Some essential bi-ideals and essential fuzzy bi-ideals in a semigroup

Nattapon Panpetch, Thanathip Muangngao, Thiti Gaketem*

Department of Mathematics, School of Science, University of Phayao, Phayao 56000, Thailand.

Abstract

In this paper, we give the concepts of essential bi-ideals and essential fuzzy bi-ideals in semigroups. In the main results, we characterized regular, left regular, intra-regular, semisimple semigroups in terms of essential fuzzy ideals and essential fuzzy bi-ideals in semigroups.

Keywords: Essential bi-ideals, minimal bi-ideals, essential minimal bi-ideals, essential fuzzy minimal bi-ideals.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh in 1965 [8]. These concepts were applied in many areas such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, topology etc. In 1979, Kuroki [3] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them.


In this paper, we give the concepts of essential bi-ideals and essential fuzzy bi-ideals in semigroups. In the main results, we characterized regular, left regular, intra-regular, semisimple semigroups in terms of essential fuzzy ideals and essential fuzzy bi-ideals in semigroups.

2. Preliminaries

In this section, we give some basic definitions and theorems that we need.

A non-empty subset I of a semigroup S is called a subsemigroup of S if $I^2 \subseteq I$. A non-empty subset I of a semigroup S is called a left (right) ideal of S if $SI \subseteq I$ ($IS \subseteq I$). An ideal I of S is a non-empty subset
which is both a left ideal and a right ideal of $S$. A subsemigroup $I$ of a semigroup $S$ is called a bi-ideal of $S$ if $ISI \subseteq I$. It well-know, every ideal of a semigroup $S$ is a bi-ideal of $S$. For any $a, b \in [0, 1]$, we have
\[ a \lor b = \max(a, b), \quad \text{and} \quad a \land b = \min(a, b). \]

A fuzzy set of a non-empty set $T$ is function from $T$ into unit closed interval $[0, 1]$ of real numbers, i.e., $f : T \rightarrow [0, 1]$. If $\supp(f) = \{ u \in T \mid f(u) \neq 0 \}$, $f \subseteq g$ if $f(u) \leq g(u)$, $(f \lor g)(u) = \max\{f(u), g(u)\} = f(u) \lor g(u)$ and $(f \land g)(u) = \min\{f(u), g(u)\} = f(u) \land g(u)$ for all $u \in T$.

For any two fuzzy sets of $f$ and $g$ of a non-empty of $T$, we defined the support of $f$ instead of $\supp(f) = \{ u \in T \mid f(u) \neq 0 \}$, $f \subseteq g$ if $f(u) \leq g(u)$, $(f \lor g)(u) = \max\{f(u), g(u)\} = f(u) \lor g(u)$ and $(f \land g)(u) = \min\{f(u), g(u)\} = f(u) \land g(u)$ for all $u \in T$.

For two fuzzy sets $f$ and $g$ in a semigroup $S$, define the product $f \circ g$ as follows : for all $u \in S$,
\[
(f \circ g)(u) = \begin{cases} \bigvee \{(f(y) \land g(z)) \mid (y, z) \in F_u\}, & \text{if } F_u \neq \emptyset, \\ 0, & \text{if } F_u = \emptyset, \end{cases}
\]
where $F_u := \{(y, z) \in S \times S \mid u = yz\}$.

A fuzzy subsemigroup of a semigroup $S$ if $f(uv) \geq f(u) \land f(v)$ for all $u, v \in S$. A fuzzy left (right) ideal of a semigroup $S$ if $f(uv) \geq f(v)$ $(f(uv) \geq f(u))$ for all $u, v \in S$. A fuzzy bi-ideal of a semigroup $S$ if $f$ is a fuzzy subsemigroup of $S$ and $f(uvw) \geq f(u) \land f(w)$ for all $u, v, w \in S$. It well-know, every fuzzy ideal of a semigroup $S$ is a fuzzy bi-ideal of $S$.

The characteristic fuzzy set $\chi_I$ of a non-empty set is defined as follows:
\[
\chi_I : T \rightarrow [0, 1], \quad u \mapsto \begin{cases} 1, & \text{if } u \in I, \\ 0, & \text{if } u \notin I. \end{cases}
\]

The following of theorems are true.

**Theorem 2.1** ([6]). Let $S$ be a semigroup. Then $I$ is a subsemigroup (left ideal right ideal, bi-ideal) of $S$ if and only if characteristic function $\chi_I$ is a fuzzy subsemigroup (left ideal right ideal, bi-ideal) of $S$.

**Theorem 2.2** ([6]). Let $I$ and $J$ be subsets of a non-empty set $S$. Then $\chi_{I \cap J} = \chi_I \land \chi_J$ and $\chi_I \circ \chi_J = \chi_{IJ}$.

**Theorem 2.3** ([6]). Let $f$ be a nonzero fuzzy set of a semigroup $S$. Then $f$ is a fuzzy subsemigroup (ideal, bi-ideal) of $S$ if and only if $\supp(f)$ is a subsemigroup (ideal, bi-ideal) of $S$.

Next, we will review of essential ideals and fuzzy essential ideals in a semigroup and properties of those.

**Definition 2.4.** An essential left (right) ideal $I$ of a semigroup $S$ if $I$ is a left (right) ideal of $S$ and $I \cap J \neq \emptyset$ for every left (right) ideal $J$ of $S$.

**Definition 2.5** ([1]). An essential ideal $I$ of a semigroup $S$ if $I$ is an ideal of $S$ and $I \cap J \neq \emptyset$ for every ideal $J$ of $S$.

**Theorem 2.6** ([1]). Let $I$ be an essential ideal of a semigroup $S$. If $I_1$ is an ideal of $S$ containing $I$, then $I_1$ is also an essential ideal of $S$.

**Theorem 2.7** ([1]). Let $I$ and $J$ be essential ideals of a semigroup $S$. Then $I \cup J$ and $I \cap J$ are essential ideals of $S$.

**Definition 2.8** ([1]). An essential fuzzy ideal $f$ of a semigroup $S$ if $f$ is a nonzero fuzzy ideal of $S$ and $f \land g \neq \emptyset$ for every nonzero fuzzy ideal $g$ of $S$.

**Theorem 2.9** ([1]). Let $I$ be an ideal of a semigroup $S$. Then $I$ is an essential ideal of $S$ if and only if $\chi_I$ is an essential fuzzy ideal of $S$.

**Theorem 2.10** ([1]). Let $f$ be a nonzero fuzzy ideal of a semigroup $S$. Then $f$ is an essential fuzzy ideal of $S$ if and only if $\supp(f)$ is an essential ideal of $S$. 
3. Essential subsemigroups and essential fuzzy subsemigroups

In this section, we will study concepts of essential subsemigroups in a semigroup and fuzzy essential subsemigroups in a semigroup and their properties.

**Definition 3.1.** An essential subsemigroup I of a semigroup S if I is a subsemigroup of S and I ∩ J ≠ ∅ for every subsemigroup J of S.

**Example 3.2.**

1. Let E be set of all even integers. Then (E, +) and (N, +) are subsemigroups of (Z, +). Thus (E, +) ∩ (N, +) ≠ ∅. Hence, (E, +) is an essential subsemigroup of (Z, +).

2. Let A = {2n | n ∈ Z} and B = {3n | n ∈ Z}. Then (A, ·) and (B, ·) are subsemigroups of (Z, ·). Thus (A, ·) ∩ (B, ·) ≠ ∅. Hence (A, ·) is an essential subsemigroup.

**Theorem 3.3.** Let I be an essential subsemigroup of a semigroup S. If I1 is an ideal of S containing I, then I1 is also an essential subsemigroup of S.

**Proof.** Suppose that I1 is a subsemigroup of S such that I1 ⊆ I and let J be any subsemigroup of S. Thus, I ∩ J ≠ ∅. Hence, I1 ∩ J ≠ ∅. Therefore I1 is an essential subsemigroup of S.

**Theorem 3.4.** Let I and J be essential subsemigroups of a semigroup S. Then I ∪ J and I ∩ J are essential subsemigroups of S.

**Proof.** Since I ⊆ I ∪ J and I is an essential subsemigroup, we have I ∪ J is an essential subsemigroup of S, by Theorem 3.3.

Since I and J are essential subsemigroups of S we have I and J are subsemigroups of S. Thus I ∩ J is a subsemigroup of S.

Let K be a subsemigroup of S. Then I ∩ K ≠ ∅. Thus there exists u, v ∈ I ∩ K. Let u, v ∈ J. Then u, v ∈ (I ∪ J) ∩ K. Thus (I ∩ J) ∩ K ≠ ∅. Hence I ∩ J is an essential subsemigroup of S.

**Definition 3.5.** An essential fuzzy subsemigroup f of a semigroup S if f is a nonzero fuzzy subsemigroup of S and f ∩ g ≠ ∅ for every nonzero fuzzy subsemigroup g of S.

**Theorem 3.6.** Let I be a subsemigroup of a semigroup S. Then I is an essential subsemigroup of S if and only if χ1 is an essential fuzzy subsemigroup of S.

**Proof.** Suppose that I is an essential subsemigroup of S and let g be a nonzero fuzzy subsemigroup of S. Then supp(g) is subsemigroup of S. By assumption we have I is a subsemigroup of S. Thus I ∩ supp(g) ≠ ∅. So there exists u ∈ I ∩ supp(g). It implies that (χ1 ∩ g)(u) ≠ 0. Hence, χ1 ∩ g ≠ 0. Therefore, χ1 is an essential fuzzy subsemigroup of S.

Conversely, assume that χ1 is an essential fuzzy subsemigroup of S and let J be a subsemigroup of S. Then χ1 is a nonzero fuzzy subsemigroup of S. Since χ1 is an essential fuzzy subsemigroup of S we have χ1 is a fuzzy subsemigroup of S. Thus, χ1 ∩ J ≠ ∅. So by Theorem 2.2, χ1 ∩ J ≠ ∅. Hence, I ∩ J ≠ ∅. Therefore I is an essential subsemigroup of S.

**Theorem 3.7.** Let f be a nonzero fuzzy subsemigroup of a semigroup S. Then f is an essential fuzzy subsemigroup of S if and only if supp(f) is an essential subsemigroup of S.

**Proof.** Assume that f is an essential fuzzy subsemigroup of S. Then supp(f) is a subsemigroup of S. Let I be a subsemigroup of S. Then by Theorem 2.1, χ1 is a subsemigroup of S. Since f is an essential fuzzy subsemigroup of S we have f is a fuzzy subsemigroup of S. Thus f ∩ χ1 ≠ ∅. So there exists u ∈ S such that f ∩ χ1(u) ≠ 0. It implies that f(u) ≠ 0 and χ1 ≠ 0. Hence, u ∈ supp(f) ∩ I so supp(f) ∩ I ≠ ∅ it implies that supp(f) is an essential subsemigroup of S.
Conversely, assume that $\text{supp}(f)$ is an essential ideal of $S$ and let $g$ be a nonzero fuzzy subsemigroup of $S$. Then $\text{supp}(g)$ is a subsemigroup of $S$. Thus $\text{supp}(f) \cap \text{supp}(g) \neq \emptyset$. So there exists $u \in \text{supp}(f) \cap \text{supp}(g)$.

This implies that $f(u) \neq 0$ and $g(u) \neq 0$ for all $u \in S$. Hence, $(f \wedge g)(u) \neq 0$ for all $u \in S$. Therefore, $f \wedge g \neq 0$. We conclude that $f$ is an essential fuzzy subsemigroup of $S$. \hfill $\square$

**Theorem 3.8.** Let $f$ be an essential fuzzy subsemigroup of a semigroup $S$. If $f_1$ is a fuzzy subsemigroup of $S$ such that $f \subseteq f_1$, then $f_1$ is also an essential fuzzy subsemigroup of $S$.

**Proof.** Let $f_1$ be a fuzzy subsemigroup of $S$ such that $f \subseteq f_1$ and let $g$ be any fuzzy subsemigroup of $S$. Thus, $f \wedge g \neq 0$. So $f_1 \wedge g \neq 0$. Hence $f_1$ is an essential fuzzy subsemigroup of $S$. \hfill $\square$

**Theorem 3.9.** Let $f_1$ and $f_2$ be essential fuzzy subsemigroups of a semigroup $S$. Then $f_1 \lor f_2$ and $f_1 \land f_2$ are essential fuzzy subsemigroups of $S$.

**Proof.** Let $f_1$ and $f_2$ be essential fuzzy subsemigroups of $S$. Then by Theorem 3.8, $f_1 \lor f_2$ is an essential fuzzy subsemigroup of $S$. Since $f_1$ and $f_2$ are essential fuzzy subsemigroups of $S$ we have $f_1 \cap f_2$ is a fuzzy subsemigroup of $S$. Let $g$ be a nonzero fuzzy subsemigroup of $S$. Then $f_1 \cap g \neq 0$. Thus there exists $u \in S$ such that $f_1(u) \neq 0$ and $(g)(u) \neq 0$. Since $f_1 \neq 0$ and let $v \in S$ such that $f_2(v) \neq 0$. Since $f_1$ and $f_2$ are fuzzy subsemigroups of $S$ we have $f_1(uv) \geq f_1(u) \wedge f_1(v) > 0$ and $f_2(uv) \geq f_2(u) \wedge f_2(v) > 0$. Thus $(f_1 \cap f_2)(uv) = f_1(uv) \wedge f_2(uv) \neq 0$. Since $g$ is a fuzzy subsemigroup of $S$ and $g(u) \neq 0$ we have $g(uv) \neq 0$ for all $u, v \in S$. Thus $[(f_1 \cap f_2) \wedge g](uv) \neq 0$. Hence $[(f_1 \cap f_2) \land g] \neq 0$. Therefore $f_1 \land f_2$ is an essential fuzzy subsemigroup of $S$. \hfill $\square$

4. Essential bi-ideals and essential fuzzy bi-ideals

In this section, we defined essential bi-ideals and essential fuzzy bi-ideal in semigroup and its integrated properties.

**Definition 4.1.** An essential bi-ideal $I$ of a semigroup $S$ if $I$ is a bi-ideal of $S$ and $I \cap J \neq \emptyset$ for every bi-ideal $J$ of $S$.

**Example 4.2.** Let $S = \{\Psi, \Omega, \Upsilon, \Pi\}$ be semigroup with the following Cayley table.

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<td>$\Upsilon$</td>
<td>$\Psi$</td>
<td>$\Psi$</td>
<td>$\Omega$</td>
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<td>$\Pi$</td>
<td>$\Psi$</td>
<td>$\Psi$</td>
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Then $\{\Psi\}$, $\{\Psi, \Omega\}$, $\{\Psi, \Omega, \Upsilon\}$, $\{\Psi, \Omega, \Pi\}$, and $\{\Psi, \Omega, \Upsilon, \Pi\}$ are bi-ideal of $S$. Thus $\{\Psi\} \cap \{\Psi, \Omega\} \neq \emptyset$ and

$\{\Psi, \Omega, \Pi\} \cap \{\Psi, \Omega, \Upsilon, \Pi\} \neq \emptyset$.

Hence $\{\Psi\}$ and $\{\Psi, \Omega, \Pi\}$ are essential bi-ideals of $S$.

**Theorem 4.3.** Let $I$ be an essential bi-ideal of a semigroup $S$. If $I_1$ is an ideal of $S$ containing $I$, then $I_1$ is also an essential bi-ideal of $S$.

**Proof.** Suppose that $I_1$ is a bi-ideal of $S$ such that $I_1 \subseteq I$ and let $J$ be any bi-ideal of $S$. Thus, $I \cap J \neq \emptyset$. Hence, $I_1 \cap J \neq \emptyset$. Therefore $I_1$ is an essential bi-ideal of $S$. \hfill $\square$

**Theorem 4.4.** Let $I$ and $J$ be essential bi-ideals of a semigroup $S$. Then $I \cup J$ and $I \cap J$ are essential bi-ideals of $S$.
Proof. Since I and J are essential bi-ideals of a semigroup S we have I and J are essential subsemigroups of a semigroup S. Thus by Theorem 3.4, I ∪ J and I ∩ J are essential subsemigroups of S. Since I ⊆ I ∪ J and I is an essential bi-ideal we have I ∪ J is an essential bi-ideal of S.

Let K be a bi-ideal of S. Then I ∩ K ≠ ∅. Thus there exists u, v and w ∈ I ∩ K. Let u, v and w ∈ J. Then uvw ∈ (I ∩ J) ∩ K. Thus (I ∩ J) ∩ K ≠ ∅. Hence I ∩ J is an essential bi-ideal of S.

Definition 4.5. An essential fuzzy bi-ideal f of a semigroup S if f is a nonzero fuzzy bi-ideal of S and f ∧ g ≠ 0 for every nonzero fuzzy bi-ideal g of S.

Theorem 4.6. Let I be a bi-ideal of a semigroup S. Then I is an essential bi-ideal of S if and only if χ₁ is an essential fuzzy bi-ideal of S.

Proof. Suppose that I is an essential bi-ideal of S and let g be a nonzero fuzzy bi-ideal of S. Then by Theorem 3.6, supp(g) is subsemigroup of S and χ₁ is an essential fuzzy subsemigroup of S. Thus there exists u, v, w ∈ I ∩ supp(g) such that (f ∧ χ₁)(uvw) ≠ 0. It implies that χ₁ ∧ g ≠ 0. Therefore, χ₁ is an essential fuzzy bi-ideal of S.

Conversely, assume that χ₁ is an essential fuzzy bi-ideal of S and let J be a bi-ideal of S. Then χ₁ is an essential fuzzy subsemigroup of S and J is a subsemigroup of S. Thus by Theorem 3.6, I is an essential subsemigroup of S. Since J be a bi-ideal of S we have χ₁ is a nonzero fuzzy bi-ideal of S. Then, χ₁ ∧ χ₁ ≠ 0. Thus, χ₁ ∧ χ₁ ≠ 0. Hence I ∩ J ≠ ∅. Therefore I is an essential bi-ideal of S.

Theorem 4.7. Let f be a nonzero fuzzy bi-ideal of a semigroup S. Then f is an essential fuzzy bi-ideal of S if and only if supp(f) is an essential bi-ideal of S.

Proof. Assume that f is an essential fuzzy bi-ideal of S. Then f is an essential fuzzy subsemigroup of S. Thus by Theorem 3.7, supp(f) is an essential subsemigroup of S. Let I be a bi-ideal of S. Then by Theorem 2.1, χ₁ is a bi-ideal of S. Thus f ∧ χ₁ ≠ 0. Then there exists u ∈ S such that (f ∧ χ₁)(u) ≠ 0. It implies that f(u) ≠ 0 and χ₁ ≠ 0. Hence, u ∈ supp(f) ∩ I so supp(f) ∩ I ≠ ∅ implies that supp(f) is an essential bi-ideal of S.

Conversely, assume that supp(f) is an essential bi-ideal of S and let g be a nonzero fuzzy bi-ideal of S. Then supp(f) is an essential bi-ideal of S. Since g be a nonzero fuzzy bi-ideal of S we have f is an essential fuzzy subsemigroup of S and supp(g) is a subsemigroup of S, by Theorem 3.7. This implies that supp(f) ∩ supp(g) ≠ ∅. So there exists u ∈ supp(f) ∩ supp(g), this implies that f(u) ≠ 0 and g(u) ≠ 0. Hence, (f ∧ g)(u) ≠ 0. Therefore, f ∧ g ≠ 0. We conclude that f is an essential fuzzy bi-ideal of S.

Theorem 4.8. Let f be an essential fuzzy bi-ideal of a semigroup S. If f₁ is a fuzzy bi-ideal of S such that f ⊆ f₁, then f₁ is also an essential fuzzy bi-ideal of S.

Proof. Let f₁ be a fuzzy bi-ideal of S such that f ⊆ f₁ and let g be any fuzzy bi-ideal of S. Thus f ∧ g ≠ 0. So f₁ ∧ g ≠ 0. Hence, f₁ is an essential fuzzy bi-ideal of S.

Theorem 4.9. Let f₁ and f₂ be essential fuzzy bi-ideals of a semigroup S. Then f₁ ∨ f₂ and f₁ ∧ f₂ are essential fuzzy bi-ideals of S.

Proof. Let f₁ and f₂ be essential fuzzy bi-ideal of S. Then by Theorem 4.8, f₁ ∨ f₂ is an essential fuzzy bi-ideal of S. Since f₁ and f₂ are essential fuzzy bi-ideals of S we have f₁ and f₂ is an essential fuzzy subsemigroup of S. Thus f₁ ∩ f₂ is an essential fuzzy subsemigroup of S. Let g be a nonzero fuzzy bi-ideal of S. Then f₁ ∧ g ≠ 0. Thus there exists u, w ∈ S such that f₁(uw) ≠ 0 and (g)(uw) ≠ 0. Since f₂ ≠ 0 and let v ∈ S such that f₂(v) ≠ 0. Since f₁ and f₂ are fuzzy subsemigroups of S we have

f₁(uw) ≥ f₁(u) ∧ f₁(w) > 0,

and

f₂(uw) ≥ f₂(u) ∧ f₂(w) > 0.
Thus \((f_1 \wedge f_2)(uvw) = f_1(uvw) \wedge f_2(uvw) \neq 0\). Since \(g\) is a fuzzy subsemigroup of \(S\) and \(g(v) \neq 0\) we have \(g(uvw) \neq 0\) for all \(u, v \in S\). Thus \([(f_1 \wedge f_2) \wedge g](uvw) \neq 0\). Hence \([(f_1 \wedge f_2) \wedge g] \neq 0\). Therefore \(f_1 \wedge f_2\) is an essential fuzzy bi-ideal of \(S\).

The following theorem we will use the basic knowledge of ideal and bi-ideal in semigroups to prove essential bi-ideal in semigroup.

**Theorem 4.10.** Every essential ideal of semigroup \(S\) is an essential bi-ideal of \(S\).

**Proof.** The proof is obvious.

**Theorem 4.11.** Every essential fuzzy ideal of semigroup \(S\) is an essential fuzzy bi-ideal of \(S\).

**Proof.** The proof is obvious.

5. Characterizing some semigroups by using essential fuzzy ideals and essential fuzzy bi-ideals

In this section, we will characterize regular, left regular, intra-regular, semisimple semigroups by using essential fuzzy ideals and essential fuzzy bi-ideals in semigroups. The following lemmas will be used to prove Theorem 5.3.

**Lemma 5.1.** Let \(S\) be a semigroup. If \(f\) is an essential fuzzy right ideal and \(g\) is an essential fuzzy left ideal of \(S\) then \(f \circ g \subseteq f \wedge g\).

**Proof.** Assume that \(f\) and \(g\) is an essential fuzzy right ideal and an essential fuzzy left ideal of \(S\) respectively. Then \(f\) and \(g\) is a fuzzy right ideal and a fuzzy left ideal of \(S\) respectively. Let \(u \in S\). If \(F_u = \emptyset\), then \((f \circ g)(u) = 0 \leq ((f(u) \wedge g(u)) = (f \wedge g)(u)).\) If \(F_u \neq \emptyset\), then

\[
(f \circ g)(u) = \bigvee_{(i,j) \in F_u} \{f(i) \wedge g(j)\} \leq \bigvee_{(i,j) \in F_u} \{f(ij) \wedge g(ij)\} = (f(u) \wedge g(u)) = (f \wedge g)(u).
\]

Hence, \((f \circ g)(u) \leq (f \wedge g)(u)\). Therefore, \(f \circ g \subseteq f \wedge g\).

**Lemma 5.2 ([6]).** A semigroup \(S\) is regular if and only if \(RL = R \cap L\) for every right ideal \(R\) and left ideal \(L\) of \(S\).

The following theorem show an equivalent conditional statement for a regular semigroup.

**Theorem 5.3.** A semigroup \(S\) is regular if and only if \(f \circ g = f \wedge g\) for every essential fuzzy right ideal \(f\) and essential fuzzy left ideal \(g\) of \(S\).

**Proof.** \((\Rightarrow):\) Let \(f\) and \(g\) be an essential fuzzy right ideal and an essential fuzzy left ideal of \(S\) respectively. Then \(f\) and \(g\) is a fuzzy right ideal and a fuzzy left ideal of \(S\) respectively. Then by Lemma 5.1, \(f \circ g \subseteq f \wedge g\). Let \(u \in S\). Then there exists \(x \in S\) such that \(u = uxu\). Thus

\[
(f \circ g)(u) = \bigvee_{(y,z) \in F_u} \{f(y) \wedge g(z)\} \leq \bigvee_{(y,z) \in F_{uxu}} \{f(y) \wedge g(z)\} = f(ux) \wedge g(u) \leq f(u) \wedge g(u) = (f \wedge g)(u).
\]

Hence, \((f \wedge g)(u) \subseteq (f \circ g)(u)\), and so \((f \wedge g)(u) \subseteq (f \circ g)(u)\). Therefore, \(f \circ g = f \wedge g\).

\((\Leftarrow):\) Let \(R\) and \(L\) be a right ideal and a left ideal of \(S\) respectively. Then by Theorem 2.1, \(\chi_R\) and \(\chi_L\) is an essential fuzzy right ideal and an essential fuzzy left ideal of \(S\) respectively. By supposition and Theorem 2.2, we have

\[
\chi_{RL}(u) = (\chi_R \circ \chi_L)(u) = (\chi_R \wedge \chi_L)(u) = \chi_{R \cap L}(u) = 1.
\]

Thus \(u \in RL\), and so \(RL = R \cap L\). It follows that by Lemma 5.2, \(S\) is regular.
Lemma 5.4 ([6]). A semigroup $S$ is regular if and only if $R_1 \cap R_2 \cap B \subseteq R_1R_2B$, for every right ideals $R_1, R_2$ and every bi-ideal $B$ of $S$.

Theorem 5.5. A semigroup $S$ is regular if and only if $f \wedge g \wedge h \subseteq f \circ g \circ h$, for every essential fuzzy right ideals $f, g$ and every essential fuzzy bi-ideal $h$ of $S$.

Proof. ($\Rightarrow$): Let $f, g$ be two essential fuzzy right ideals, $h$ be an essential fuzzy bi-ideal of $S$. Then $f, g$ be two fuzzy right ideals, $h$ is a fuzzy bi-ideal of $S$. Let $u \in S$ Since $S$ is regular, there exists $x \in S$ such that $u = xu^2$. Thus

$$\begin{align*}
(f \circ g \circ h)(u) &= \left( \bigvee_{(i,j) \in F_u} \{ f(i) \wedge (g \circ h)(j) \} \right) = \left( \bigvee_{(i,j) \in F_{uxu}} \{ f(i) \wedge (g \circ h)(j) \} \right) \\
&\geq \left( f(ux) \wedge (g \circ h)(u) \right) = f(ux) \wedge \left( \bigvee_{(p,q) \in F_u} \{ (g(p) \wedge h(q)) \} \right) \\
&= f(ux) \wedge \left( \bigvee_{(p,q) \in F_{uxu}} \{ (g(p) \wedge h(q)) \} \right) \geq f(ux) \wedge (g(ux) \wedge h(u)) \\
&= f(u) \wedge (g(u) \wedge h(u)) = (f \wedge g \wedge h)(u).
\end{align*}$$

Hence, $(f \wedge g \wedge h)(u) \subseteq (f \circ g \circ h)(u)$. Therefore, $f \wedge g \wedge h \subseteq f \circ g \circ h$.

($\Leftarrow$): Let $R_1, R_2$ be two right ideals and let $B$ be a bi-ideal of $S$. Then by Theorem 2.1, $\chi_{R_1}$ and $\chi_{R_2}$ are essential fuzzy right ideals and $\chi_B$ is an essential fuzzy bi-ideal of $S$. Thus $\chi_{R_1}$ and $\chi_{R_2}$ are fuzzy right ideals and $\chi_B$ is a fuzzy bi-ideal of $S$. By supposition and Lemma 2.2, we have

$$1 = (\chi_{R_1 \cap R_2 \cap B})(u) = (\chi_{R_1}) \wedge (\chi_{R_2}) \wedge (\chi_B)(u) \subseteq (\chi_{R_1} \circ \chi_{R_2} \circ \chi_B)(u) = \chi_{R_1R_2B}(u).$$

Thus, $u \in R_1R_2B$ and so, $R_1 \cap R_2 \cap B \subseteq R_1R_2B$. It follows that by Lemma 5.4, $S$ is regular.

Definition 5.6 ([6]). A semigroup $S$ called left regular if for each element $u \in S$, there exists an element $x \in S$ such that $u = xu^2$.

Lemma 5.7 ([6]). A semigroup $S$ is left regular if and only if $I \cap B \subseteq IB$, for every ideal $I$ of $S$ and every bi-ideal $B$ of $S$.

Theorem 5.8. A semigroup $S$ is left regular if and only if $f \wedge g \subseteq f \circ g$, for every essential fuzzy ideal $f$ and every essential fuzzy bi-ideal $g$ of $S$.

Proof. ($\Rightarrow$): Assume that $f$ and $g$ is an essential fuzzy ideals and an essential fuzzy bi-ideal of $S$ respectively. Then $f$ and $g$ is a fuzzy ideals and a fuzzy bi-ideal of $S$ respectively. Let $u \in S$. Since $S$ is left regular, there exist $x \in S$ such that $u = xu^2$. Thus

$$\begin{align*}
(f \circ g)(u) &= \left( \bigvee_{(i,j) \in F_u} \{ f(i) \wedge g(j) \} \right) = \left( \bigvee_{(i,j) \in F_{uxu}} \{ f(i) \wedge g(j) \} \right) \\
&\geq f(ux) \wedge g(u) \geq f(u) \wedge g(u) = (f \wedge g)(u).
\end{align*}$$

Hence, $(f \wedge g)(u) \subseteq (f \circ g)(u)$. Therefore, $f \wedge g \subseteq f \circ g$.

($\Leftarrow$): Let $I$ and $B$ be an ideal and a bi-ideal of $S$ respectively. Then by Theorem 2.1, $\succ_I$ and $\succ_J$ is an essential fuzzy ideal and an essential fuzzy bi-ideal of $S$ respectively. Thus $\succ_I$ and $\succ_J$ is a fuzzy ideal and a fuzzy bi-ideal of $S$ respectively. By supposition and Lemma 2.2, we have

$$\chi_{I \cap B}(u) = (\chi_I \wedge \chi_B)(u) \subseteq (\chi_I \circ \chi_B)(u) = \chi_{IB}(u) = 1.$$
The following definition and lemma will be used to prove in Theorem 5.11.

**Definition 5.9 ([6]).** A semigroup $S$ is called *intra-regular* if for each $u \in S$, there exist $a, b \in S$ such that $u = au^2b$.

**Lemma 5.10 ([6]).** A semigroup $S$ is intra-regular if and only if $L \cap R \subseteq LR$, for every left ideal $L$ and every right ideal $R$ of $S$.

**Theorem 5.11.** A semigroup $S$ is intra-regular if and only if $f \land g \subseteq f \circ g$, for every essential left ideal $f$ and essential right ideal $g$ of $S$.

**Proof.** $(\Rightarrow)$: Assume that $f$ and $g$ is an essential fuzzy left ideal and an essential right ideal of $S$ respectively. Then $f$ and $g$ is a left ideal and a right ideal of $S$ respectively. Let $u \in S$. Since $S$ is intra-regular, there exist $a, b \in S$ such that $u = au^2b$. Thus
\[
(f \circ g)(u) = \bigvee_{(i,j) \in F_u} \{f(i) \land g(j)\} = \bigvee_{(i,j) \in F_{au^2b}} \{f(i) \land g(j)\} \\
\geq f(au) \land g(ub) \geq f(u) \land g(u) = (f \land g)(u).
\]
It implies that, $(f \circ g)(u) \leq (f \land g)(u)$. Hence, $f \land g \subseteq f \circ g$.

$(\Leftarrow)$: Let $R$ and $L$ be a right ideal and a left ideal of $S$ respectively. Then by Theorem 2.1, $\chi_R$ and $\chi_L$ is an essential fuzzy right ideal and an essential fuzzy left ideal of $S$ respectively. Thus $\chi_R$ and $\chi_L$ is a fuzzy right ideal and a fuzzy left ideal of $S$. By supposition and Lemma 2.2, we have
\[
\chi_{R \cap L}(u) = (\chi_R \land \chi_L)(u) \geq \chi_{RL}(u) = (\chi_R \circ \chi_L)(u) = 1.
\]
Thus $u \in LR$, and so $L \cap R \subseteq LR$. It follows that by Lemma 5.10, $S$ is intra-regular. □

The following definition and lemma will be used to prove in Theorem 5.15.

**Definition 5.12 ([6]).** A semigroup $S$ is called *semisimple* if every ideal of $S$ is idempotent.

**Remark 5.13.** A semigroup $S$ is semisimple if and only if $u \in (SuS)(SuS)$ for every $u \in S$, that is there exist $w, y, z \in S$ such that $u = wuyuz$.

**Lemma 5.14 ([6]).** A semigroup $S$ is semisimple if and only if $I \cap J = IJ$, for every ideals $I$ and $J$ of $S$.

**Theorem 5.15.** A semigroup $S$ is semisimple if and only if $f \land g = f \circ g$, for every essential fuzzy ideals $f$ and $g$ of $S$.

**Proof.** $(\Rightarrow)$ Assume that $f$ and $g$ are essential fuzzy ideals of $S$. Then $f$ and $g$ are fuzzy ideals of $S$. Then by Theorem 5.1, $f \circ g \subseteq f \land g$. Let $u \in S$. Since $S$ is semisimple, there exist $w, x, y, z \in S$ such that $u = (xuy)(wuz)$. Thus
\[
(f \circ g)(u) = \bigvee_{(i,j) \in F_u} \{f(i) \land g(j)\} = \bigvee_{(i,j) \in F_{(xuy)(wuz)}} \{f(i) \land g(j)\} \\
\geq f(xuy) \land g(wuz) \geq f(xu) \land g(uz) \\
\geq f(u) \land g(u) = (f \land g)(u).
\]
Hence, $(f \land g)(u) \leq (f \circ g)(u)$, and so $f \land g \subseteq f \circ g$. Therefore, $f \land g = f \circ g$.

$(\Leftarrow)$: Let $I$ and $J$ be ideals of $S$. Then by Theorem 2.1, $\chi_I$ and $\chi_J$ are essential fuzzy ideals of $S$. Thus $\chi_I$ and $\chi_J$ are fuzzy ideals of $S$. By supposition and Lemma 2.2, we have
\[
\chi_{IJ}(u) = (\chi_I \circ \chi_J)(u) = (\chi_I \land \chi_J)(u) = \chi_{I \cap J}(u) = 1.
\]
Thus $u \in IJ$, and so $IJ = I \cap J$. It follows that by Lemma 5.14, $S$ is semisimple. □
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