty}\left(\left[{b+p-1 \atop b}\right]x_b\right)^{y_m}\sum_{m=b}^{\infty}\left(\frac{1}{\left[{p+m \atop m}\right]}\right)^{y_m} \\ &\leqslant |\sup_{m=b}^{\infty}\left(p!\left[{b+p-1 \atop b}\right]x_b\right)^{y_m}\sum_{m=0}^{\infty}\left(\frac{1}{m+1}\right)^{y_0 p} < \infty. \end{aligned}$$

Hence $\overline{e_b} \in (\Gamma^F_p(x,y))_\varpi$, for every $b \in \mathbb{N}_0$.

(a2) and (iv). Suppose $|\overline{i_a}| \leqslant |\overline{j_a}|$, for all $a \in \mathbb{N}_0$ and $|\bar{j}| \in (\Gamma^F_p(x,y))_\varpi$, then

$$\varpi(|\bar{i}|) = |\sum_{m=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{m}\left[{a+p-1 \atop a}\right]x_a|\overline{i_a}|,\bar{0}\right)}{\left[{p+m \atop m}\right]}\right)^{y_m} \leqslant |\sum_{m=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{m}\left[{a+p-1 \atop a}\right]x_a|\overline{j_a}|,\bar{0}\right)}{\left[{p+m \atop m}\right]}\right)^{y_m} = \varpi(|\bar{j}|) < \infty,$$

so $|\bar{i}| \in (\Gamma^F_p(x,y))_\varpi$.

(a3) and (v). Suppose $(|\overline{j_a}|) \in (\Gamma^F_p(x,y))_\varpi$, under $(y_b) \in \uparrow \cap \ell_\infty$ and $\left(\left[{a+p-1 \atop a}\right]x_a\right)_{a=0}^{\infty} \in \downarrow$, we get

$$\begin{aligned} \varpi(|\overline{j_{[\frac{a}{2}]}}|) &= |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_{[\frac{a}{2}]}}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} \\ &= |\sum_{b=0,2,4,\cdots}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_{[\frac{a}{2}]}}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} + |\sum_{b=1,3,5,\cdots}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_{[\frac{a}{2}]}}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} \\ &= |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{2b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_{[\frac{a}{2}]}}|,\bar{0}\right)}{\left[{p+2b \atop 2b}\right]}\right)^{y_{2b}} + |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{2b+1}\left[{a+p-1 \atop a}\right]x_a|\overline{j_{[\frac{a}{2}]}}|,\bar{0}\right)}{\left[{p+2b+1 \atop 2b+1}\right]}\right)^{y_{2b+1}} \\ &\leqslant |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{2b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_{[\frac{a}{2}]}}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} + |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{2b+1}\left[{a+p-1 \atop a}\right]x_a|\overline{j_{[\frac{a}{2}]}}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} \\ &\leqslant |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\left[{2b+p-1 \atop 2b}\right]x_{2b}|\overline{j_b}| + \sum_{a=0}^{b}\left(\left[{2a+p-1 \atop a}\right]x_{2a} + \left[{2a+p \atop 2a+1}\right]x_{2a+1}\right)|\overline{j_a}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} \\ &\quad + |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\left(\left[{2a+p-1 \atop a}\right]x_{2a} + \left[{2a+p \atop 2a+1}\right]x_{2a+1}\right)|\overline{j_a}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} \\ &\leqslant 2^{\hbar-1}\left(|\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_a}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} + |\sum_{b=0}^{\infty}\left(\frac{2\bar{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_a}|,\bar{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}\right) \end{aligned}$$

$$+|\sum_{b=0}^{\infty}\left(\frac{2\overline{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a|\overline{j_a}|,\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} \leqslant D_0\varpi(|\overline{j}|) < \infty,$$

where $D_0 \geqslant (2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar}) \geqslant 1$. Hence $(|\overline{j_{[\frac{a}{2}]}}|) \in (\Gamma_p^F(x,y))_\varpi$.

(vi). Obviously, the closure of $\mathcal{F} = \Gamma_p^F(x,y)$.

(vii). We have $0 < \gamma \leqslant \sup_l |\beta|^{y_b-1}$ so that $\varpi(\overline{\beta},\overline{0},\overline{0},\overline{0},\ldots) \geqslant \gamma|\beta|\varpi(\overline{1},\overline{0},\overline{0},\overline{0},\ldots)$, for all $\beta \neq 0$ and $\gamma > 0$, when $\beta = 0$. Hence $(\Gamma_p^F(x,y))_\varpi$ is a pre-modular $\mathfrak{pssff}$. To show that $(\Gamma_p^F(x,y))_\varpi$ is a Banach space, suppose $\overline{A^m} = (\overline{A_a^m})_{a=0}^{\infty}$ is a Cauchy sequence in $(\Gamma_p^F(x,y))_\varpi$, then for every $\gamma \in (0,1)$, one has $m_0 \in \mathbb{N}_0$ with $m,n \geqslant m_0$, then

$$\varpi(\overline{A^m}-\overline{A^n}) = |\sum_{b=0}^{\infty}\left(\frac{\overline{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a\left(\overline{A_a^m}-\overline{A_a^n}\right),\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} < \gamma^{\hbar}.$$

Therefore, $\overline{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a\left(\overline{A_a^m}-\overline{A_a^n}\right),\overline{0}\right) < \gamma$. As $(\mathcal{R}([0,1]),\overline{\tau})$ is a complete metric space, hence $(\overline{A_a^n})$ is a Cauchy sequence in $\mathcal{R}([0,1])$, for constant $a \in \mathbb{N}_0$. Therefore, it is convergent to $\overline{A_a^0} \in \mathcal{R}([0,1])$. So $\varpi(\overline{A^m}-\overline{A^0}) < \gamma^{\hbar}$, for every $m \geqslant m_0$. Clearly, By part (iii) that $\overline{A^0} \in (\Gamma_p^F(x,y))_\varpi$. □

4. Multiplication operators on $(\Gamma_p^F(x,y))_\varpi$

In this section, we present some properties of the multiplication operator acting on $(\Gamma_p^F(x,y))_\varpi$, supposing that the conditions of Theorem 3.11 are confirmed. Assume $(\text{Range}(B))^c$ is the complement of $\text{Range}(B)$ and $\mathfrak{I}$ is the space of all sets with finite number of elements and ℓ_∞^F is the space of bounded sequences of fuzzy functions.

Definition 4.1. If $\mathcal{E}_\varpi^F$ is a pre-quasi normed $\mathfrak{pssff}$ and $\psi = (\psi_k) \in \mathcal{R}^{\mathbb{N}_0}$, the operator $G_\psi : \mathcal{E}_\varpi^F \to \mathcal{E}_\varpi^F$ is called a multiplication operator on $\mathcal{E}_\varpi^F$, when $G_\psi\overline{j} = \left(\psi_a\overline{j_a}\right) \in \mathcal{E}_\varpi^F$, for every $j \in \mathcal{E}_\varpi^F$. The multiplication operator is called generated by ψ, when $G_\psi \in \mathcal{L}(\mathcal{E}_\varpi^F)$.

Theorem 4.2.
(1) $\psi \in \ell_\infty \Longleftrightarrow G_\psi \in \mathcal{L}((\Gamma_p^F(x,y))_\varpi)$.
(2) $|\psi_a| = 1$, *for every* $a \in \mathbb{N}_0$, *if and only if,* G_ψ *is an isometry.*
(3) $G_\psi \in \mathfrak{P}((\Gamma_p^F(x,y))_\varpi) \Longleftrightarrow (\psi_a)_{a=0}^{\infty} \in c_0$.
(4) $G_\psi \in \mathcal{C}((\Gamma_p^F(x,y))_\varpi) \Longleftrightarrow (\psi_b)_{b=0}^{\infty} \in c_0$.
(5) $\mathcal{C}((\Gamma_p^F(x,y))_\varpi) \subsetneqq \mathcal{L}((\Gamma_p^F(x,y))_\varpi)$.
(6) $0 < \alpha < |\psi_a| < \eta$, *for all* $a \in (\ker(\psi))^c$, *if and only if,* $\text{Range}(G_\psi)$ *is closed.*
(7) $0 < \alpha < |\psi_a| < \eta$, *for every* $a \in \mathbb{N}_0$, *if and only if,* $G_\psi \in \mathcal{L}((\Gamma_p^F(x,y))_\varpi)$ *is invertible.*
(8) G_ψ *is Fredholm operator, if and only if,*
   (g1) $\ker(\psi) \subsetneqq \mathbb{N}_0 \cap \mathfrak{I}$;
   (g2) $|\psi_a| \geqslant \rho$, *for every* $a \in (\ker(\psi))^c$.

Proof.
(1). Assume $\psi \in \ell_\infty$, we have $\nu > 0$ so that $|\psi_a| \leqslant \nu$, for every $a \in \mathbb{N}_0$. Suppose $\overline{j} \in (\Gamma_p^F(x,y))_\varpi$, one has

$$\varpi(G_\psi\overline{j}) = \varpi(\psi\overline{j}) = |\sum_{b=0}^{\infty}\left(\frac{\overline{\tau}\left(\sum_{a=0}^{b}\psi_a\left[{a+p-1 \atop a}\right]x_a\overline{j_a},\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}$$

$$\leqslant \sup_l \nu^{y_b}|\sum_{b=0}^{\infty}\left(\frac{\overline{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a\overline{j_a},\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b} = \sup_l \nu^{y_b}\varpi(\overline{j}).$$

Hence $G_\psi \in \mathcal{L}((\Gamma_p^F(x,y))_\varpi)$.

Next, assume $G_\psi \in \mathcal{L}((\Gamma_p^F(x,y))_\varpi)$ and $\psi \notin \ell_\infty$. We have $q_d \in \mathbb{N}_0$, for all $d \in \mathbb{N}_0$ so that $\psi_{q_d} > d$. Hence

$$\begin{aligned}\varpi(G_\psi \overline{e_{q_d}}) = \varpi(\psi \overline{e_{q_d}}) &= \Big| \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \psi_a \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \overline{(e_{q_d})_a}, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \\ &= \Big| \sum_{b=q_d}^{\infty} \left(\frac{\psi_{(q_d)} \begin{bmatrix} q_d+p-1 \\ q_d \end{bmatrix} x_{q_d}}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} > \Big| \sum_{b=q_d}^{\infty} \left(\frac{d \begin{bmatrix} q_d+p-1 \\ q_d \end{bmatrix} x_{q_d}}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} > d^{y_0} \varpi(\overline{e_{q_d}}).\end{aligned}$$

So $G_\psi \notin \mathcal{L}((\Gamma_p^F(x,y))_\varpi)$. So $\psi \in \ell_\infty$.

(2). Suppose $\bar{j} \in (\Gamma_p^F(x,y))_\varpi$ and $|\psi_b| = 1$, for all $b \in \mathbb{N}_0$. We have

$$\begin{aligned}\varpi(G_\psi \bar{j}) = \varpi(\psi \bar{j}) &= \Big| \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \psi_a \overline{j_a}, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \\ &= \Big| \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \overline{j_a}, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} = \varpi(\bar{j}),\end{aligned}$$

hence G_ψ is an isometry.

Next assume for some $d = d_0$ that $|\psi_d| < 1$, we have

$$\begin{aligned}\varpi(G_\psi \overline{e_{d_0}}) = \varpi(\psi \overline{e_{d_0}}) &= \Big| \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \psi_a \overline{(e_{d_0})_a}, \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \\ &= \Big| \sum_{b=d_0}^{\infty} \left(\frac{|\psi_{d_0}| \begin{bmatrix} d_0+p-1 \\ d_0 \end{bmatrix} x_{d_0}}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} < \Big| \sum_{b=d_0}^{\infty} \left(\frac{\begin{bmatrix} d_0+p-1 \\ d_0 \end{bmatrix} x_{d_0}}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} = \varpi(\overline{e_{d_0}}).\end{aligned}$$

If $|\psi_{d_0}| > 1$, so $\varpi(G_\psi \overline{e_{d_0}}) > \varpi(\overline{e_{d_0}})$. Hence $|\psi_a| = 1$, for every $a \in \mathbb{N}_0$.

(3). Let $G_\psi \in \mathfrak{P}((\Gamma_p^F(x,y))_\varpi)$, hence $G_\psi \in \mathcal{C}((\Gamma_p^F(x,y))_\varpi)$. Assume $\lim_{b\to\infty} \psi_b \neq 0$. We get $\rho > 0$ so that $K_\rho = \{a \in \mathbb{N}_0 : |\psi_a| \geqslant \rho\} \nsubseteq \mathfrak{I}$. If $\{\lambda_q\}_{q\in\mathbb{N}_0} \subset K_\rho$, one has $\{\overline{e_{\lambda_q}} : \lambda_q \in K_\rho\} \in \ell_\infty^F$ is an infinite set in $(\Gamma_p^F(x,y))_\varpi$. For every $\lambda_q, \lambda_r \in K_\rho$, we have

$$\begin{aligned}\varpi(G_\psi \overline{e_{\lambda_q}} - G_\psi \overline{e_{\lambda_r}}) &= \varpi(\psi \overline{e_{\lambda_q}} - \psi \overline{e_{\lambda_r}}) \\ &= \Big| \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \psi_a \left(\overline{(e_{\lambda_q})_a} - \overline{(e_{\lambda_r})_a}\right), \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \\ &\geqslant \Big| \sum_{b=0}^{\infty} \left(\frac{\bar{\tau}\left(\sum_{a=0}^{b} \begin{bmatrix} a+p-1 \\ a \end{bmatrix} x_a \rho \left(\overline{(e_{\lambda_q})_a} - \overline{(e_{\lambda_r})_a}\right), \bar{0}\right)}{\begin{bmatrix} p+b \\ b \end{bmatrix}} \right)^{y_b} \geqslant \inf_b \rho^{y_b} \varpi(\overline{e_{\lambda_q}} - \overline{e_{\lambda_r}}).\end{aligned}$$

Therefore, $\{\overline{e_{\lambda_r}} : \lambda_r \in K_\rho\} \in \ell_\infty^F$ has not a convergent subsequence under G_ψ. Hence $G_\psi \notin \mathcal{C}((\Gamma_p^F(x,y))_\varpi)$. So $G_\psi \notin \mathfrak{P}((\Gamma_p^F(x,y))_\varpi)$, this is a contradiction. Hence $\lim_{b\to\infty} \psi_b = 0$. Next, assume $\lim_{a\to\infty} \psi_a = 0$. Therefore for all $\rho > 0$, one has $K_\rho = \{b \in \mathbb{N}_0 : |\psi_b| \geqslant \rho\} \subset \mathfrak{I}$. Hence for every $\rho > 0$, we have $\dim\left(\left((\Gamma_p^F(x,y))_\varpi\right)_{K_\rho}\right) = \dim\left(\mathcal{R}^{K_\rho}\right) < \infty$. Then $G_\psi \in \mathbb{I}\left(\left((\Gamma_p^F(x,y))_\varpi\right)_{K_\rho}\right)$, if $\psi_q \in \mathcal{R}^{\mathbb{N}_0}$, for every $q \in \mathbb{N}_0$, where

$$(\psi_q)_r = \begin{cases} \psi_r, & r \in K_{\frac{1}{q+1}}, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, $G_{\psi_a} \in \mathbb{I}\left(\left((\Gamma_p^F(x,y))_{\varpi}\right)_{K_{\frac{1}{q+1}}}\right)$, as $\dim\left(\left((\Gamma_p^F(x,y))_{\varpi}\right)_{K_{\frac{1}{q+1}}}\right) < \infty$, for all $q \in \mathbb{N}_0$. Since $(y_b) \in \uparrow \cap \ell_\infty$ so that $y_0 > \frac{1}{p}$, one has

$$
\begin{aligned}
&\varpi((G_\psi - G_{\psi_q})\bar{j}) \\
&= \varpi\Big(\Big((\psi_r - (\psi_q)_r)\overline{j_r}\Big)_{r=0}^{\infty}\Big) \\
&= |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a-(\psi_q)_a)\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&= |\sum_{b=0, b\in K_{\frac{1}{q+1}}}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a-(\psi_q)_a)\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} + |\sum_{b=0, b\notin K_{\frac{1}{q+1}}}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a-(\psi_q)_a)\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&= |\sum_{b=0, b\in K_{\frac{1}{q+1}}}^{\infty}\left(\frac{\bar{\tau}\left(|\sum_{a=0, a\in K_{\frac{1}{q+1}}}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a-(\psi_q)_a)\overline{j_a} + |\sum_{a=0, a\notin K_{\frac{1}{q+1}}}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a-(\psi_q)_a)\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&\quad + |\sum_{b=0, b\notin K_{\frac{1}{q+1}}}^{\infty}\left(\frac{\bar{\tau}\left(|\sum_{a=0, a\in K_{\frac{1}{q+1}}}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a-(\psi_q)_a)\overline{j_a} + |\sum_{a=0, a\notin K_{\frac{1}{q+1}}}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a-(\psi_q)_a)\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&= |\sum_{b=0, b\in K_{\frac{1}{q+1}}}^{\infty}\left(\frac{\bar{\tau}\left(|\sum_{a=0, a\notin K_{\frac{1}{q+1}}}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a\psi_a\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} + |\sum_{b=0, b\notin K_{\frac{1}{q+1}}}^{\infty}\left(\frac{\bar{\tau}\left(|\sum_{a=0, a\notin K_{\frac{1}{q+1}}}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a\psi_a\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&\leqslant 2|\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(|\sum_{a=0, a\notin K_{\frac{1}{q+1}}}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a\psi_a\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&< \frac{2}{(q+1)^{y_0}}|\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a\overline{j_a},\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} = \frac{2}{(q+1)^{y_0}}\varpi(\bar{j}).
\end{aligned}
$$

Hence $\|G_\psi - G_{\psi_q}\| \leqslant \frac{2}{(q+1)^{y_0}}$. We get G_ψ is a limit of finite rank operators.

(4). Since $\mathfrak{P}((\Gamma_p^F(x,y))_{\varpi}) \subsetneqq \mathcal{C}((\Gamma_p^F(x,y))_{\varpi})$, the proof follows.

(5). As $I = I_\psi$, where $\psi = (1,1,\dots)$, one has $I \notin \mathcal{C}((\Gamma_p^F(x,y))_{\varpi})$ and $I \in \mathcal{L}((\Gamma_p^F(x,y))_{\varpi})$.

(6). If the sufficient setups are verified, we have $\rho > 0$ so that $|\psi_a| \geqslant \rho$, for all $a \in (\ker(\psi))^c$. To prove that $\text{Range}(G_\psi)$ is closed, when $\bar{g}$ is a limit point of $\text{Range}(G_\psi)$, we have $G_\psi\overline{j_b} \in (\Gamma_p^F(x,y))_{\varpi}$, for every $b \in \mathbb{N}_0$ so that $\lim_{b\to\infty} G_\psi\overline{j_b} = \bar{g}$. Obviously, $G_\psi\overline{j_b}$ is a Cauchy sequence. As $(y_b) \in \uparrow \cap \ell_\infty$, one can find $c > 0$ such that

$$
\begin{aligned}
\varpi(G_\psi\overline{j_q} - G_\psi\overline{j_r}) &= |\sum_{b=0}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a\overline{(j_q)_a} - \psi_a\overline{(j_r)_a}),\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&= |\sum_{b=0, b\in(\ker(\psi))^c}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a\overline{(j_q)_a} - \psi_a\overline{(j_r)_a}),\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b} \\
&\quad + |\sum_{b=0, b\notin(\ker(\psi))^c}^{\infty}\left(\frac{\bar{\tau}\left(\sum_{a=0}^{b}\begin{bmatrix}a+p-1\\a\end{bmatrix}x_a(\psi_a\overline{(j_q)_a} - \psi_a\overline{(j_r)_a}),\bar{0}\right)}{\begin{bmatrix}p+b\\b\end{bmatrix}}\right)^{y_b}
\end{aligned}
$$

$$\begin{aligned}
&\geqslant |\sum_{b=0,b\in(\ker(\psi))^c}^{\infty}\left(\frac{\overline{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a(\psi_a\overline{(j_q)_a}-\psi_a\overline{(j_r)_a}),\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}\\
&\geqslant c|\sum_{b=0,b\in(\ker(\psi))^c}^{\infty}\left(\frac{\overline{\tau}\left(|\sum_{a=0,a\in(\ker(\psi))^c}^{b}\left[{a+p-1 \atop a}\right]x_a(\psi_a\overline{(j_q)_a}-\psi_a\overline{(j_r)_a}),\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}\\
&= c|\sum_{b=0}^{\infty}\left(\frac{\overline{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a(\psi_a\overline{(u_q)_a}-\psi_a\overline{(u_r)_a}),\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}\\
&> c|\sum_{b=0}^{\infty}\left(\frac{\overline{\tau}\left(\rho\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a(\overline{(u_q)_a}-\overline{(u_r)_a}),\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}\geqslant \inf_b c\rho^{y_b}\varpi\left(\overline{u_q}-\overline{u_r}\right),
\end{aligned}$$

where

$$\overline{(u_q)_k}=\begin{cases}\overline{(j_q)_k}, & k\in(\ker(\psi))^c,\\ 0, & k\notin(\ker(\psi))^c.\end{cases}$$

Hence $\{\overline{u_q}\}$ is a Cauchy sequence in $(\Gamma_p^F(x,y))_\varpi$. As $(\Gamma_p^F(x,y))_\varpi$ is complete, one has $\overline{j}\in(\Gamma_p^F(x,y))_\varpi$ with $\lim_{b\to\infty}\overline{u_b}=\overline{j}$. Since $G_\psi\in\mathcal{L}((\Gamma_p^F(x,y))_\varpi)$, one has $\lim_{b\to\infty}G_\psi\overline{u_b}=G_\psi\overline{j}$. Since $\lim_{b\to\infty}G_\psi\overline{u_b}=\lim_{b\to\infty}G_\psi\overline{j_b}=\overline{g}$, therefore, $G_\psi\overline{j}=\overline{g}$. So $\overline{g}\in\mathrm{Range}(G_\psi)$, i.e., $\mathrm{Range}(G_\psi)$ is closed. Next, assume the necessity condition is satisfied. We have $\rho>0$ with $\varpi(G_\psi\overline{j})\geqslant\rho\varpi(\overline{j})$ and $\overline{j}\in\left((\Gamma_p^F(x,y))_\varpi\right)_{(\ker(\psi))^c}$. Let $K=\left\{b\in(\ker(\psi))^c:|\psi_b|<\rho\right\}\neq\emptyset$, then for $q_0\in K$, one gets

$$\begin{aligned}
\varpi(G_\psi\overline{e_{q_0}})=\varpi\left(\left(\psi_b\overline{(e_{q_0})_b}\right)_{b=0}^{\infty}\right)&=|\sum_{b=0}^{\infty}\left(\frac{\overline{\tau}\left(\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a\psi_a\overline{(e_{q_0})_a},\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}\\
&<|\sum_{b=0}^{\infty}\left(\frac{\overline{\tau}\left(\rho\sum_{a=0}^{b}\left[{a+p-1 \atop a}\right]x_a\overline{(e_{q_0})_a},\overline{0}\right)}{\left[{p+b \atop b}\right]}\right)^{y_b}\leqslant\sup_l\rho^{y_b}\varpi(\overline{e_{q_0}}),
\end{aligned}$$

this gives a contradiction. Hence $K=\phi$, then $|\psi_a|\geqslant\rho$, for every $a\in(\ker(\psi))^c$.

(7). First, if $\beta\in\mathcal{R}^{\mathbb{N}_0}$ with $\beta_a=\frac{1}{\psi_a}$, from Theorem 4.2 part (1), one has $G_\psi,G_\beta\in\mathcal{L}((\Gamma_p^F(x,y))_\varpi)$. We get $G_\psi.G_\beta=G_\beta.G_\psi=I$. So $G_\beta=G_\psi^{-1}$. Second, assume G_ψ is invertible. Then $\mathrm{Range}(G_\psi)=\left((\Gamma_p^F(x,y))_\varpi\right)_{\mathbb{N}_0}$. Hence $\mathrm{Range}(G_\psi)$ is closed. From Theorem 4.2 part (5), we have $\alpha>0$ so that $|\psi_a|\geqslant\alpha$, for every $a\in(\ker(\psi))^c$. Hence $\ker(\psi)=\emptyset$, if $\psi_{a_0}=0$, where $a_0\in\mathbb{N}_0$, so $e_{a_0}\in\ker(G_\psi)$, which is a contradiction, as $\ker(G_\psi)$ is trivial. Therefore, $|\psi_a|\geqslant\alpha$, for every $a\in\mathbb{N}_0$. Since $G_\psi\in\ell_\infty$, by Theorem 4.2 part (1), we have $\eta>0$ with $|\psi_a|\leqslant\eta$, for every $a\in\mathbb{N}_0$. Hence $\alpha\leqslant|\psi_a|\leqslant\eta$, for every $a\in\mathbb{N}_0$.

(8). First, assume $\ker(\psi)\subsetneqq\mathbb{N}_0$ and $\ker(\psi)\notin\mathfrak{I}$, we obtain $\overline{e_a}\in\ker(G_\psi)$, for every $a\in\ker(\psi)$. Since $\overline{e_a}$'s are linearly independent, one gets $\dim(\ker(G_\psi))=\infty$, this is a contradiction. So $\ker(\psi)\subsetneqq\mathbb{N}_0\in\mathfrak{I}$. The setup (g2) comes from Theorem 4.2 part (6). Next, if the setups (g1) and (g2) are confirmed, by Theorem 4.2 part (6), the setup (g2) implies that $\mathrm{Range}(G_\psi)$ is closed. The setup (g1) gives that $\dim((\mathrm{Range}(G_\psi))^c)<\infty$ and $\dim(\ker(G_\psi))<\infty$. Hence G_ψ is Fredholm. □

5. Conclusion

A new general solution space for numerous stochastic nonlinear dynamical systems are presented. We have defined and examined some topological, geometric properties of $\left(\Gamma_p^F(x,y)\right)_\varpi$ and the multiplication operators acting on it.

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