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# Modeling the dynamics of business cycle with general investment and variable depreciation rate of capital stock

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## Abstract

In this paper, we develop a business cycle model with general investment, variable depreciation rate of capital stock and two delays. The first delay describes the time lag between the decision of investment and its implementation, while the second one models the time lag for investment to be produced. The well-posedness and the existence of economic equilibrium are carefully investigated. Moreover, the stability of the economic equilibrium and the existence of Hopf bifurcation are established. The case when the two delays are equal is rigorously studied.

**Keywords:** Business cycle, depreciation, time delays, stability, Hopf bifurcation.

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## 1. Introduction

Business cycles are recurrent economic phenomena often called economic cycles and which are defined as a type of fluctuations in macroeconomic variables caused by the instability of endogenous economic factors. In the literature, several mathematical models have been proposed to understand these business cycles. In 1940, Kaldor [5] demonstrated that fluctuations mainly due to investment. Kalecki [6] introduced the idea of the existence of a delay between the investment decision and its implementation. In 1999, Krawiec and Sydlowski [7] incorporated Kalecki's idea into Kaldor's model presented in [5]. In 2017, Hattaf et al. [4] introduced a second delay in the model of [7] in order to describe the time for the investment to be productive. Then they proposed the following business cycle model:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), K(t)) - \gamma Y(t)], \\ \frac{dK}{dt} = I(Y(t - \tau_1), K(t - \tau_2)) - \delta K(t), \end{cases}$$

where $Y(t)$ and $K(t)$ denote the gross product and capital stock at time $t$, respectively. The first delay $\tau_1$ is the time lag between the decision of investment and its implementation. The second delay $\tau_2$ is the

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time lag for the investment to be productive. Moreover, the coefficient $\gamma \in (0,1)$ is a saving constant. The investment function is represented by $I(Y,K)$ and it is assumed to be continuously differentiable in $\mathbb{R}^2$ with $\frac{\partial I}{\partial Y} > 0$ and $\frac{\partial I}{\partial K} < 0$. Finally, the positive parameters $\alpha$ and $\delta$ are, respectively, the adjustment coefficient in the goods market and the depreciation rate of the capital stock.

On the other hand, economists have noticed that people gain experience when they are working, which leads to improve work efficiency. Therefore, the rate of depreciation of stock capital is not constant, but will decrease as people's experience increases. In addition, Chatterjee [1] remarked that the rate of depreciation depends on the degree of the capital stock use. He assumed that the rate of depreciation of capital is an increasing function of the rate of utilization. Further, a higher utilization rate causes a faster depreciation of the capital stock, either because wear and tear increase with use or because less time can devoted to maintenance [2]. In 2020, Mao and Liu [8] investigated the dynamics of a Kaldor business cycle model by assuming that the depreciation rate of the capital stock $\delta$ is a decreasing convex function of capital stock, where $0 < \lim_{K \to +\infty} \delta(K) = \delta_1 < \delta_0 = \delta(0) < 1$. For simplicity, they assumed that the depreciation rate of the capital stock has the form of $\delta(K) = \delta_1 + \frac{\delta_0 - \delta_1}{1+K}$.

Based on the above mathematical and economical considerations, we propose the following model:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), K(t)) - \gamma Y(t)], \\ \frac{dK}{dt} = I(Y(t-\tau_1), K(t-\tau_2)) - \delta(K(t))K(t), \end{cases} \tag{1.1}$$

where $\delta(K)$ is the depreciation rate function depending on K.

The rest of this paper is outlined as follows. In Section 2, we present some preliminary results including the existence, the uniqueness and the uniform boundedness of solutions of our model (1.1), as well as the existence of economic equilibrium. Section 3 deals with the stability analysis and Hopf bifurcation. Finally, the paper ends with a conclusion given in Section 4.

## 2. Preliminary results

Let $\tau = \max\{\tau_1, \tau_2\}$ and $C = C([-\tau, 0], \mathbb{R}^2)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^2$ equipped with the sup-norm $\|\varphi\| = \sup_{-\tau \leqslant \theta \leqslant 0} |\varphi(\theta)|$ for $\varphi \in C$. As in [4], we assume that the general investment function $I(Y, K)$ satisfies the following hypotheses.

**(H$_1$)** There exist two constants $L > 0$ and $\bar{q} \geqslant 0$ such that $|I(Y,K) + \bar{q}K| \leqslant L$ for all $Y, K \in \mathbb{R}$.

**(H$_2$)** There exists a $\delta_1 > 0$ such that $\delta(K) \geqslant \delta_1$ for all $K \in \mathbb{R}$.

First, we have the following result.

**Theorem 2.1.** *Assume* (H$_1$) *and* (H$_2$) *hold. For any initial condition* $(\phi_1, \phi_2) \in C$, *there exists a unique solution of system* (1.1) *defined on* $[0, +\infty)$ *and this solution is uniformly bounded.*

*Proof.* By the standard theory of functional differential equations [3], we know that for any initial condition $(\phi_1, \phi_2) \in C$, there exists a unique local solution of system (1.1) on $[0, T_{max})$, where $T_{max}$ is the maximal existence time for solution of system (1.1).

From the second equation of system (1.1), we have

$$\frac{dK}{dt} + (\delta(K(t)) + \bar{q})K(t) = I(Y(t-\tau_1), K(t-\tau_2)) + \bar{q}K(t).$$

Then

$$\frac{d}{dt}\left(K(t)e^{\int_0^t \delta(K(s))ds + \bar{q}t}\right) = [I(Y(t-\tau_1), K(t-\tau_2)) + \bar{q}K(t)] e^{\int_0^t \delta(K(s))ds + \bar{q}t}.$$

By integrating the above formula from 0 to t, we get

$$K(t)e^{\int_0^t \delta(K(s))ds+\bar{q}t} - K(0) = \int_0^t [I(Y(\xi-\tau_1), K(\xi-\tau_2)) + \bar{q}K(\xi)] e^{\int_0^\xi \delta(K(s))ds+\bar{q}\xi} d\xi.$$

Hence,

$$K(t) = e^{-(\int_0^t \delta(K(s))ds+\bar{q}t)}\phi_2(0) + \int_0^t e^{-\int_\xi^t \delta(K(s))ds-\bar{q}(t-\xi)}[I(Y(\xi-\tau_1), K(\xi-\tau_2)) + \bar{q}K(\xi)]d\xi.$$

From the assumptions $(H_1)$ and $(H_2)$, we deduce that

$$|K(t)| \leqslant e^{-(\delta_1+\bar{q})t}|\phi_2(0)| + L\int_0^t e^{-(\delta_1+\bar{q})(t-\xi)}d\xi \leqslant e^{-(\delta_1+\bar{q})t}|\phi_2(0)| + \frac{L}{\delta_1+\bar{q}}.$$

As $\lim_{t\to+\infty} e^{-(\delta_1+\bar{q})t}|\phi_2(0)| = 0$, then there exists a $t_0 > 0$ such that $e^{-(\delta_1+\bar{q})t}|\phi_2(0)| \leqslant 1$ for all $t \in [t_0, T_{max})$. Thus, $|K(t)| \leqslant 1 + \frac{L}{\delta_1+\bar{q}}$, which means that $K(t)$ is uniformly bounded for bound $A_1 = 1 + \frac{L}{\delta_1+\bar{q}}$.

Now, we show the uniform boundedness of Y. Based on the first equation of system (1.1), we obtain

$$Y(t) = Y(t_0)e^{-\alpha\gamma(t-t_0)} + \alpha\int_{t_0}^t e^{-\alpha\gamma(t-\xi)}I(Y(\xi), K(\xi))d\xi, \ t \geqslant t_0.$$

Hence,

$$|Y(t)| \leqslant |Y(t_0)|e^{-\alpha\gamma(t-t_0)} + \alpha\int_{t_0}^t e^{-\alpha\gamma(t-\xi)}\left(L+\bar{q}|K(\xi)|\right)d\xi \leqslant |Y(t_0)|e^{-\alpha\gamma(t-t_0)} + \frac{L+\bar{q}A_1}{\gamma}.$$

In the same way, this implies that the function $Y(t)$ is uniformly bounded for bound $A_2 = 1 + \frac{L+\bar{q}A_1}{\gamma}$. Based on the above and the continuity of the functions $Y(t)$ and $K(t)$, we deduce that $Y(t)$ and $K(t)$ are uniformly bounded $[0, T_{max})$. Therefore, $T_{max} = +\infty$. □

To study the existence of equilibria of (1.1), we need the following hypotheses.

$(H_3)$ $I(0,0) > 0$.

$(H_4)$ $\frac{\delta'(K)K+\delta(K)}{\gamma}\frac{\partial I}{\partial Y}(Y,K) - \delta'(K)K - \delta(K) + \frac{\partial I}{\partial K}(Y,K) < 0$ for all $(Y,K) \in \mathbb{R}^2$.

Then, we get following result.

**Theorem 2.2.** *If $(H_1)$-$(H_4)$ hold, then system (1.1) has a unique economic equilibrium of the form* $E^*(\frac{\delta(K^*)K^*}{\gamma}, K^*)$, *where $K^*$ is the unique positive solution of the equation* $I\left(\frac{\delta(K)K}{\gamma}, K\right) - \delta(K)K = 0$.

*Proof.* Economic equilibrium is the solution of the following system:

$$\begin{cases} \alpha[I(Y,K) - \gamma Y] = 0, \\ I(Y,K) - \delta(K)K = 0. \end{cases} \tag{2.1}$$

Then

$$Y = \frac{\delta(K)K}{\gamma}. \tag{2.2}$$

Replacing (2.2) into the first equation of (2.1), we find

$$I\left(\frac{\delta(K)K}{\gamma}, K\right) - \delta(K)K = 0.$$

Let V be the function defined on the interval $[0, +\infty)$ by

$$V(K) = I\left(\frac{\delta(K)K}{\gamma}, K\right) - \delta(K)K.$$

Using assumptions $(H_1)$-$(H_4)$, we have $V(0) = I(0,0) > 0$, $\lim\limits_{K \to +\infty} V(K) = -\infty$, and

$$V'(K) = \frac{\delta'(K)K + \delta(K)}{\gamma}\frac{\partial I}{\partial Y} - \delta'(K)K - \delta(K) + \frac{\partial I}{\partial K} < 0.$$

Therefore, there is a unique economic equilibrium $E^*(Y^*, K^*)$, where $K^*$ is the solution of the equation $V(K) = 0$ and $Y^* = \frac{\delta(K^*)K^*}{\gamma}$. $\qquad\qquad\square$

## 3. Stability analysis and Hopf bifurcation

In this section, we focus on the local stability of the economic equilibrium $E^*(Y^*, K^*)$ and the existence of Hopf bifurcation. Let $y = Y - Y^*$ and $k = K - K^*$. By substituting $y$ and $k$ into system (1.1) and linearizing of the system in a neighbourhood of the equilibrium $E^*(Y^*, K^*)$, we obtain

$$\begin{cases} \frac{dy}{dt} = \alpha[ay(t) + \beta k(t) - \gamma y(t)], \\ \frac{dk}{dt} = ay(t - \tau_1) + \beta k(t - \tau_2) - \bar{\delta}k(t), \end{cases}$$

where $a = \frac{\partial I}{\partial Y}(Y^*, K^*) > 0$, $\beta = \frac{\partial I}{\partial K}(Y^*, K^*) < 0$, and $\bar{\delta} = K^*\delta'(K^*) + \delta(K^*) > 0$. Then, we can know that the characteristic equation at $E^*$ is

$$\lambda^2 - [\alpha(a - \gamma) - \bar{\delta}]\lambda - \alpha\beta a e^{-\lambda\tau_1} + \beta[\alpha(a - \gamma) - \lambda]e^{-\lambda\tau_2} - \alpha\bar{\delta}(a - \gamma) = 0. \tag{3.1}$$

Similarly to the works presented in [4, 11], we distinguish three cases.

*3.1. The case $\tau_1 = \tau_2 = 0$*
When $\tau_1 = \tau_2 = 0$, Eq. (3.1) becomes

$$\lambda^2 - \lambda[\alpha(a - \gamma) + \beta - \bar{\delta}] + \alpha(a - \gamma)(\beta - \bar{\delta}) - \alpha\beta a = 0. \tag{3.2}$$

Hence, all roots of Eq. (3.2) have negative real parts if and only if $\alpha(a - \gamma) + \beta - \bar{\delta} < 0$ and $(a - \gamma)(\beta - \bar{\delta}) - \beta a > 0$. This is equivalent to

$$a - \gamma < \min\left\{\frac{\bar{\delta} - \beta}{\alpha}, \frac{-a\beta}{\bar{\delta} - \beta}\right\}. \tag{3.3}$$

Thus, the economic equilibrium $E^*$ is locally asymptotically stable when (3.3) holds.

*3.2. The case $\tau_1 \neq 0$, $\tau_2 = 0$*
In this case, Eq. (3.1) takes the following form

$$\lambda^2 - [\alpha(a - \gamma) + \beta - \bar{\delta}]\lambda + \alpha(a - \gamma)(\beta - \bar{\delta}) - \alpha\beta a e^{-\lambda\tau_1} = 0. \tag{3.4}$$

Let $i\omega$ $(\omega > 0)$ be a root of (3.4). Then

$$\begin{cases} -\omega^2 + \alpha(a - \gamma)(\beta - \bar{\delta}) = \alpha a\beta\cos(\omega\tau_1), \\ \omega[\alpha(a - \gamma) + \beta - \bar{\delta}] = \alpha a\beta\sin(\omega\tau_1), \end{cases}$$

which implies that

$$\omega^4 + [\alpha^2(a - \gamma)^2 + (\beta - \bar{\delta})^2]\omega^2 + \alpha^2[(a - \gamma)^2(\beta - \bar{\delta})^2 - a^2\beta^2] = 0. \tag{3.5}$$

Let $u = \omega^2$. So, Eq. (3.5) becomes

$$F(u) = u^2 + [\alpha^2(a - \gamma)^2 + (\beta - \bar{\delta})^2]u + \alpha^2[(a - \gamma)^2(\beta - \bar{\delta})^2 - a^2\beta^2] = 0.$$

As $\alpha^2(a - \gamma)^2 + (\beta - \bar{\delta})^2 > 0$, we get the following result.

**Lemma 3.1.**

  (i) *If* $|a - \gamma|(\bar{\delta} - \beta) \geqslant -a\beta$, *then Eq.* (3.5) *has no positive root.*

  (ii) *If* $|a - \gamma|(\bar{\delta} - \beta) < -a\beta$, *then Eq.* (3.5) *has a unique positive root given by*

$$\omega_0 = \frac{\sqrt{2}}{2}\left(\sqrt{\Delta} - \alpha^2(a - \gamma)^2 - (\beta - \bar{\delta})^2\right)^{\frac{1}{2}},$$

*where* $\Delta = [\alpha^2(a - \gamma)^2 + (\beta - \bar{\delta})^2]^2 - 4\alpha^2[(a - \gamma)^2(\beta - \bar{\delta})^2 - a^2\beta^2].$

According to above analysis and [9], we obtain the following results.

**Theorem 3.2.** *For* $\tau_2 = 0$, *we deduce the following conclusions.*

  (i) *If* $(a - \gamma)(\bar{\delta} - \beta) < a\beta$, *then the economic equilibrium* $E^*$ *is locally asymptotically stable for all* $\tau_1 \geqslant 0$.

  (ii) *If* $(a - \gamma)(\bar{\delta} - \beta) > -a\beta$ , *then* $E^*$ *is unstable for all* $\tau_1 \geqslant 0$.

  (iii) *If* $\frac{a\beta}{\bar{\delta} - \beta} < a - \gamma < \min\left\{\frac{\bar{\delta} - \beta}{\alpha}, \frac{-a\beta}{\bar{\delta} - \beta}\right\}$, *then system* (1.1) *undergoes Hopf bifurcation at* $E^*$ *when* $\tau_1 = \tau_{1,j}$, $j \in \mathbb{N}$. *In addition, the economic equilibrium* $E^*$ *is locally asymptotically stable for* $\tau_1 < \tau_{1,0}$ *and unstable for* $\tau_1 > \tau_{1,0}$, *where*

$$\tau_{1,j} = \frac{1}{\omega_0}\arccos\left(\frac{-\omega_0^2 + \alpha(a - \gamma)(\beta - \bar{\delta})}{\alpha a\beta}\right) + \frac{2j\pi}{\omega_0}.$$

*3.3. The case $\tau_1 \neq 0$, $\tau_2 \neq 0$*

Here, we study Eq. (3.1) with $\tau_2 > 0$ and $\tau_1$ in the stable regions. We consider $\tau_2$ as a parameter of bifurcation. According to Ruan and Wei [10], we get the following lemma.

**Lemma 3.3.** *If all roots of equation* (3.4) *have negative real parts for* $\tau_1 > 0$, *then there exists a* $\tau_2^*(\tau_1) > 0$, *such that when* $0 \leqslant \tau_2 < \tau_2^*(\tau_1)$ *all roots of equation* (3.1) *have negative real parts.*

*Proof.* The left hand side of Eq. (3.1) is analytic in $\lambda$ and $\tau_2$. It follows from [10] that when $\tau_2$ varies, the sum of the multiplicities of zeros of the left hand side of Eq. (3.1) in the open right half-plane can change only if a zero on or cross the imaginary axis. □

**Theorem 3.4.** *For $\tau_1$ in the stable regions and $\tau_2 > 0$, we have*

  (i) *if* $(a - \gamma)(\bar{\delta} - \beta) < a\beta$, *then for* $\tau_1 \geqslant 0$, *there exists a* $\tau_2^*(\tau_1)$ *such that the economic equilibrium* $E^*$ *is locally asymptotically stable, when* $\tau_2 \in [0, \tau_2^*(\tau_1))$;

  (ii) *if* $\frac{a\beta}{\bar{\delta} - \beta} < a - \gamma < \min\left\{\frac{\bar{\delta} - \beta}{\alpha}, \frac{-a\beta}{\bar{\delta} - \beta}\right\}$, *then for any* $\tau_1 \in [0, \tau_{1,0})$, *there exists a* $\tau_2^*(\tau_1)$ *such that the economic equilibrium* $E^*$ *is locally asymptotically stable, when* $\tau_2 \in [0, \tau_2^*(\tau_1))$.

*Proof.* The proof of (i) follows immediately from Lemma 3.1 (i), Lemma 3.3, and Theorem 3.2.

Assume that $\frac{a\beta}{\bar{\delta} - \beta} < a - \gamma < \min\left\{\frac{\bar{\delta} - \beta}{\alpha}, \frac{-a\beta}{\bar{\delta} - \beta}\right\}$. Based on Theorem 3.2, we deduce that the economic equilibrium is locally asymptotically stable for $\tau_1 \in [0, \tau_{1,0})$. Therefore, all roots of Eq. (3.4) have negative real parts. It follows from Lemma 3.3, that there exists a $\tau_2^*(\tau_1) > 0$, such that when $0 \leqslant \tau_2 < \tau_2^*(\tau_1)$ all roots of equation (3.1) have negative real parts. Thus, the economic equilibrium $E^*$ is locally asymptotically stable if $\tau_2 \in [0, \tau_2^*(\tau_1))$. □

*3.4. Study of special case*

When the two delays are equal, system (1.1) becomes:

$$\begin{cases} \frac{dY}{dt} = \alpha[I(Y(t), K(t)) - \gamma Y(t)], \\ \frac{dK}{dt} = I(Y(t - \tau), K(t - \tau)) - \delta(K(t))K(t). \end{cases} \tag{3.6}$$

According to Theorems 2.1 and 2.2, we get the following results.

**Corollary 3.5.**

(i) *For any initial condition $(\phi_1, \phi_2) \in C$, there exists a unique solution of system (3.6) defined on $[0, +\infty)$ and this solution is uniformly bounded if $(H_1)$ holds.*

(ii) *The system (3.6) has a unique economic equilibrium of the form $E^*(\frac{\delta(K^*)K^*}{\gamma}, K^*)$, where $K^*$ is the unique positive solution of the equation $I\left(\frac{\delta(K)K}{\gamma}, K\right) - \delta(K)K = 0$, if $(H_1)$-$(H_4)$ hold.*

On the other hand, Eq. (3.1) becomes

$$\lambda^2 - \lambda[\alpha(a - \gamma) - \bar{\delta}] - \alpha\bar{\delta}(a - \gamma) - (\alpha\gamma + \lambda)\beta e^{-\lambda\tau} = 0. \tag{3.7}$$

When $\tau = 0$, all roots of Eq. (3.7) have negative real parts if the condition (3.3) holds. Thus, $E^*$ is locally asymptotically stable.

For $\tau > 0$, let $i\omega$ ($\omega > 0$) is a root of (3.7), then we get

$$\begin{cases} -\omega^2 - \alpha\bar{\delta}(a - \gamma) = \alpha\beta\gamma\cos(\omega\tau) + \beta\omega\sin(\omega\tau), \\ \omega[\alpha(a - \gamma) - \bar{\delta}] = \alpha\beta\gamma\sin(\omega\tau) - \beta\omega\cos(\omega\tau), \end{cases}$$

which implies that

$$\omega^4 + [\alpha^2(a - \gamma)^2 + \bar{\delta}^2 - \beta^2]\omega^2 + \alpha^2[\bar{\delta}^2(a - \gamma)^2 - \beta^2\gamma^2] = 0. \tag{3.8}$$

Let $A = \alpha^2(a - \gamma)^2 + \bar{\delta}^2 - \beta^2$, $B = \alpha^2[\bar{\delta}^2(a - \gamma)^2 - \beta^2\gamma^2]$ and $\overline{\Delta} = A^2 - 4B$. As in [4], we easily get the following lemma.

**Lemma 3.6.**

(i) *If $A \geqslant 0$ and $B \geqslant 0$, then Eq. (3.8) has no positive roots. All roots of Eq. (3.7) have negative real part for $\tau \geqslant 0$.*

(ii) *If $B < 0$, then Eq. (3.8) has a unique positive root $\overline{\omega}_0$ and there exists one sequence of critical values of $\tau$ given by*

$$\tau_j^0 = \frac{1}{\overline{\omega}_0}\arccos\left(\frac{\overline{\omega}_0^2(\bar{\delta} - \alpha a) - \alpha^2\gamma\bar{\delta}(a - \gamma)}{\beta(\overline{\omega}_0^2 + \alpha^2\gamma^2)}\right) + \frac{2j\pi}{\overline{\omega}_0}, \quad \text{where } j \in \mathbb{N}.$$

(iii) *If $A < 0$, $B > 0$ and $\overline{\Delta} > 0$, then Eq. (3.8) has two positive roots $\overline{\omega}_\pm$ and there exist two sequences of critical values of $\tau$ given by*

$$\tau_j^\pm = \frac{1}{\overline{\omega}_\pm}\arccos\left(\frac{\overline{\omega}_\pm^2(\bar{\delta} - \alpha a) - \alpha^2\gamma\bar{\delta}(a - \gamma)}{\beta(\overline{\omega}_\pm^2 + \alpha^2\gamma^2)}\right) + \frac{2j\pi}{\overline{\omega}_\pm}, \quad \text{where } j \in \mathbb{N}.$$

We take $\lambda(\tau) = \sigma(\tau) + i\omega(\tau)$ as the root of Eq. (3.7) satisfying $\sigma(\tau_j^\pm) = 0$ and $\omega(\tau_j^\pm) = \overline{\omega}_\pm$. Differentiating Eq. (3.7) with respect to $\tau$, we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda - \alpha(a - \gamma) + \bar{\delta} - \beta e^{-\lambda\tau}}{-(\lambda + \alpha\gamma)\beta\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

It is not hard to have that

$$\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_j^\pm} = \frac{2\overline{\omega}_\pm^2 + \alpha^2(a - \gamma)^2 + \bar{\delta}^2 - \beta^2}{\beta^2(\overline{\omega}_\pm^2 + \alpha^2\gamma^2)} = \frac{\pm\overline{\Delta}}{\beta^2(\overline{\omega}_\pm^2 + \alpha^2\gamma^2)}.$$

As $\overline{\Delta} > 0$, we get

$$\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_j^+} > 0, \quad \text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_j^-} < 0.$$

In the same way, we have $\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\bigg|_{\tau=\tau_j^0} > 0$. Thus, the transversally condition is satisfied. According to Lemma 3.6, we have the following results.

**Theorem 3.7.** *Suppose that* $\bar{\delta} \geqslant -\beta$. *We have*

(i) *if* $a - \gamma < \dfrac{\gamma\beta}{\bar{\delta}}$, *then the economic equilibrium* $E^*$ *is locally asymptotically stable for all* $\tau \geqslant 0$ *and it is unstable for all* $\tau \geqslant 0$, *when* $a - \gamma > \dfrac{-\gamma\beta}{\bar{\delta}}$;

(ii) *if* $\dfrac{\gamma\beta}{\bar{\delta}} < a - \gamma < \min\left\{\dfrac{-\gamma\beta}{\bar{\delta}}, \dfrac{\bar{\delta} - \beta}{\alpha}\right\}$, *then system* (3.6) *undergoes a Hopf bifurcation at* $E^*$, *when* $\tau = \tau_j^0$, $j \in$ . *In addition,* $E^*$ *is locally asymptotically stable for all* $\tau < \tau_0^0$ *and unstable for all* $\tau > \tau_0^0$.

**Theorem 3.8.** *Suppose that* $\bar{\delta} < -\beta$. *We have*

(i) *if* $a - \gamma < \min\left\{\dfrac{\gamma\beta}{\bar{\delta}}, \dfrac{-\sqrt{\beta^2 - \bar{\delta}^2}}{\alpha}\right\}$, *then the economic equilibrium* $E^*$ *is locally asymptotically stable for all* $\tau \geqslant 0$, *and it is unstable for all* $\tau \geqslant 0$, *when* $a - \gamma > \dfrac{-\gamma\beta}{\bar{\delta}}$;

(ii) *if* $\dfrac{\gamma\beta}{\bar{\delta}} < a - \gamma < \min\left\{\dfrac{-\gamma\beta}{\bar{\delta}}, \dfrac{\bar{\delta} - \beta}{\alpha}\right\}$, *then system* (3.6) *undergoes a Hopf bifurcation at* $E^*$, *when* $\tau = \tau_j^0$, $j \in$ , *further,* $E^*$ *is locally asymptotically stable for all* $\tau < \tau_0^0$ *and unstable for all* $\tau > \tau_0^0$;

(iii) *if* $\dfrac{-\sqrt{\beta^2 - \bar{\delta}^2}}{\alpha} < a - \gamma < \dfrac{\gamma\beta}{\bar{\delta}}$ *and* $\overline{\Delta} > 0$, *then there is a positive integer* $n$ *such that the economic equilibrium* $E^*$ *is locally asymptotically stable, when* $\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup \cdots \cup (\tau_{n-1}^-, \tau_n^+)$, *and it is unstable when* $\tau \in [\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \cdots \cup (\tau_n^+, \tau_n^-) \cup (\tau_n^+, +\infty)$, *furthermore, system* (3.6) *undergoes a Hopf bifurcation at* $E^*$, *when* $\tau = \tau_j^{\pm}$, $j \in \mathbb{N}$.

## 4. Conclusion

In this work, we have proposed and analyzed the dynamics of a business cycle model with general investment and variable depreciation rate of capital stock. We have firstly proved that the proposed model is mathematically and economically well-posed. Moreover, we have established the local stability of the economic equilibrium and the existence of Hopf bifurcation. On the other hand, the recent business cycle models presented in [4, 8] are improved and generalized.

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