Conformal quasi-bi-slant Riemannian maps

Sushil Kumar\textsuperscript{a}, Sumeet Kumar\textsuperscript{b}, Deepak Kumar\textsuperscript{c},\textsuperscript{*}

\textsuperscript{a}Shri Jai Narain Post Graduate College, Lucknow, India.
\textsuperscript{b}Dr. S. K. S. Women’s College Motihari, B.R.Ambedkar Bihar University, India.
\textsuperscript{c}T. P. Varma College Narkatiyaganj, B.R.Ambedkar Bihar University, India.

Abstract

Conformal maps or horizontally conformal maps are very useful for characterization of harmonic morphisms. Nowadays, many medical problems (directly or indirectly) such as brain imaging (brain surface mapping, [Y. L. Wang, L. M. Lui, X. F. Gu, K. M. Hayashi, T. F. Chan, A. W. Toga, P. M. Thompson, S.-T. Yau, IEEE Transactions on Medical Imaging, 26 (2007), 853–865], [Y. L. Wang, X. F. Gu, K. M. Hayashi, T. F. Chan, P. M. Thompson, S.-T. Yau, IEEE Transactions on Medical Imaging, 23 (2004), 949–958]) computer graphics ([X. F. Gu, Y. L. Wang, T. F. Chan, P. M. Thompson, S.-T. Yau, Tenth IEEE International Conference on Computer Vision (ICCV’05), 2005 (2005), 1061–1066]) etc. can be solved using conformal Riemannian maps. In this paper, as a generalization of conformal Riemannian maps and conformal bi-slant submersions, we introduce conformal quasi-bi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. We study the geometry of leaves of distributions which are involved in the definition of the conformal quasi bi-slant Riemannian maps. We work out conditions for such maps to be integrable, totally geodesic and pluriharmonic. We present two examples for the introduced notion.

Keywords: Almost Hermitian manifolds, Riemannian maps, conformal quasi bi-slant Riemannian maps.

1. Introduction

In Riemannian geometry, the theory of smooth maps between Riemannian manifolds is a fascinating topic that continually generates new ideas which are very helpful in comparing geometric structures between manifolds. In this point of view, isometric immersions and submersions are basic such maps studied by O’Neill [18] and Gray [10]. In 1992, Fischer introduced the notion of Riemannian maps [8] as a generalization of isometric immersions and Riemannian submersions. More precisely, a smooth map $\pi: (B_1, g_{B_1}) \to (B_2, g_{B_2})$ between Riemannian manifolds such that $0 < \text{rank}\pi < \min\{m, n\}$, where $\dim B_1 = m$ and $\dim B_2 = n$. It satisfies the equation:

$$g_{B_1}(V_1, V_2) = g_{B_2}(\pi_* V_1, \pi_* V_2), \text{ for } V_1, V_2 \in \Gamma(\ker \pi_*)^\perp.$$
It follows that isometric immersions and Riemannian submersions are particular cases of Riemannian maps with \( \ker \pi_* = \{0\} \) and \( (\text{range } \pi_*)_\perp = \{0\} \), respectively. If we denote the kernel space of \( \pi_* \) by \( \ker \pi_* \) and the orthogonal complementary space of \( \ker \pi_* \) by \( (\ker \pi_*)_\perp \) in \( T\pi_1 \), then the \( T\pi_1 \) has the following orthogonal decomposition:

\[
T\pi_1 = \ker \pi_* \oplus (\ker \pi_*)_\perp.
\]

Also, if we denote the range of \( \pi_* \) by \( \text{range } \pi_* \) and for a point \( p \in B_1 \) the orthogonal complementary space of \( \text{range } \pi_* |_{\pi(p)} \) by \( (\text{range } \pi_* |_{\pi(p)})_\perp \) in \( T_{\pi(p)} \), then the tangent space \( T_{\pi(p)} \) has the following orthogonal decomposition:

\[
T_{\pi(p)} = (\text{range } \pi_* |_{\pi(p)}) \oplus (\text{range } \pi_* |_{\pi(p)})_\perp.
\]

A differentiable map \( \pi : (B_1, g_{B_1}) \to (B_2, g_{B_2}) \) is called a Riemannian map at \( p \in B_1 \) if the horizontal restriction \( \pi^h_\pi : (\ker \pi_* |_{\pi(p)})_\perp \to (\text{range } \pi_* |_{\pi(p)})_\perp \) is linear isometry between the inner product space \( ((\ker \pi_* |_{\pi(p)})_\perp, (g_{B_1} |_{\pi(p)})(\ker \pi_* |_{\pi(p)})_\perp) \) and \( (\text{range } \pi_* |_{\pi(p)}, (g_{B_2} |_{\pi(p)})_{\text{range } \pi_* |_{\pi(p)}}) \) (for details see [5]).

Fischer showed that such maps could be used to solve the generalized eikonal equation, i.e., it satisfies the generalized eikonal equation \( \| \pi_* \|^2 = \text{rank } \pi \). Since \( \text{rank } \pi \) is an integer valued function and \( \| \pi_* \|^2 \) is continuous function on the Riemannian manifold so the equality implies that \( \text{rank } \pi \) is locally constant and globally constant on connected components. Since energy density \( 2\varepsilon(\pi) = \| \pi_* \|^2 = \text{rank } \pi, i.e., \) density is quantized to integer if the Riemannian manifold is connected. Thus the eikonal equation is a bridge between geometric optics and physical optics. On the other hand, horizontally conformal maps were defined by Fuglede [9] and Ishihara [14] and these maps are useful for characterization of harmonic morphisms. Horizontally conformal maps (conformal maps) have applications in mathematics as well as in physics. Especially, within the Yang-Mills theory [6], Kaluza-Klein theory [12], supergravity and superstring theories ([7],[13]) redundant robotic chains [4] etc. Thus, the notion of Riemannian maps deserves through study from different perspectives.

Furthermore, Sahin [28] introduced the notion of conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition Theorems. After that, several kinds of conformal Riemannian maps were introduced and studied, some of them are like: conformal Riemannian maps ([28, 31]), conformal anti-invariant Riemannian maps [1], conformal semi-invariant Riemannian maps ([2, 32]), conformal slant Riemannian maps ([3]), etc. Likewise, these maps have been studied widely by many geometers (see also [15, 19–25, 27, 29, 30]) etc.

The present article is organized as follows. Section 2 contains some basic definitions needed throughout this paper. In Section 3, we define conformal quasi bi-slant Riemannian map from almost Hermitian manifolds to Riemannian manifolds and obtain some results on conformal quasi bi-slant Riemannian map from Kähler manifold to Riemannian manifold. In Section 4, some examples for this notion are provided.

2. Preliminaries

An almost Hermitian manifold \( (N_1, g_1, J) \) is called a Kähler manifold [36] if

\[
(\nabla_{W_1} J)W_2 = 0,
\]

for \( W_1, W_2 \in \Gamma(TN_1) \) with almost complex structure \( J \) and almost Hermitian metric \( g_1 \) on \( N_1 \).

Watson introduced the fundamental tensors of a submersion in [35]. It is known that the fundamental tensor play similar role to that of the second fundamental form of a submersion [16]. O’Neill’s tensors \( \mathcal{A} \) and \( \mathcal{J} \) [18], for vector fields \( V_1, V_2 \in \Gamma(TN_1) \), are defined as

\[
\mathcal{A}_{V_1} V_2 = \nabla_{\nabla_{V_1} J} V_2 + J(\nabla_{V_1}V_2), \quad \mathcal{J}_{V_1} V_2 = J(\nabla_{V_1} V_2) + \nabla_{\nabla_{V_1} J} V_2,
\]

where \( \nabla \) and \( J \) are the vertical and horizontal projections and \( \nabla \) is Levi-Civita connection \( N_1 \). On the other hand, from (2.2), we have

\[
\nabla_{Y_1} Y_2 = \mathcal{J}_{Y_1} Y_2 + \nabla_{Y_1} Y_2,
\]
for \(Y_1, Y_2 \in \Gamma(\ker \pi_*)\) and \(U_1, U_2 \in \Gamma(\ker \pi_*)\), where \(\nabla Y_i Y_2 = \nabla Y_i Y_2\). If \(U_1\) is basic, then \(A_{Y_1} U_1 = \mathcal{H}\nabla U_1 Y_1\).

It is seen that for \(p \in N_1, Y_1 \in \mathcal{V}_p\) and \(U_1 \in \mathcal{H}_p\) the linear operators \(A_{U_1}, \mathcal{V}_{Y_1} : T_p N_1 \to T_p N_1\) are skew-symmetric, that is

\[
g_1(A_{U_1} Z_1, Z_2) = -g_1(Z_1, A_{U_1} Z_2) \quad \text{and} \quad g_1(\mathcal{V}_{Y_1} Z_1, Z_2) = -g_1(Z_1, \mathcal{V}_{Y_1} Z_2),
\]

for each \(Z_1, Z_2 \in \Gamma(T_p N_1)\).

Let \(\pi : (N_1, g_1) \to (N_2, g_2)\) is a smooth map between Riemannian manifolds. Then the differential \(\pi_*\) of \(\pi\) can be observed a section of the bundle \(\text{Hom}(TN_1, \pi^{-1}TN_2) \to N_1\), where \(\pi^{-1}TN_2\) is the bundle which has fibres \((\pi^{-1}TN_2)_p = T_{\pi(p)}N_2\), has a connection \(\nabla\) induced from the Riemannian connection and \(\nabla^N_1\) pullback connection. Then the second fundamental form of \(\pi\) is given by

\[
(\nabla \pi_*)(V_1, V_2) = \nabla^N_1 \pi_* (V_2) - \pi_*(\mathcal{H}\nabla^N_1 V_1),
\]

for any vector fields \(V_1, V_2 \in \Gamma(TN_1)\), where \(\nabla^\pi\) is the pullback connection. We recollection that a differentiable map \(\pi\) between two Riemannian manifolds is called totally geodesic if

\[
(\nabla \pi_*)(V_1, V_2) = 0, \quad \text{for} \quad Y_1, Y_2 \in \Gamma(TN_1).
\]

**Definition 2.1.** Let \((N_1, g_1)\) and \((N_2, g_2)\) are two Riemannian manifolds with dimensions \(m\) and \(n\), respectively. If \(\pi : (N_1, g_1) \to (N_2, g_2)\) is a smooth map, then \(\pi\) is a conformal Riemannian map at \(p \in N_1\) if \(0 < \text{rank} \kappa_{\pi, p} < \min\{m, n\}\) and \(\pi_*\) maps \(\mathcal{H}_p = (\ker \pi_*^p)^\perp\) conformally onto \(\text{range}(\pi_*^p)\), i.e., there exists a number \(\lambda^2(p) \neq 0\) such that

\[
\lambda^2(p) g_1(V_1, V_2) = g_2(\pi_* V_1, \pi_* V_2),
\]

for \(V_1, V_2 \in (\ker \pi_*)^\perp\). \(\pi\) is called conformal Riemannian map if \(\pi\) is a conformal map at each point \(p \in N_1\). A conformal Riemannian map \(\pi\) is proper if \(\lambda \neq 1\).

On the other hand, let \(\pi : (N_1, g_1) \to (N_2, g_2)\) be a conformal map between Riemannian manifolds. Then, we get

\[
(\nabla \pi_*)(V_1, V_2) |_{\text{range}^\perp} = V_1(\text{ln} \lambda) \pi_* (V_2) + V_2(\text{ln} \lambda) \pi_* (V_1) - g_1(V_1, V_2) \pi_* (\text{grad} \text{ln} \lambda),
\]

where \(V_1, V_2 \in (\ker \pi_*^p)^\perp\). From equation (2.8), we get

\[
\nabla^N_1 \pi_* (V_2) = \pi_* (\mathcal{H}\nabla^N_1 V_1) + V_1(\text{ln} \lambda) \pi_* (V_2) + V_2(\text{ln} \lambda) \pi_* (V_1) - g_1(V_1, V_2) \pi_* (\text{grad} \text{ln} \lambda) + (\nabla \pi_*)^\perp (V_1, V_2),
\]

where \((\nabla \pi_*)^\perp (V_1, V_2)\) is the component of \((\nabla \pi_*)(V_1, V_2)\) on \((\text{range}^\perp)^\perp\) for \(V_1, V_2 \in (\ker \pi_*^p)^\perp\). Thus if we denote the \((\text{range}^\perp)^\perp\) component of \((\nabla \pi_*)(V_1, V_2)\) by \((\nabla \pi_*)(V_1, V_2) |_{\text{range}^\perp}\), we can write \((\nabla \pi_*)(V_1, V_2)\) as

\[
(\nabla \pi_*)(V_1, V_2) = (\nabla \pi_*)(V_1, V_2) |_{\text{range}^\perp} + (\nabla \pi_*)(V_1, V_2) |_{\text{range}^\perp^\perp},
\]

for \(V_1, V_2 \in (\ker \pi_*^p)^\perp\).

**Definition 2.2.** Let \((N_1, g_1, J)\) be an almost Hermitian manifold and \((N_2, g_2)\) be a Riemannian manifold with dimension \(m\) and \(n\), respectively. A map \(\pi\) from an almost Hermitian manifold \((N_1, g_1, J)\) to Riemannian manifold \((N_2, g_2)\) is pluriharmonic map [17] if

\[
(\nabla \pi_*)(Z_1, Z_2) + (\nabla \pi_*)(JZ_1, JZ_2) = 0,
\]

for \(Z_1, Z_2 \in \Gamma(TN_1)\).
3. Conformal quasi bi-slant Riemannian maps

**Definition 3.1.** Let \((N_1, g_1, J)\) be an almost Hermitian manifold and \((N_2, g_2)\) be a Riemannian manifold. A Riemannian map \(\pi: (N_1, g_1, J) \rightarrow (N_2, g_2)\) is called a conformal quasi bi-slant Riemannian map if there exist three mutually orthogonal distributions \(D, D_1,\) and \(D_2\) such that

(i) \(\ker \pi_* = D \oplus D_1 \oplus D_2;\)
(ii) \(J(D) = D,\) i.e., \(D\) is invariant;
(iii) \(J(D_1) \perp D_2\) and \(J(D_2) \perp D_1;\)
(iv) for any non-zero vector field \(Y_1 \in (D_1)_p, p \in N_1,\) the angle \(\theta_1\) between \(JY_1\) and \((D_1)_p\) is constant and independent of the choice of point \(p\) and \(Y_1\) in \((D_1)_p;\)
(v) for any non-zero vector field \(Y_2 \in (D_2)_q, q \in N_1,\) the angle \(\theta_2\) between \(JY_2\) and \((D_2)_q\) is constant and independent of the choice of point \(q\) and \(Z_2\) in \((D_2)_q.\)

These angles \(\theta_1\) and \(\theta_2\) are called slant angles of the Riemannian map.

Let \(\pi\) be conformal quasi bi-slant Riemannian map from an almost Hermitian manifold \((N_1, g_1, J)\) to a Riemannian manifold \((N_2, g_2)\). Then, we have

\[ TN_1 = \ker \pi_* \oplus (\ker \pi_*)^\perp. \]

Now, for any vector field \(U_1 \in \Gamma(\ker \pi_*),\) we have

\[ U_1 = PU_1 + QU_1 + RU_1, \quad (3.1) \]

where \(P, Q\) and \(R\) are projection morphisms of \(\ker \pi_*\) onto \(D, D_1\) and \(D_2\), respectively. For \(W_1 \in \Gamma(\ker \pi_*),\) we get

\[ JW_1 = \phi W_1 + \omega W_1, \quad (3.2) \]

where \(\phi W_1 \in (\Gamma \ker \pi_*)\) and \(\omega W_1 \in (\Gamma \ker \pi_*)^\perp.\) From equations \((3.1)\) and \((3.2),\) we have

\[ JU_1 = J(PU_1) + J(QU_1) + J(RU_1) = \phi(PU_1) + \omega(QU_1) + \phi(QU_1) + \omega(QU_1) + \phi(RU_1) + \omega(RU_1). \]

Since \(JD = D,\) we get \(\omega PU_1 = 0.\) Therefore, above equation reduces to

\[ JU_1 = \phi(PU_1) + \omega QU_1 + \phi QU_1 + \omega RU_1 + \omega RU_1. \]

Now, we have the following decomposition

\[ J(\ker \pi_*) = D \oplus (\phi D_1 \oplus \phi D_2) \oplus (\omega D_1 \oplus \omega D_2) , \]

where \(\oplus\) denotes orthogonal direct sum. Further, let \(V_1 \in \Gamma(D_1)\) and \(V_2 \in \Gamma(D_2).\) Then, we get

\[ g_1(V_1, V_2) = 0, \quad g_1(JV_1, V_2) = g_1(V_1, JV_2) = 0, \quad g_1(\phi V_1, V_2) = 0, \quad g_1(V_1, \phi V_2) = 0. \]

If \(W_1 \in \Gamma(D_1), W_2 \in \Gamma(D_1)\) and \(W_3 \in \Gamma(D_2),\) then

\[ g_1(\phi W_1, W_2) = 0, \quad g_1(\phi W_1, W_3) = 0, \quad g_1(\phi W_2, \phi W_3) = 0, \quad g_1(\omega W_2, \omega W_3) = 0. \]

So, we can write \(\phi D_1 \cap \phi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}.\) Since \(\omega D_1 \subseteq (\ker \pi_*)^\perp, \omega D_2 \subseteq (\ker \pi_*)^\perp,\) so we can write

\[ (\ker \pi_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mu, \]

where \(\mu\) is orthogonal complement of \((\omega D_1 \oplus \omega D_2)\) in \((\ker \pi_*)^\perp.\) Also, for any non-zero vector field \(X_1 \in (\ker \pi_*)^\perp,\) we have

\[ JX_1 = BX_1 + CX_1, \quad (3.3) \]

where \(BX_1 \in \Gamma(\ker \pi_*)\) and \(CX_1 \in \Gamma(\mu).\)
Lemma 3.2. If $\pi$ is a conformal quasi bi-slant Riemannian map, then
$$
\phi^2 U_1 + B\omega U_1 = -U_1, \omega \phi U_1 + C\omega U_1 = 0, \quad \omega B U_2 + C^2 U_2 = -U_2, \phi B U_2 + B C U_2 = 0,
$$
for $U_1 \in \Gamma(\ker \pi_*)$ and $U_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (3.2) and (3.3), we have Lemma 3.4. \hfill $\Box$

The proof of the following Lemma is exactly the same as that one for quasi bi-slant submersion, see Lemma 3.2 of [26]. So, we omit it.

Lemma 3.3. If $\pi$ is a conformal quasi-bi-slant Riemannian map, then
(i) $\phi^2 Y_1 = -(\cos^2 \theta_1) Y_1$;
(ii) $g_1(\phi Y_1, \phi Y_2) = \cos^2 \theta_1 g_1(Y_1, Y_2)$;
(iii) $g_1(\omega Y_1, \omega Y_2) = \sin^2 \theta_1 g_1(Y_1, Y_2)$, for $Y_1, Y_2 \in \Gamma(D_1)$, where $i = 1, 2$.

Lemma 3.4. If $\pi$ is a conformal quasi-bi-slant Riemannian map, then
$$
\begin{align*}
\nabla Y_1 \phi Y_2 + \mathcal{J} Y_1 \omega Y_2 &= \phi \nabla Y_1 Y_2 + B \mathcal{J} Y_1 Y_2, & (3.4) \\
\mathcal{J} Y_1 \phi Y_2 + \mathcal{H} \nabla Y_1 \omega Y_2 &= \omega \nabla Y_1 Y_2 + C \mathcal{J} Y_1 Y_2, & (3.5) \\
\nabla U_1, BU_2 + A U_1, CU_2 &= \phi A U_1, U_2 + B \mathcal{J} \nabla U_1, U_2, & (3.6) \\
A U_1, BU_2 + \mathcal{H} \nabla U_1, CU_2 &= \omega A U_1, U_2 + C \mathcal{J} \nabla U_1, U_2, & (3.7) \\
\nabla Y_1, BU_1 + \mathcal{J} Y_1, CU_1 &= \phi \mathcal{J} Y_1, U_1 + B \mathcal{J} \nabla Y_1, U_1, & (3.8) \\
\mathcal{J} Y_1, BU_1 + \mathcal{H} \nabla Y_1, CU_1 &= \omega \mathcal{J} Y_1, U_1 + C \mathcal{J} \nabla Y_1, U_1, & (3.9) \\
\nabla U_1, \phi Y_1 + A U_1, \omega Y_1 &= B A U_1, Y_1 + \phi \nabla U_1 Y_1, & (3.10) \\
A U_1, \phi Y_1 + \mathcal{J} \nabla U_1, \omega Y_1 &= C A U_1, Y_1 + \omega \nabla U_1 Y_1, & (3.11)
\end{align*}
$$
for any $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (2.1), (2.3), (2.4), (2.7), (2.8), (3.2), and (3.3), we get equations (3.4)-(3.11). \hfill $\Box$

Now, we define
$$
(\nabla V_1, \phi) V_2 = \nabla V_1, \phi V_2 - \phi \nabla V_1 V_2, \quad (\nabla V_1, \omega) V_2 = \mathcal{J} (\nabla V_1, \omega V_2 - \omega \nabla V_1 V_2),
$$
for any $V_1, V_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$.

Lemma 3.5. If $\pi$ is a conformal quasi-bi-slant Riemannian map, then
$$
\begin{align*}
(\nabla W_1, \phi) W_2 &= B \mathcal{J} W_1, W_2 - \mathcal{J} W_1, \omega W_2, & (3.12) \\
(\nabla Z_1, C) Z_2 &= \omega A Z_1, Z_2 - A Z_1, B Z_2, & (3.13)
\end{align*}
$$
for any vectors $W_1, W_2 \in \Gamma(\ker \pi_*)$ and $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (3.4), (3.5), (3.6), (3.7), and (3.12), we get all equations of Lemma 3.5. \hfill $\Box$

If the tensors $\phi$ and $\omega$ are parallel with respect to the linear connection $\nabla$ on $N_1$, respectively, then
$$
B \mathcal{J} Y_1 Y_2 = \mathcal{J} Y_1 Y_2, \quad C \mathcal{J} Y_1 Y_2 = \mathcal{J} Y_1 Y_2,
$$
for any $Y_1, Y_2 \in \Gamma(TN_1)$. 
Theorem 3.6. D is integrable if and only if

\[ ∀V_1 J V_2 - ∀V_2 J V_1 ∈ \Gamma(D), \phi(\mathcal{J}_{V_1 - V_2} \omega Z_1) ∈ \Gamma(D_1 + D_2), \]

for \( V_1, V_2 ∈ \Gamma(D) \) and \( Z_1 ∈ \Gamma(D_1 + D_2) \).

Proof. Using equations (2.1), (2.3), (2.4), and (3.2), we have

\[
g_1([V_1, V_2], Z_1) = g_1(∀V_1 J V_2, JZ_1) - g_1(∀V_2 J V_1, JZ_1)
\]

\[
= g_1(∀V_1 J V_2, φZ_1) - g_1(∀V_2 J V_1, φZ_1) + g_1(∀V_1 J V_2, ωZ_1) - g_1(∀V_2 J V_1, ωZ_1)
\]

for \( V_1, V_2 ∈ \Gamma(D) \) and \( Z_1 ∈ \Gamma(D_1 + D_2) \), which completes the proof.

Theorem 3.7. D_1 is integrable if and only if

\[
\frac{1}{\lambda^2} \left\{ g_2((\nabla π_*)(U_1, PW_1), π_*(ω φ U_2)) - g_2((\nabla π_*)(U_2, PW_1), π_*(ω φ U_1)) + g_2((\nabla π_*)(U_1, U_2) - (∇ π_*)(U_2, ω U_1), π_*(ω RW_1)) \right. \\
\left. - g_2(π_*(ω U_2), π_*(U_1, JPW_1)) + g_2(π_*(ω U_1), π_*(U_2, JPW_1)) \right\}
\]

for \( U_1, U_2 ∈ \Gamma(D_1) \) and \( W_1 ∈ \Gamma(D_1 + D_2) \).

Proof. For \( U_1, U_2 ∈ \Gamma(D_1) \) and \( W_1 ∈ \Gamma(D_1 + D_2) \), we have

\[
g_1([U_1, U_2], W_1) = g_1(∀U_1, U_2, W_1) - g_1(∀U_1, U_1, W_1).
\]

Using equations (2.1), (2.3), (2.4), (3.1), (3.2), and Lemma 3.3, we have

\[
g_1([U_1, U_2], W_1) = g_1(∀U_1, U_2, J W_1) - g_1(∀U_1, JU_1, J W_1)
\]

\[
= \cos^2 θ_1 g_1(∀U_1, U_2, PW_1) - \cos^2 θ_1 g_1(∀U_1, U_1, PW_1) + g_1(ω φ U_2, TJ_1 U_1, PW_1) \]

\[
- g_1(ω φ U_1, TJ_1 U_1, PW_1) + g_1(∀U_1, φ U_2 - ∀U_1, φ U_1, φ RW_1) + g_1(∀U_1, ω U_2 - TJ_1 U_1, φ RW_1) \]

\[
+ g_1(∀U_1, U_2 - TJ_1 U_1, φ RW_1) + g_1(∀U_1, ω U_2 - TJ_1 U_1, φ RW_1).
\]

Since \( π \) is conformal Riemannian map, using equations (2.8) and (2.9), we have

\[
g_1([U_1, U_2], W_1) - \cos^2 θ_1 g_1([U_1, U_2], W_1)
\]

\[
= g_1(∀U_1, ω U_2 - J(∀U_2 - ∀U_1, ω RW_1)) + g_1(∀U_1, φ U_2 - ∀U_1, φ U_1, φ RW_1)
\]

\[
- \frac{1}{λ^2} g_2((∀ π_*)(U_1, PW_1), π_*(ω φ U_2)) + \frac{1}{λ^2} g_2((∀ π_*)(U_2, PW_1), π_*(ω φ U_1))
\]

\[
- \frac{1}{λ^2} g_2((∀ π_*)(U_1, ω U_2) - (∇ π_*)(U_2, ω U_1), π_*(ω RW_1)) + \frac{1}{λ^2} g_2(∀ π_*(ω U_2), π_*(U_1, JPW_1))
\]

\[
- \frac{1}{λ^2} g_2(∀ π_*(ω U_1), π_*(U_2, JPW_1)) + g_1(∀ U_1, ω U_2 - TJ_1 U_1, φ RW_1) + g_1(∀ U_1, ω U_2 - TJ_1 U_1, φ RW_1),
\]

which completes the proof.
Theorem 3.10. \( D_2 \) is integrable if and only if
\[
\frac{1}{\lambda^2} \left( g_2(\nabla \pi_1(Y_1, PX_1), \pi_\ast(\omega \phi Y_2)) - g_2(\nabla \pi_1(Y_2, PX_1), \pi_\ast(\omega \phi Y_1)) \right) \\
+ g_2(\nabla \pi_1(Y_1, \omega Y_2) - (\nabla \pi_1)(Y_2, \omega Y_1), \pi_\ast(\omega \phi Y_1)) \\
- g_2(\pi_\ast(\omega Y_2), \pi_\ast(\omega Y_1, \omega Y_2, \pi_\ast(\omega \phi Y_1)) \\
= g_1(\mathcal{H}(\nabla Y_1, \omega Y_2 - \mathcal{H}(\nabla Y_1, \omega Y_2, \omega \phi Y_1)) + g_1(\nabla Y_1, \omega Y_2 - \nabla Y_1, \phi Y_1, \omega \phi Y_1) \\
+ g_1(\mathcal{H}(\nabla Y_1, \omega Y_1, \phi Y_1) \\
for Y_1, Y_2 \in \Gamma(D^2) and X_1 \in \Gamma(D \oplus D_1).
\]

Theorem 3.9. \((\ker \pi_\ast)^\perp\) is integrable if and only if
\[
g_1(\nabla \eta_2, \omega \eta_1) = g_1(\nabla \eta_2, \omega \eta_1) \\
- g_1(\nabla \eta_2, \omega \eta_1) \\
g_1(\nabla \eta_2, \omega \eta_1) \\
= g_1(\mathcal{H}(\nabla Y_1, \omega Y_2, \omega \eta_1) \\
\]
Proof. For $Z_1, Z_2 \in \Gamma(\ker \pi_\ast)$ and $\eta \in \Gamma(\ker \pi_\ast)$, using equations (2.1), (3.2), and Lemma 3.3, we have
\[
g_1(\nabla_{Z_1}Z_2, \eta) = g_1(\nabla_{Z_1}JZ_2, JP\eta) + g_1(\nabla_{Z_1}JZ_2, JQ\eta) + g_1(\nabla_{Z_1}JRZ_2, JR\eta),
\]
\[
= g_1(\nabla_{Z_1}Z_2, P\eta + \cos^2 \theta_1 Q\eta + \cos^2 \theta_2 R\eta)
- g_1(\nabla_{Z_1}Z_2, \omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta) + g_1(\nabla_{Z_1}JZ_2, \omega Q\eta + \omega R\eta).
\]
Since $\omega P\eta + \omega Q\eta + \omega R\eta = \omega \eta$, $\omega P\eta = 0$ and using equations (2.5) and (2.6), we get
\[
g_1(\nabla_{Z_1}Z_2, \eta) = g_1(A_{Z_1}Z_2, P\eta + \cos^2 \theta_1 Q\eta + \cos^2 \theta_2 R\eta) + g_1(A_{Z_1}BZ_2, \omega \eta)
- g_1(J(\nabla_{Z_1}Z_2, \omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta) + g_1(J(\nabla_{Z_1}CZ_2, \omega \eta).
\]
Since $\pi$ is conformal Riemannian map, using equations (2.8) and (2.9), we have
\[
g_1(\nabla_{Z_1}Z_2, \eta) = g_1(A_{Z_1}Z_2, P\eta + \cos^2 \theta_1 Q\eta + \cos^2 \theta_2 R\eta) + g_1(A_{Z_1}BZ_2, \omega \eta)
- \frac{1}{\lambda^2} g_2(\nabla_{Z_1}z, Z_2, \pi_\ast(\omega \phi P\eta + \omega\phi Q\eta + \omega\phi R\eta))
+ \frac{1}{\lambda^2} g_2(\nabla_{\nabla_{Z_1}z}, \pi_\ast(\omega \phi Q\eta + \omega\phi R\eta))
+ \frac{1}{\lambda^2} g_2(\nabla_{\nabla_{Z_1}z}, \pi_\ast(\omega \phi R\eta))
- \frac{1}{\lambda^2} g_2(\nabla_{\nabla_{Z_1}z}, \omega \phi Q\eta + \omega\phi R\eta)
- \frac{1}{\lambda^2} g_2(\nabla_{\nabla_{Z_1}z}, \omega \phi R\eta),
\]
which completes the proof. $\square$

Theorem 3.11. \((\ker \pi_\ast)\) defines a totally geodesic foliation on $N_1$ if and only if
\[
\mathcal{J}W_1PW_2 + \cos^2 \theta_1 J_{W_1}QW_2 + \cos^2 \theta_2 J_{W_1}RW_2 - \mathcal{H}\nabla_{W_1}\omega\phi W_2 - \omega J_{W_1}wW_2 - C\mathcal{H}\nabla_{W_1}\omega W_2 = 0,
\]
for $W_1, W_2 \in \Gamma(\ker \pi_\ast)$.

Proof. For $W_1, W_2 \in \Gamma(\ker \pi_\ast)$, using equations (2.1), (2.3), (2.4), (2.8), (3.2), (3.3), and Lemma 3.3, we have
\[
(\nabla_{\pi_\ast})(W_1, W_2) = \pi_\ast(J_{W_1}W_2),
\]
\[
= \pi_\ast(J_{W_1}W_1, \omega PW_2 + J_{W_1}wW_2 + J_{W_1}QW_2 + J_{W_1}wQW_2
+ J_{W_2}wRw_2),
\]
\[
= \pi_\ast(-J_{W_1}PW_2 - \nabla_{W_1}PW_2 - \cos^2 \theta_1 J_{W_1}QW_2 - \cos^2 \theta_1 J_{W_1}QW_2 - \cos^2 \theta_2 J_{W_1}IW_2
- \cos^2 \theta_2 J_{W_1}QW_2 + J_{W_1}w\phi PW_2 + J_{W_1}w\phi QW_2 + J_{W_1}w\phi RW_2
+ J_{W_2}w\phi QW_2 + J_{W_1}w\phi QW_2 + \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ C\mathcal{J}\nabla_{W_1}wQW_2 + \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2
+ \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2.
\]
Since $PW_2 + QW_2 + RW_2 = W_2, \omega PW_2 + \omega QW_2 + \omega RW_2 = \omega W_2$ and $\omega PW_2 = 0$, we get
\[
(\nabla_{\pi_\ast})(W_1, W_2) = \pi_\ast(-J_{W_1}PW_2 - \nabla_{W_1}PW_2 - \cos^2 \theta_1 J_{W_1}QW_2 - \cos^2 \theta_1 J_{W_1}QW_2
- \cos^2 \theta_2 J_{W_1}RW_2 - \cos^2 \theta_2 J_{W_1}QW_2 + J_{W_1}w\phi PW_2 + J_{W_1}w\phi QW_2
+ J_{W_2}w\phi QW_2 + J_{W_1}w\phi QW_2 + \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ C\mathcal{J}\nabla_{W_1}wQW_2 + \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2
+ \phi J_{W_1}wRw_2 + \omega J_{W_1}wQW_2 + \mathcal{B}(\nabla_{W_1}wQW_2.
\]
the proof follows from the above equation. $\square$

Theorem 3.12. D defines a totally geodesic foliation on $N_1$ if and only if
\[
\omega \nabla_{Y_1} J_{Y_2} + C\mathcal{J}Y_1 J_{Y_2} = 0,
\]
for $Y_1, Y_2 \in \Gamma(D)$.
From equations (3.13) and (3.14), we have
\[(\nabla \pi_*)(Y_1, Y_2) = \pi_*(\nabla Y_1) Y_2 = \pi_*(\phi \nabla Y_1) Y_2 + \omega \nabla Y_1 JY_2 + B \nabla Y_1 JY_2 + C JY_1 JY_2),\]
for all \(Y_1, Y_2 \in \Gamma(D)\), which completes the proof.

**Theorem 3.13.** \(D_1\) defines a totally geodesic foliation on \(N_1\) if and only if
\[\mathcal{H} \nabla Y_1 \omega \phi Y_2 + C \mathcal{H} \nabla Y_1 \omega Y_2 + \omega JY_1 \omega Y_2 = 0,\]
for all \(Y_1, Y_2 \in \Gamma(D_1)\).

**Proof.** For all \(Y_1, Y_2 \in \Gamma(D_1)\), using equations (2.1), (2.3), (2.8), (2.9), and Lemma 3.3, we have
\[(\nabla \pi_*)(Y_1, Y_2) = \pi_*(\nabla Y_1 \omega \phi Y_2 + \nabla Y_1 \omega Y_2) = \pi_*(\phi \nabla Y_1 \omega Y_2 + J \nabla Y_1 \omega Y_2) + J \nabla Y_1 \omega Y_2).
\]
Now, using equations (2.4) and (3.3), we have
\[\sin^2 \theta_1 (\nabla \pi_*)(Y_1, Y_2) = \pi_*([\mathcal{H} \nabla Y_1 \omega \phi Y_2 + \mathcal{J} \omega Y_2 + B \mathcal{H} \nabla Y_1 \omega Y_2 + C \mathcal{H} \nabla Y_1 \omega Y_2 + \phi \mathcal{J} \omega Y_2 + \omega \mathcal{J} \omega Y_2)],\]
which completes the proof.

**Theorem 3.14.** \(D_2\) defines a totally geodesic foliation on \(N_1\) if and only if
\[\mathcal{H} \nabla U_1 \omega \phi U_2 + C \mathcal{H} \nabla U_1 \omega U_2 + \omega \mathcal{J}U_1 \omega U_2 = 0,\]
for all \(U_1, U_2 \in \Gamma(D_2)\).

**Proof.** The proof of the above theorem follows the similar approach as the proof of Theorem 3.14.

**Theorem 3.15.** Let \(\pi\) be a conformal quasi-bi-slant Riemannian map from a Kähler manifold \((N_1, g_1, J)\) to a Riemannian manifold \((N_2, g_2)\). Then, any two of following assertions imply the third one:

(i) the horizontal distribution \((\ker \pi_*)^\perp\) defines totally geodesic foliation on \(N_1\);

(ii) the map \(\pi\) is a horizontally homothetic map;

(iii) \(\nabla X, \pi_*(CX_2) = \pi_*(J[X_1, X_2]) + (\nabla X_1)(CX_1, CX_2) + \pi_*(A_{CX_2} BX_1 + A_{CX_1} BX_2 + \mathcal{J} BX_1 BX_2), \) for \(X_1, X_2 \in \ker \pi_*\)^{\perp}.

**Proof.** For all \(X_1, X_2 \in \ker \pi_*\)^{\perp}, using equations (2.1), (2.3), (2.5), (2.8), (2.9), and (3.3), we get
\[\pi_*(\nabla X_1 JX_2) = \nabla X_1 \pi_*(CX_2) - (\nabla \pi_*)(CX_1, BX_2) - (\nabla \pi_*)(BX_1, BX_2)\]
\[= \nabla X_1 \pi_*(CX_2) - (\nabla \pi_*)(CX_1, CX_2) - \pi_*(A_{CX_2} BX_1 + A_{CX_1} BX_2 + \mathcal{J} BX_1 BX_2)\]
\[= - CX_1 [\ln \lambda] \pi_*(CX_2) - CX_2 [\ln \lambda] \pi_*(CX_1) + g_1(CX_1, CX_2) \pi_*(\text{grad} \ln \lambda).\] (3.13)

On the other hand, we get
\[\nabla X_1 JX_2 = J[X_1, X_2] + J \nabla X_2 JX_1, \quad \nabla X_1 X_2 = J[X_1, X_2] - \nabla X_1 JX_2.\] (3.14)

From equations (3.13) and (3.14), we have
\[\pi_*(\nabla X_1 X_1) = \pi_*(J[X_1, X_2]) - \nabla \pi_*(CX_2) + (\nabla \pi_*)(CX_1, CX_2)\]
\[= \pi_*(A_{CX_2} BX_1 + A_{CX_1} BX_2 + \mathcal{J} BX_1 BX_2) + \pi_*(CX_1 [\ln \lambda] \pi_*(CX_1) + g_1(CX_1, CX_2) \pi_*(\text{grad} \ln \lambda).\] (3.15)
Now, taking assertions (i) and (ii) in equation (3.15), we get the (iii). Taking assertions (ii) and (iii) and equation (3.15), we get \( \pi_{*}(\nabla_{X_{1}}X_{1}) = 0 \). Hence, the horizontal distribution \( (\ker \pi_{*})^\perp \) defines totally geodesic foliation on \( N_{1} \). Further, using assertions (i) and (iii) and equation (3.15), we have

\[
CX_{1}(\ln \lambda)\pi_{*}(CX_{2}) + CX_{2}(\ln \lambda)\pi_{*}(CX_{1}) - g_{1}(CX_{1}, CX_{2})\pi_{*}(\text{grad} \ln \lambda) = 0,
\]

for \( CX_{1} \in \Gamma(\mu) \). Taking \( X_{2} = X_{1} \) in the above equation, we obtain

\[
CX_{1}(\ln \lambda)\pi_{*}(CX_{1}) + CX_{1}(\ln \lambda)\pi_{*}(CX_{1}) - g_{1}(CX_{1}, CX_{1})\pi_{*}(\text{grad} \ln \lambda) = 0.
\]

(3.16)

Taking inner product in equation (3.16) with \( \pi_{*}(CX_{1}) \), we get

\[
\lambda^{2}CX_{1}(\ln \lambda)g_{1}(CX_{1}, CX_{1}) = 0.
\]

(3.17)

It gives \( \lambda \) is a constant on \( \mu \), for all \( Y_{1} \in \Gamma(\ker \pi_{*}) \) and \( \omega Y_{1} \in \Gamma(\omega D_{1} \oplus \omega D_{2}) \). Similarly, taking inner product in equation (3.16) with \( \pi_{*}(\omega Y_{1}) \), we have

\[
\lambda^{2}\omega Y_{1}(\ln \lambda)g_{1}(CX_{1}, CX_{1}) = 0.
\]

(3.18)

It means \( \lambda \) is a constant on \( \Gamma(\omega D_{1} \oplus \omega D_{2}) \). Therefore, \( \lambda \) is constant on horizontal distribution. Thus, from equations (3.17) and (3.18), we obtain (iii) one.

**Theorem 3.16.** Let \( \pi \) be a conformal quasi-bi-slant Riemannian map from a Kähler manifold \((N_{1}, g_{1}, J)\) to a Riemannian manifold \((N_{2}, g_{2})\). Then the map \( \pi_{*} \) defines totally geodesic foliations on \( N_{1} \) if and only if

(i) the map \( \pi \) is a horizontally homothetic map; and

(ii) \( \nabla_{X}^{N_{2}}\pi_{*}(X_{2}) + (\nabla \pi_{*})^{\perp}(X_{1}, X_{2}) = \pi_{*}(A_{X_{1}}Y_{2} + J_{Y_{1}}Y_{2} + J(\nabla_{X_{1}}^{N_{1}}X_{2}) + \nabla_{X_{1}}^{N_{1}}\pi_{*}(X_{2})) \), provided for \( X, Y \in \Gamma(TN_{1}) \), where \( X_{1}, X_{2} \) and \( Y_{1}, Y_{2} \) are horizontal and vertical parts of \( X \) and \( Y \), respectively.

**Proof.** From equations (2.4), (2.5), (2.6), and (2.8), we get

\[
(\nabla \pi_{*})(X, Y) = \nabla_{X}^{N_{2}}\pi_{*}(X_{2}) - \pi_{*}(\nabla_{X_{1}}^{N_{1}}Y_{2} + \nabla_{Y_{1}}^{N_{1}}X_{2} + \nabla_{X_{1}}^{N_{1}}\pi_{*}(X_{2}) + \nabla_{X_{1}}^{N_{1}}\pi_{*}(X_{2}))
\]

\[
= \nabla_{X}^{N_{2}}\pi_{*}(X_{2}) - \pi_{*}(A_{X_{1}}Y_{2} + J_{Y_{1}}Y_{2} + J(\nabla_{X_{1}}^{N_{1}}X_{2}) - \nabla_{X_{1}}^{N_{1}}\pi_{*}(X_{2})),
\]

for \( X, Y \in \Gamma(TN_{1}) \), \( X_{1}, X_{2} \) and \( Y_{1}, Y_{2} \) are horizontal and vertical parts of \( X \) and \( Y \), respectively. Using equation (2.9), we get

\[
(\nabla \pi_{*})(X, Y) = \nabla_{X}^{N_{2}}\pi_{*}(X_{2}) - \pi_{*}(A_{X_{1}}Y_{2} + J_{Y_{1}}Y_{2} + J(\nabla_{X_{1}}^{N_{1}}X_{2}) - \nabla_{X_{1}}^{N_{1}}\pi_{*}(X_{2}))
\]

\[
+ X_{1}(\ln \lambda)\pi_{*}(X_{2}) + X_{2}(\ln \lambda)\pi_{*}(X_{1}) - g_{1}(X_{1}, X_{2})\pi_{*}(\text{grad} \ln \lambda) + (\nabla \pi_{*})^{\perp}(X_{1}, X_{2}).
\]

(3.19)

Since \( \pi \) defines totally geodesic foliation on \( N_{1} \), we have (3.19). When we take \( \pi \) is a horizontally homothetic map then from equation (3.19), we get

\[
X_{1}(\ln \lambda)\pi_{*}(X_{2}) + X_{2}(\ln \lambda)\pi_{*}(X_{1}) - g_{1}(X_{1}, X_{2})\pi_{*}(\text{grad} \ln \lambda) = 0.
\]

(3.20)

From equation (3.20), we have

\[
\lambda^{2}X_{2}(\ln \lambda)g_{1}(X_{1}, X_{1}) = 0,
\]

(3.21)

for \( X_{1} \in \Gamma(\ker \pi_{*})^{\perp} \). From equation (3.21), \( \lambda \) is a constant on horizontal distribution. Since \( \pi \) is a horizontally homothetic map, we get assertion (i). Further, from equation (3.19), we have

\[
\nabla_{X}^{N_{2}}\pi_{*}(X_{2}) = \pi_{*}(A_{X_{1}}Y_{2} + J_{Y_{1}}Y_{2} + J(\nabla_{X_{1}}^{N_{1}}X_{2}) + \nabla_{X_{1}}^{N_{1}}\pi_{*}(X_{2}) - (\nabla \pi_{*})^{\perp}(X_{1}, X_{2}).
\]

(3.22)

From equation (3.22), we get (ii).
Theorem 3.17. Let π be a conformal quasi-bi-slant Riemannian map from a Kähler manifold \((N_1, g_1, J_1)\) to a Riemannian manifold \((N_2, g_2)\). If π is a D-pluriharmonic map then one of the below assertions imply the second one:

(i) D defines totally geodesic foliation on \(N_1\);
(ii) \(\mathcal{C}TX_1X_2 + \omega \nabla \nabla JX_1X_2 = 0\), for \(X_1, X_2 \in \Gamma(D)\).

Proof. By definition of pluriharmonic map, we have

\[0 = (\nabla \pi_*)(X_1, X_2) + (\nabla \pi_*)(JX_1, JX_2),\]

for \(X_1, X_2 \in \Gamma(D)\). From equations (2.1), (2.4), (2.5), (2.8), (3.2), and (3.3), we get

\[\pi_*(\nabla_{X_1}X_2) = -\pi_*(J(\nabla_{JX_1}X_2)),\]
\[\pi_*(\nabla_{X_1}X_2) = -\pi_*(J(\mathcal{C}TX_1X_2 + \nabla \nabla JX_1X_2)),\]
\[\pi_*(\nabla_{X_1}X_2) = -\pi_*(B\mathcal{C}TX_1X_2 + \mathcal{C}TX_1X_2 + \phi \nabla \nabla JX_1X_2 + \omega \nabla \nabla JX_1X_2).\]  \hspace{1cm} (3.23)

Taking assertion (i) in equation (3.23), we obtain (ii) as, \(\mathcal{C}TX_1X_2 + \omega \nabla \nabla JX_1X_2 = 0\). Similarly, taking assertion (ii) in equation (3.23), we get (i) one. \(\square\)

Theorem 3.18. Let π be a conformal quasi-bi-slant Riemannian map from a Kähler manifold \((N_1, g_1, J_1)\) to a Riemannian manifold \((N_2, g_2)\). If π is a D\(_1\)-pluriharmonic map then any two of the following assertions imply the third one:

(i) \(D_1\) defines totally geodesic foliation on \(N_1\);
(ii) \(\lambda\) is constant on \(\omega D_1\) and \((\nabla \pi_*)^{-1}(\omega Y_1, \omega Y_2) = 0\);
(iii) \(\cos^2 \theta_1(\mathcal{C}TX_1Y_1 + \omega \nabla \nabla JX_1Y_1) = (\mathcal{C}TX_1Y_1 - \omega JX_1Y_1 - \omega JX_1Y_1)\), for \(Y_1, Y_2 \in \Gamma(D_1)\).

Proof. By definition of pluriharmonic map, we get

\[0 = (\nabla \pi_*)(Y_1, Y_2) + (\nabla \pi_*)(JY_1, JY_2),\]

for \(Y_1, Y_2 \in \Gamma(D_1)\). Using equations (2.1), (2.5), (2.9), and Lemma 3.3, we get

\[\pi_*(\nabla_{Y_1}Y_2) = -\pi_*(\nabla_{Y_1}Y_2) - \pi_*(\nabla_{Y_1}Y_1) - \pi_*(\nabla_{Y_1}Y_2) - \pi_*(\nabla_{Y_1}Y_2),\]
\[\pi_*(\nabla_{Y_1}Y_2) = \pi_*(J\nabla_{Y_1}Y_2) - \pi_*(\nabla_{\omega Y_1}Y_1) + (\nabla \pi_*)^{-1}(\omega Y_1, \omega Y_2) + \omega Y_1(\ln \lambda) \pi_*(\omega Y_2) + \omega Y_2(\ln \lambda) \pi_*(\omega Y_1)\]
\[\pi_*(\nabla_{Y_1}Y_2) = -\cos^2 \theta_1 \pi_*(B\mathcal{C}TX_1Y_1 + \mathcal{C}TX_1Y_1 + \phi \nabla \nabla JX_1Y_1 + \omega \nabla \nabla JX_1Y_1)\]
\[\pi_*(\nabla_{Y_1}Y_2) = \pi_*(B\mathcal{C}TX_1Y_1 + \mathcal{C}TX_1Y_1 + \phi \nabla \nabla JX_1Y_1 + \omega \nabla \nabla JX_1Y_1)\]
\[\pi_*(\nabla_{Y_1}Y_2) = \pi_*(\nabla_{\omega Y_1}Y_1) + (\nabla \pi_*)^{-1}(\omega Y_1, \omega Y_2) + \omega Y_1(\ln \lambda) \pi_*(\omega Y_2) + \omega Y_2(\ln \lambda) \pi_*(\omega Y_1)\]
\[\pi_*(\nabla_{Y_1}Y_2) = -\cos^2 \theta_1 \pi_*(B\mathcal{C}TX_1Y_1 + \mathcal{C}TX_1Y_1 + \phi \nabla \nabla JX_1Y_1 + \omega \nabla \nabla JX_1Y_1)\]  \hspace{1cm} (3.24)
Now, taking assertions (i) and (ii) in equation (3.24), we get
\[ \pi_*(\nabla_{Y_1} Y_2) = 0, \]
\[ \omega Y_1 (\ln \lambda) \pi_*(\omega Y_2) + \omega Y_2 (\ln \lambda) \pi_*(\omega Y_1) - g_1(\omega Y_1, \omega Y_2) \pi_*(\text{grad } \ln \lambda) = 0, \]
\[ (\nabla \pi_*)^\perp(\omega Y_1, \omega Y_2) = 0, \]
respectively. We obtain (iii) as
\[ \cos^2 \theta(C\mathcal{T}_{\phi Y_1} Y_2 + \omega \nabla_{\phi Y_1} Y_2) = (C\mathcal{J}(\mathcal{V}^N_{\phi Y_1} \omega \phi Y_2 + \omega \mathcal{T}_{\phi Y_1} \omega \phi Y_2) - (A_{\omega Y_2} \omega Y_1 + A_{\omega Y_1} \phi Y_2). \]
Taking assertions (ii) and (iii) in equation (3.24), we get (i). Lastly, suppose that (i) and (iii) are satisfied in equation (3.24). Then, we get
\[ \omega Y_1 (\ln \lambda) \pi_*(\omega Y_2) + \omega Y_2 (\ln \lambda) \pi_*(\omega Y_1) - g_1(\omega Y_1, \omega Y_2) \pi_*(\text{grad } \ln \lambda) = 0. \] (3.25)
Taking inner product in equation (3.25) with \( \pi_*(\omega Y_1) \), we get
\[ \lambda^2 \omega Y_2 (\ln \lambda) g_1(\omega Y_1, \omega Y_1) = 0, \] (3.26)
for all \( \omega Y_1 \in \Gamma(D_1) \). We have \( \omega Y_2 (\ln \lambda) = 0 \), from equation (3.26), i.e., \( \omega D_1 (\ln \lambda) = 0 \). Hence, we obtain assertion (ii). Thus, we complete the proof.

In a similar way as above theorem, we obtain the following theorem.

**Theorem 3.19.** Let \( \pi \) be a conformal quasi-bi-slant Riemannian map from a Kähler manifold \((N_1, g_1, J)\) to a Riemannian manifold \((N_2, g_2)\). If \( \pi \) is a \( D_2 \)-pluri-harmonic map then any two of the following assertions imply the third one:

(i) \( D_2 \) defines totally geodesic foliation on \( N_1 \);  
(ii) \( \lambda \) is constant on \( \omega D_2 \) and \( (\nabla \pi_*) (\omega Z_1, \omega Z_2) = 0 \);  
(iii) \[ \cos^2 \theta(C\mathcal{T}_{\phi Z_1} Z_2 + \omega \nabla_{\phi Z_1} Z_2) = (C\mathcal{J}(\mathcal{V}^N_{\phi Z_1} \omega \phi Z_2 + \omega \mathcal{T}_{\phi Z_1} \omega \phi Z_2) - (A_{\omega Z_2} \omega Z_1 + A_{\omega Z_1} \phi Z_2), \]
for \( Z_1, Z_2 \in \Gamma(D_2) \).

**Theorem 3.20.** Let \( \pi \) be a conformal quasi-bi-slant Riemannian map from a Kähler manifold \((N_1, g_1, J)\) to a Riemannian manifold \((N_2, g_2)\). If \( \pi \) is a \( (\ker \pi_*)^\perp \)-pluri-harmonic map then any two of the following assertions imply the third one:

(i) \( (\ker \pi_*)^\perp \) defines totally geodesic foliation on \( N_1 \);  
(ii) \( \lambda \) is a constant on \( \mu \);  
(iii) \[ \nabla^a_{X_1} \pi_*(X_2) = \pi_*([\mathcal{B}_X, X_1]B_{X_2} + A_{C_{X_1}}B_{X_2} + A_{C_{X_2}}B_{X_1}) + (\nabla \pi_*)^\perp(C_{X_1}, C_{X_2}), \]
for \( X_1, X_2 \in \Gamma(\ker \pi_*)^\perp \).

**Proof.** By definition of pluri-harmonic map, we have
\[ (\nabla \pi_*) (X_1, X_2) + (\nabla \pi_*) (JX_1, JX_2) = 0, \]
for \( X_1, X_2 \in \Gamma(\ker \pi_*)^\perp \). Now, using equations (2.1), (2.3), (2.5), and (2.9), we get
\[ 0 = \nabla^a_{X_1} \pi_*(X_2) - \pi_*(\nabla^a_{X_1} X_2) - \pi_*([\mathcal{B}_X, X_1]B_{X_2} + V \nabla B_{X_1} B_{X_2} \]
\[ + A_{C_{X_1}}B_{X_2} + A_{C_{X_2}}B_{X_1} + V \nabla_{C_{X_1}} B_{X_2} + V \nabla_{C_{X_2}} B_{X_1}) \]
\[ + C_{X_1} (\ln \lambda) \pi_*(C_{X_2}) + C_{X_2} (\ln \lambda) \pi_*(C_{X_1}) - g_1(C_{X_1}, C_{X_2}) \pi_*(\text{grad } \ln \lambda) + (\nabla \pi_*)^\perp(C_{X_1}, C_{X_2}). \] (3.27)
Thus, where which is a conformal quasi-bi-slant Riemannian map, such that with quasi-bi-slant angle

When (i) and (ii) are satisfied, from equation (3.27), we get

\[ \pi_* (\nabla X_1 X_2) = 0, \]
\[ CX_1 (\ln \lambda) \pi_* (CX_2) + CX_2 (\ln \lambda) \pi_* (CX_1) - g_1 (CX_1, CX_2) \pi_* (\text{grad} \ln \lambda) = 0. \]  

(3.28)

So, we get assertion (iii).

When assertions (ii) and (iii) are satisfied in equation (3.27), we obtain \( \pi_* (\nabla X_1 X_2) = 0 \), which means that \( (\ker \pi_*)^\perp \) defines totally geodesic foliation on \( N_1 \), for all \( X_1, X_2 \in \Gamma (\ker \pi_*)^\perp \). Hence, assertions (ii) and (iii) imply assertion (i). Further, when assertions (i) and (iii) are satisfied in equation (3.27), we obtain equation (3.28). From equation (3.27), we get

\[ \lambda^2 CX_1 (\ln \lambda) g_1 (CX_2, CX_2) + \lambda^2 CX_2 (\ln \lambda) g_1 (CX_2, CX_1) - \lambda^2 g_1 (CX_1, CX_2) CX_2 (\ln \lambda) = 0, \]
\[ \lambda^2 CX_1 (\ln \lambda) g_1 (CX_2, CX_2) = 0, \]

for all \( CX_1, CX_2 \in \Gamma (\mu) \). Here, we have \( CX_1 (\ln \lambda) = 0 \) which implies that \( \lambda \) is a constant on \( \mu \). So we have assertion (ii), which completes the proof.

\( \square \)

4. Example

Note that given an Euclidean space \( \mathbb{R}^{2k} \) with coordinates \( (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) \), we can canonically choose an almost complex structure \( J \) on \( \mathbb{R}^{2k} \) as follows:

\[ J (a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_{2k-1} \frac{\partial}{\partial x_{2k-1}} + a_{2k} \frac{\partial}{\partial x_{2k}}) = -a_2 \frac{\partial}{\partial x_1} + a_1 \frac{\partial}{\partial x_2} + \cdots - a_{2k-1} \frac{\partial}{\partial x_{2k-1}} + a_{2k} \frac{\partial}{\partial x_{2k}}, \]

where \( a_1, a_2, \ldots, a_{2k} \) are \( C^\infty \) functions defined on \( \mathbb{R}^{2k} \). Throughout this section, we will use this notation.

**Example 4.1.** Define a map \( \pi : \mathbb{R}^{10} \to \mathbb{R}^6 \) by

\[ \pi (x_1, x_2, \ldots, x_{10}) = e^5 (x_1 - x_3, x_4, 2021, x_6 + x_8, x_7, 2022), \]

which is a conformal quasi-bi-slant Riemannian map, such that

\[ X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_5}, \quad X_4 = \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8}, \quad X_5 = \frac{\partial}{\partial x_9}, \quad X_6 = \frac{\partial}{\partial x_{10}}, \quad \ker \pi_* = D \oplus D_1 \oplus D_2, \]

where

\[ D = < X_5 = \frac{\partial}{\partial x_9}, X_6 = \frac{\partial}{\partial x_{10}}, >, \]
\[ D_1 = < X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_2}, >, \]
\[ D_2 = < X_3 = \frac{\partial}{\partial x_5}, X_4 = \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8}, >, \]

\( (\ker \pi_*)^\perp = < H_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}, H_2 = \frac{\partial}{\partial x_4}, H_3 = \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}, H_4 = \frac{\partial}{\partial x_7}, >, \)

\[ \pi_* H_1 = e^5 \frac{\partial}{\partial v_1}, \quad \pi_* H_2 = e^5 \frac{\partial}{\partial v_2}, \quad \pi_* H_3 = e^5 \frac{\partial}{\partial v_3}, \quad \pi_* H_4 = e^5 \frac{\partial}{\partial v_4}, \]

with quasi-bi-slant angle \( \theta_1 = \frac{\pi}{3} \) and \( \theta_2 = \frac{\pi}{4} \). Hence, we have

\[ g_2 (\pi_* H_1, \pi_* H_2) = (e^5)^2 g_1 (H_1, H_1), \quad g_2 (\pi_* H_2, \pi_* H_2) = (e^5)^2 g_1 (H_2, H_2), \]
\[ g_2 (\pi_* H_3, \pi_* H_3) = (e^5)^2 g_1 (H_3, H_3), \quad g_2 (\pi_* H_4, \pi_* H_4) = (e^5)^2 g_1 (H_4, H_4). \]

Thus, \( \pi \) is a conformal quasi-bi-slant Riemannian map with \( \lambda = e^5 \).
Example 4.2. Let $\pi : (\mathbb{R}^{10}, g_{10} = e^{x_5}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2 + dx_{10}^2)) \rightarrow (\mathbb{R}^4, g_4 = (dv_1^2 + dv_2^2 + dv_3^2 + dv_4^2))$ defined by

$$\pi(x_1, \ldots, x_{10}) = \left( \frac{x_3 - \sqrt{3}x_5}{2}, \frac{x_6}{\sqrt{2}}, \frac{x_7 + x_9}{2} \right),$$

which is a conformal quasi-bi-slant Riemannian map such that

$$\ker \pi_* = D \oplus D_1 \oplus D_2,$$

where

$$D = \langle X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2} >, \quad D_1 = \langle X_3 = \frac{1}{2}(\sqrt{3} \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}), X_4 = \frac{\partial}{\partial x_4} >, \quad D_2 = \langle X_5 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9}), X_6 = \frac{\partial}{\partial x_8}, X_7 = \frac{\partial}{\partial x_{10}} >,$$

$$\ker \pi_*^\perp = \langle H_1 = \frac{1}{2}(\sqrt{3} \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}), H_2 = \frac{\partial}{\partial x_6}, H_3 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}) >.$$

with conformal quasi-bi-slant angle $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \frac{\pi}{4}$. Hence, we have

$$g_2(\pi_* H_1, \pi_* H_1) = e^{-2x_5} g_1(H_1, H_1), \quad g_2(\pi_* H_2, \pi_* H_2) = e^{-2x_5} g_1(H_2, H_2), \quad g_2(\pi_* H_3, \pi_* H_3) = e^{-2x_5} g_1(H_3, H_3).$$

Thus, $\pi$ is a conformal quasi-bi-slant Riemannian map with $\lambda = e^{-x_5}$.

The defined map $\pi$ in above examples satisfies Lemma 3.3 for any $Y_1, Y_2 \in \Gamma(D)$. One can easily observe that $(\nabla \pi_*)^\perp(Y_1, Y_2) = 0$, i.e., $D$ defines totally geodesic foliation on $\mathbb{R}^{10}$. So, the map $\pi$ satisfies Theorems 3.13 and 3.18.

Similarly the distribution $D_1$ satisfies Theorems 3.14 and 3.19 and the distribution $D_2$ satisfies Theorems 3.15 and 3.20.

References


