J. Math. Computer Sci., 28 (2023), 192–202

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

New oscillation criteria and some refinements for second-order neutral delay dynamic equations on time scales

A. M. Hassan*, S. E. Affan

Department of Mathematics, Faculty of Science, Benha University, Benha-Kalubia 13518, Egypt.

Abstract

In this paper, we present more effective criteria for oscillation of second-order half-linear neutral dynamic equations with delayed arguments. Our results essentially improve, complement, and simplify several related ones in the literature. Some examples are given to illustrate our main results.

Keywords: Second order, nonlinear dynamic equations, oscillation, Riccati transformation.

2020 MSC: 34C11, 34K11.

1. Introduction

This paper aims to study the oscillation problem of second-order half-linear neutral delay dynamic equations

$$(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} + q(t)x^{\alpha}(\delta(t)) = 0, \quad t \geqslant t_0, \tag{1.1}$$

where $\alpha > 0$ is a ratio of odd positive integers and $z(t) := x(t) + p(t)x(\tau(t))$. Throughout, the following assumptions are satisfied

(I1) $r \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty)), r^{\Delta}(t) \geqslant 0$ and $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s = \infty$;

(I2) $p, q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}),\ 0 \leqslant p(t) \leqslant p_0 \leqslant 1,\ q(t) \geqslant 0$, and $q(t)$ not identically zero for large t;

(I3) $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T}), \delta^{\Delta} \geqslant 0,\ \tau(t) \leqslant t,\ \delta(t) \leqslant t$, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$.

Furthermore for sufficiently large t_1, we assume

$$R(t) = \int_{t_1}^{\infty} r^{-1/\alpha}(s)\Delta s. \tag{1.2}$$

*Corresponding author

Email addresses: ahmed.mohamed@fsc.bu.edu.eg (A. M. Hassan), samy.affan@fsc.bu.edu.eg (S. E. Affan)

doi: 10.22436/jmcs.028.02.07

Received: 2021-07-08 Revised: 2022-03-16 Accepted: 2022-03-16

By a solution of (1.1), we mean a function $x \in C_{rd}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [t_0, \infty)_{\mathbb{T}}$ which has the property $r(z^{\Delta})^{\alpha} \in C^1_{rd}[T_x, \infty)_{\mathbb{T}}$ and satisfies (1.1) on $[T_x, \infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup |x(t)| : t \in [T_x, \infty)_{\mathbb{T}} > 0$ for all $T \in [T_x, \infty)_{\mathbb{T}}$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

In recent years, there has been a renewed interest in studying and establishing criteria for the oscillatory and asymptotic behavior of various classes of differential equations, partial differential equations, and dynamic equations on time scales, see; [5–9, 11–15, 17, 27, 28, 31, 33–35, 37, 38].

For instance, the authors in [37] established sufficient conditions for the oscillation of all solutions of the dynamic equation

$$\left(r(t)\left((y(t)+p(t)y(\tau(t)))^{\Delta}\right)^{\gamma}\right)^{\Delta} + f(t, y(\delta(t))) = 0,$$

where $\gamma > 0$. In the particular case of (1.1) when $\mathbb{T} = \mathbb{R}$, Bohner et al. [6] studied the oscillatory behavior of the solutions of differential equation

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\alpha})' + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geqslant t_0, \tag{1.3}$$

where $\alpha > 0$ is a quotient of odd positive integers, with the condition $\int_{t_0}^{\infty} r^{-1/\alpha}(s)ds < \infty$.

Grace et al. [17] established more sharper oscillation criteria for (1.3), under the assumption that $\int_{t_0}^{\infty} r^{-1/\alpha}(s)ds = \infty$. This study essentially improved the well-known results reported in the literature, including those for non-neutral differential equations, this study based on taking into account such part of the overall impact of the delay. However, it seems that there are no known results for the oscillation of second-order neutral half-linear delay dynamic equations takes into account impact of the delay.

Now, we refer the reader to the particular case of (1.1)

$$\left(r(t)\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta} + p(t)x^{\gamma}(t) = 0. \tag{1.4}$$

Saker [30] examined oscillation for (1.4), where $\gamma > 1$ is an odd positive integer and Agarwal [3] studied oscillation for the same (1.4), where $\gamma > 1$ is the quotient of odd positive integers which cannot be applied when $0 < \gamma \leqslant 1$. Therefore it will be of great interest to solve this problem and improve Agarwal's and Saker's results.

The following studies play an essential role in our paper, this will allow us to refine classical Riccati transformation techniques by taking into account the impact of the delay that has been neglected in the earlier results. O'Regan [28] studied the distributions of zeros of solutions of first-order delay dynamic equation on time scales

$$x^{\Delta}(t) + p(t)x(\tau(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

and established lower bound for the quotient $x(\tau(t))/x(t)$ using the sequences $\{f_n(\rho)\}_{n=0}^{\infty}$ defined as

$$f_n(\rho) = \begin{cases} f_0(\rho) = 1, f_1(\rho) = 1/\rho, \\ f_{n+2}(\rho) = \frac{f_n(\rho)}{f_n(\rho)+1-e^{(1-\rho)f_n(\rho)}}, & n = 0, 1, 2, 3, \dots, \end{cases} \tag{1.5}$$

where $0 \leqslant \rho < 1$ is a positive constant.

In [38] Zhang and Zhou obtained the particular case for (1.5) when $\mathbb{T} = \mathbb{R}$, the sequences $\{f_n(\rho)\}_{n=0}^{\infty}$, defined as

$$f_n(\rho) = \begin{cases} f_0(\rho) = 1, \\ f_{n+1}(\rho) = e^{(\rho)f_n(\rho)}, & n = 0, 1, 2, 3, \dots, \end{cases}$$

where ρ is a positive constant.

They showed that, for $\rho \in (0, 1/e]$, the sequence is increasing and bounded above and $\lim_{n\to\infty} f_n(\rho) = f(\rho) \in [1, e]$, where $f(\rho)$ is a real root of the equation

$$f(\rho) = e^{\rho f(\rho)}.$$

In fact, equation (1.1) has numerous applications in mathematical, theoretical, and chemical physics. So, there has been much research activities concerning oscillatory behavior of various classes of dynamic equations and its particular cases. We refer the reader to [1, 2, 16, 18, 20–26]. See also [26, 32] for models from mathematical biology where delay and oscillatory-type effects phenomena are idealized by external sources of evolutive PDE's.

Motivated by this observation, our aim in this paper is to present sufficient conditions for the oscillatory behaviour of solutions of (1.1), under the condition $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s = \infty$. We introduce two different techniques, the first is the comparison theorems and the second is classical Riccati transformation techniques by taking into account the impact of the delay. Also, interesting examples that illustrate the importance of our results are included.

2. Main Results

For simplicity, through this paper, we consider the following:

$$Q(t) := (1-p(\delta(t)))^{\alpha} q(t),$$
$$\tilde{R}(t) := R(t) + \frac{1}{\alpha}\int_{t_1}^{t} R(s)R^{\alpha}(\delta(s))Q(s)\Delta s.$$

In what follows, we present some results, which will be useful in the proof of our main results.

Theorem 2.1 ([10]). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta} = (w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta},$$

where $\tilde{\Delta}$ denotes to the derivative on $\tilde{\mathbb{T}}$.

Lemma 2.2. *Let (I1)-(I3) hold. If $x(t)$ be an eventually positive solution of (1.1), then $z(t)$ satisfies*

$$z(t) > 0, z^{\Delta}(t) > 0, z^{\Delta\Delta}(t) \leqslant 0, \quad \textit{for} \;\; t \in [t_0, \infty)_{\mathbb{T}}.$$

Proof. Since $x(t)$ be an eventually positive solution of (1.1), then by (I1)-(I3) there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large such that $x(t) > 0, x(\delta(t)) > 0$, and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Moreover, we get

$$(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \leqslant -Q(t)z^{\alpha}(\delta(t)) \leqslant 0. \tag{2.1}$$

Hence $r(t)(z^{\Delta}(t))^{\alpha}$ is a non-increasing function and is eventually of one sign. We claim that $z^{\Delta}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. If not, then there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) \leqslant 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. Since q is not identically zero eventually, we may assume that $z^{\Delta}(t) < 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. From (2.1) we have

$$(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \leqslant -c < 0, \quad \text{for} \;\; t \in [t_2, \infty)_{\mathbb{T}},$$

where $c := -r(t_2)(z^{\Delta}(t_2))^{\alpha} > 0$, then

$$z^{\Delta}(t) \leqslant -\frac{c^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(t)} \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.2}$$

Integrating (2.2) on $[t_2, t)_{\mathbb{T}} \subseteq [t_1, t)_{\mathbb{T}}$, we obtain

$$z(t) \leqslant z(t_2) - c^{\frac{1}{\alpha}} \int_{t_2}^{t} \frac{\Delta\xi}{r^{\frac{1}{\alpha}}(\xi)} \text{ for all } t \in [t_2, \infty)_{\mathbb{T}}.$$

Letting $t \to \infty$, then it follows that $\lim_{t\to\infty} z(t) = -\infty$. Therefore, $\lim_{t\to\infty} z(t) = -\infty$, which is a contradiction. Consequently, $z^{\Delta}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Now, since

$$[r(t)((z^{\Delta}(t))^{\alpha})]^{\Delta} = r(t)((z^{\Delta}(t))^{\alpha})^{\Delta} + r^{\Delta}(t)(z^{\Delta}(\sigma(t)))^{\alpha} \text{ for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.3}$$

Applying the Pötzsche chain rule and Theorem 2.1, we get

$$\left(\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \geqslant \begin{cases} \alpha\left(z^{\Delta}(t)\right)^{\alpha-1} z^{\Delta\Delta}(t), & \alpha \geqslant 1, \\ \alpha\left(z^{\Delta}(\sigma(t))\right)^{\alpha-1} z^{\Delta\Delta}(t), & 0 < \alpha < 1. \end{cases}$$

This with the non-increasing fact of $z^{\Delta}(t)$, leads to

$$\left(\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \geqslant \alpha\left(z^{\Delta}(t)\right)^{\alpha-1} z^{\Delta\Delta}(t). \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\alpha r(t)(z^{\Delta}(t))^{\alpha-1} z^{\Delta\Delta}(t) + r^{\Delta}(t)(z^{\Delta}(\sigma(t)))^{\alpha} \leqslant 0.$$

Since $r(t) > 0, r^{\Delta}(t) \geqslant 0, z^{\Delta}(t) > 0$, we obtain

$$z(t) > 0, z^{\Delta}(t) > 0, z^{\Delta\Delta}(t) \leqslant 0, \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

which completes the proof. □

The following theorem presents oscillation criteria for (1.1) using a comparison theorem.

Theorem 2.3. *Assume that* (1.2) *holds. If the first-order dynamic equation*

$$y^{\Delta}(t) + Q(t)\tilde{R}^{\alpha}(\delta(t))y(\delta(t)) = 0, \tag{2.5}$$

is oscillatory, then (1.1) *is also oscillatory.*

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \geqslant t_0$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\delta(t)) > 0$, for $t \geqslant t_1$. By the definition of $z(t)$, we obtain

$$x(t) \geqslant z(t) - p(t)x(\tau(t)) \geqslant z(t) - p(t)z(\tau(t)) \geqslant (1 - p(t))z(t).$$

This combined with (1.1) provides

$$(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \leqslant -Q(t)z^{\alpha}(\delta(t)). \tag{2.6}$$

Also, from the definition of $z(t)$, we get

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s)z^{\Delta}(s)\Delta s \geqslant R(t)r^{1/\alpha}(t)z^{\Delta}(t). \tag{2.7}$$

On the other hand

$$\begin{aligned} [(z(t) - R(t)r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta} &= z^{\Delta}(t) - [R(t)r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta} \\ &= z^{\Delta}(t) - [R(t)(r^{1/\alpha}(t)z^{\Delta}(t))^{\Delta} + R^{\Delta}(t)(r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t)))] \\ &= z^{\Delta}(t) - R(t)(r^{1/\alpha}(t)z^{\Delta}(t))^{\Delta} - \left(\frac{r(\sigma(t))}{r(t)}\right)^{1/\alpha} z^{\Delta}(\sigma(t)). \end{aligned} \tag{2.8}$$

Since $(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \leqslant 0$, then $r(\sigma(t))(z^{\Delta}(\sigma(t)) \leqslant r(t)(z^{\Delta}(t))^{\alpha}$. Hence, (2.8) takes the form

$$[z(t) - R(t)r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta} \geqslant z^{\Delta}(t) - R(t)(r^{1/\alpha}(t)z^{\Delta}(t))^{\Delta} - \left(\frac{z^{\Delta}(t)}{z^{\Delta}(\sigma(t))}\right)z^{\Delta}(\sigma(t)),$$

i.e.,

$$[z(t) - R(t)r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta} \geqslant -R(t)[r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta}. \tag{2.9}$$

Applying chain rule, we get

$$R(t)[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta} \geqslant \alpha R(t)\begin{cases}[r^{1/\alpha}(t)z^{\Delta}(t)]^{\alpha-1}[r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta}, & \alpha \geqslant 1,\\ [r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]^{\alpha-1}[r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta}, & 0<\alpha<1.\end{cases} \tag{2.10}$$

Since $(r(t)(z^{\Delta}(t))^{\alpha})$ is a nonincreasing function, then (2.10) takes the form

$$R(t)[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta} \geqslant \alpha R(t)[r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]^{\alpha-1}[r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta}.$$

This with (2.6), leads to

$$-R(t)[r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta} \geqslant \frac{1}{\alpha}R(t)[r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]^{1-\alpha}Q(t)z^{\alpha}(\delta(t)). \tag{2.11}$$

From (2.9) and (2.11), we get

$$[z(t) - R(t)r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta} \geqslant \frac{1}{\alpha}R(t)[r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]^{1-\alpha}Q(t)z^{\alpha}(\delta(t)). \tag{2.12}$$

Combining (2.7) with (2.12), and taking in your account that $(r(t)(z^{\Delta}(t))^{\alpha})$ is non-increasing, we conclude that

$$\begin{aligned}[z(t) - R(t)r^{1/\alpha}(t)z^{\Delta}(t)]^{\Delta} &\geqslant \frac{1}{\alpha}R(t)[r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]^{1-\alpha}Q(t)R^{\alpha}(\delta(t))r(\delta(t))(z^{\Delta}(\delta(t)))^{\alpha}\\ &\geqslant \frac{1}{\alpha}R(t)R^{\alpha}(\delta(t))[r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]^{1-\alpha}Q(t)r(t)(z^{\Delta}(t))^{\alpha}\\ &\geqslant \frac{1}{\alpha}R(t)R^{\alpha}(\delta(t))[r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]^{1-\alpha}Q(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}\\ &\geqslant \frac{1}{\alpha}R(t)R^{\alpha}(\delta(t))[r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))]Q(t).\end{aligned} \tag{2.13}$$

Integrating (2.13) from t_1 to t and considering the non-increasing fact of the function $(r(t)(z^{\Delta}(t))^{\alpha})$, we get

$$\begin{aligned}z(t) &\geqslant R(t)r^{1/\alpha}(t)z^{\Delta}(t) + \frac{1}{\alpha}\int_{t_1}^{t}[r^{1/\alpha}(\sigma(s))z^{\Delta}(\sigma(s))]R(s)R^{\alpha}(\delta(s))Q(s)\Delta s\\ &\geqslant R(t)r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t)) + \frac{1}{\alpha}\int_{t_1}^{t}[r^{1/\alpha}(\sigma(s))z^{\Delta}(\sigma(s))]R(s)R^{\alpha}(\delta(s))Q(s)\Delta s\\ &\geqslant r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))\left(R(t) + \frac{1}{\alpha}\int_{t_1}^{t}R(s)R^{\alpha}(\delta(s))Q(s)\Delta s\right)\\ &\geqslant r^{1/\alpha}(\sigma(t))z^{\Delta}(\sigma(t))\tilde{R}(t).\end{aligned}$$

Thus, we have

$$z(\delta(t)) \geqslant r^{1/\alpha}(\sigma(\delta(t)))z^{\Delta}(\sigma(\delta(t)))\tilde{R}(\delta(t)) \geqslant r^{1/\alpha}(\delta(t))z^{\Delta}(\delta(t))\tilde{R}(\delta(t)). \tag{2.14}$$

Substituting (2.14) into (2.6), we get

$$y^{\Delta}(t)+Q(t)\tilde{R}^{\alpha}(\delta(t))y(\delta(t))\leqslant 0, \tag{2.15}$$

where $y(t):=r(t)(z^{\Delta}(t))^{\alpha}$, is a positive solution of the first order delay dynamic inequality (2.14). By [19, Theorem 3.1], (2.5) also presents a nonoscillatory solution. This contradiction proves that (1.1) is oscillatory. □

In view of Theorem 2.3 and [19, Theorems 1 and 2], we introduce the following Corollary.

Corollary 2.4. *Let $\delta^{\sigma}(t)\leqslant t$ and $\delta^{\Delta}(t)\geqslant 0$. If*

$$\limsup_{t\to\infty}\left(\frac{\int_{\delta(t)}^{\sigma(t)}Q(s)\tilde{R}^{\alpha}(\delta(s))\Delta s}{1-[1-\mu(\delta(t))Q(\delta(t))\tilde{R}^{\alpha}(\delta(\delta(t)))]\mu(\sigma(t))Q(\sigma(t))\tilde{R}^{\alpha}(\delta(\sigma(t)))}\right)>1,$$

then every solution of (2.5) is oscillatory.

The following lemma plays a major role throughout the proofs of our main results.

Lemma 2.5. *Let condition (1.2) holds and assume that $\delta(t)$ is strictly increasing and (1.1) has a positive solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Then for every $n>0$ and $y(t):=r(t)(z^{\Delta}(t))^{\alpha}$ satisfies*

$$\frac{y(\delta(t))}{y(t)}\geqslant f_n(\rho), \tag{2.16}$$

for t large enough and $f_n(\rho)$ defined by (1.5).

Proof. Assume that (1.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we can suppose that there exists a $t_1\geqslant t_0$, such that $x(t)>0, x(\tau(t))>0$, and $x(\delta(t))>0$ for $t\geqslant t_1$. We deduce that $y(t):=r(t)(z^{\Delta}(t))^{\alpha}$ is a positive solution of the first-order delay dynamic inequality (2.15). Proceeding similarly as in the proof of [28, Theorem 3.1], we get (2.16) holds. □

Now, we present new oscillation criteria for (1.1) using the Riccati substitution technique.

Theorem 2.6. *Assume that (I1)-(I3) and (1.2) hold, and $\delta^{\Delta}>0$. If there exists a function $\varphi\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, such that*

$$\limsup_{t\to\infty}\int_{t_2}^{s}\left(\varphi(s)Q(s)-\frac{(\varphi_+^{\Delta}(s))^{\alpha+1}r(\delta(s))r(\sigma(s))}{(\alpha+1)^{\alpha+1}f_n(\rho)\varphi^{\alpha}(s)(\delta^{\Delta}(s))^{\alpha}r(s)}\right)\Delta s=\infty, \tag{2.17}$$

holds for some sufficiently large $t_2\geqslant t_1$ and for some $n\geqslant 0$, where $\varphi_+^{\Delta}(t)=\max\{0,\varphi^{\Delta}(t)\}$, then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0, x(\tau(t))>0$, and $x(\delta(t))>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Define the Riccati substitution

$$\omega(t)=\varphi(t)r(t)\left(\frac{z^{\Delta}(t)}{z(\delta(t))}\right)^{\alpha}.$$

It is clear that $\omega(t)>0$ and

$$\begin{aligned}\omega^{\Delta}(t)=&\frac{\varphi(t)}{z(\delta(t))^{\alpha}}[r(t)\ (z(\delta(t)))^{\alpha}]^{\Delta}+\left(\frac{\varphi(t)}{z(\delta(t))^{\alpha}}\right)^{\Delta}[r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}]\\ =&\frac{\varphi(t)}{(z(\delta(t)))^{\alpha}}[r(t)\ (z(\delta(t)))^{\alpha}]^{\Delta}+\frac{\varphi^{\Delta}(t)}{(z(\delta(\sigma(t))))^{\alpha}}[r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}]\\ &-\frac{\varphi(t)(z^{\alpha}(\delta(t)))^{\Delta}}{z^{\alpha}(\delta(t))z^{\alpha}(\delta(\sigma(t)))}[r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}]\\ \leqslant&-Q(t)\varphi(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))-\frac{\varphi(t)}{\varphi(\sigma(t))}\frac{(z^{\alpha}(\delta(t)))^{\Delta}}{z^{\alpha}(\delta(t))}\omega(\sigma(t)).\end{aligned} \tag{2.18}$$

Applying the Pötzsche chain rule and Theorem 2.1, we get

$$(z^{\alpha}(\delta(t)))^{\Delta} \geqslant \begin{cases} \alpha z^{\alpha-1}(\delta(t))z^{\Delta}(\delta(t))\delta^{\Delta}(t), & \alpha \geqslant 1, \\ \alpha z^{\alpha-1}(\delta(\sigma(t)))z^{\Delta}(\delta(t))\delta^{\Delta}(t), & 0 < \alpha < 1. \end{cases}$$

This with the increasing property of $z(t)$, leads to

$$(z^{\alpha}(\delta(t)))^{\Delta} \geqslant \alpha z^{\alpha-1}(\delta(\sigma(t)))z^{\Delta}(\delta(t))\delta^{\Delta}(t).$$

Then, (2.18) takes the form

$$\begin{aligned} \omega^{\Delta}(t) &\leqslant - Q(t)\varphi(t) + \frac{\varphi_{+}^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) - \alpha\frac{\varphi(t)}{\varphi(\sigma(t))}\frac{z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{z(\delta(\sigma(t)))}\omega(\sigma(t)) \\ &\leqslant - Q(t)\varphi(t) + \frac{\varphi_{+}^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) - \alpha\frac{\varphi(t)\delta^{\Delta}(t)}{\varphi(\sigma(t))}\frac{z^{\Delta}(\delta(t))}{z^{\Delta}(t)}\frac{z^{\Delta}(t)}{z(\delta(\sigma(t)))}\omega(\sigma(t)). \end{aligned} \tag{2.19}$$

From Lemma 2.5, we get

$$\frac{z^{\Delta}(\delta(t))}{z^{\Delta}(t)} \geqslant \left(\frac{r(t)f_n(\rho)}{r(\delta(t))}\right)^{1/\alpha}.$$

This combined with (2.19) provides

$$\omega^{\Delta}(t) \leqslant -Q(t)\varphi(t) + \frac{\varphi_{+}^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) - \alpha\frac{\varphi(t)\delta^{\Delta}(t)}{\varphi(\sigma(t))}\left(\frac{r(t)f_n(\rho)}{r(\delta(t))}\right)^{1/\alpha}\frac{z^{\Delta}(t)}{z(\delta(\sigma(t)))}\omega(\sigma(t)). \tag{2.20}$$

Since $z^{\Delta\Delta}(t) \leqslant 0$, then for t and $\sigma(t)$,

$$z^{\Delta}(t) \geqslant z^{\Delta}(\sigma(t)).$$

This with (2.20), leads to

$$\begin{aligned} \omega^{\Delta}(t) &\leqslant - Q(t)\varphi(t) + \frac{\varphi_{+}^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) - \alpha\frac{\varphi(t)\delta^{\Delta}(t)}{\varphi(\sigma(t))}\left(\frac{r(t)f_n(\rho)}{r(\delta(t))}\right)^{1/\alpha}\frac{z^{\Delta}(\sigma(t))}{z(\delta(\sigma(t)))}\omega(\sigma(t)) \\ &\leqslant - Q(t)\varphi(t) + \frac{\varphi_{+}^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t))) - \alpha\frac{\varphi(t)\delta^{\Delta}(t)}{(\varphi(\sigma(t)))^{\alpha+1/\alpha}}\left(\frac{r(t)f_n(\rho)}{r(\delta(t))r(\sigma(t))}\right)^{1/\alpha}(\omega(\sigma(t)))^{\alpha+1/\alpha}. \end{aligned}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\alpha+1}{\alpha}} \leqslant \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},$$

with $B = \frac{\varphi_{+}^{\Delta}(t)}{\varphi(\sigma(t))}$ and $A = \frac{\alpha\varphi(t)\delta^{\Delta}(t)}{(\varphi(\sigma(t)))^{\alpha+1/\alpha}}\left(\frac{r(t)f_n(\rho)}{r(\delta(t))r(\sigma(t))}\right)^{1/\alpha}$, we get

$$\omega^{\Delta}(t) \leqslant -\varphi(t)Q(t) + \frac{(\varphi_{+}^{\Delta}(t))^{\alpha+1}r(\delta(t))r(\sigma(t))}{(\alpha+1)^{\alpha+1}f_n(\rho)\varphi^{\alpha}(t)(\delta^{\Delta}(t))^{\alpha}r(t)}. \tag{2.21}$$

Integrating (2.21) from $t_2 \in [t_1, \infty)_{\mathbb{T}}$ to t, we have

$$\int_{t_2}^{t}\left(\varphi(s)Q(s) - \frac{(\varphi_{+}^{\Delta}(s))^{\alpha+1}r(\delta(s))r(\sigma(s))}{(\alpha+1)^{\alpha+1}f_n(\rho)\varphi^{\alpha}(s)(\delta^{\Delta}(s))^{\alpha}r(s)}\right)\Delta s \leqslant \omega(t_2),$$

which contradicts (2.17). This completes the proof. □

Letting $\varphi(t) = R^{\alpha}(\delta(t))$, then Theorem 2.6 yields the following result.

Corollary 2.7. *Assume that (I1)-(I3) and (1.2) hold, $\delta^{\Delta} > 0$, and suppose that $\varphi(t) = R^{\alpha}(\delta(t))$. If*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left(R^{\alpha}(\delta(s))Q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{r(\sigma(s))\delta^{\Delta}(s)}{f_n(\rho)R(\delta(s))r^{1/\alpha}(\delta(s))r(s)} \right)\Delta s = \infty, \tag{2.22}$$

holds for some sufficiently large $t_2 \geqslant t_1$ and for some $n \geqslant 0$, then (1.1) is oscillatory.

Theorem 2.8. *Assume that (I1)-(I3) and (1.2) hold, and there exists a function $\psi \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$. If*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left(\psi(s)Q(s) - \frac{(\psi_+^{\Delta}(s))^{\alpha+1} r(\sigma(s))}{(\alpha+1)^{\alpha+1}(\psi(s))^{\alpha}} \right)\Delta s = \infty, \tag{2.23}$$

holds for some sufficiently large $t_2 \geqslant t_1$, where $\psi_+^{\Delta}(t) = \max\{0, \psi^{\Delta}(t)\}$, then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0, x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$.

Now, define the Riccati substitution

$$\omega(t) = \psi(t)r(t)\left(\frac{z^{\Delta}(t)}{z(t)}\right)^{\alpha},$$

then,

$$\begin{aligned}
\omega^{\Delta}(t) =& \psi^{\Delta}(t)\left[r(t)\left(\frac{z^{\Delta}(t)}{z(t)}\right)^{\alpha}\right]^{\sigma} + \psi(t)\frac{(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta}}{z^{\alpha}(t)} + \psi(t)r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}\left(\frac{1}{z^{\alpha}(t)}\right)^{\Delta} \\
\leqslant& \frac{\psi_+^{\Delta}(t))}{\psi(\sigma(t))}\omega(\sigma(t)) + \psi(t)\frac{(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta}}{z^{\alpha}(t)} - \alpha\psi(t)r(\sigma(t))\left(\frac{z^{\Delta}(\sigma(t))}{z(t)}\right)^{\alpha}\left(\frac{z^{\Delta}(t)}{z(\sigma(t))}\right),
\end{aligned}$$

since $z^{\Delta}(t)$ is nonincreasing and $z(t)$ is increasing, then for t and $\sigma(t)$,

$$\begin{aligned}
\omega^{\Delta}(t) \leqslant& \frac{\psi_+^{\Delta}(t)}{\psi(\sigma(t))}\omega(\sigma(t)) + \psi(t)\frac{(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta}}{z^{\alpha}(t)} - \alpha\psi(t)r(\sigma(t))\left(\frac{z^{\Delta}(\sigma(t))}{z(\sigma(t))}\right)^{\alpha+1} \\
\leqslant& \frac{\psi_+^{\Delta}(t)}{\psi(\sigma(t))}\omega(\sigma(t)) - \psi(t)Q(t) - \alpha\psi(t)r(\sigma(t))\left(\frac{z^{\Delta}(\sigma(t))}{z(\sigma(t))}\right)^{\alpha+1} \\
\leqslant& \frac{\psi_+^{\Delta}(t)}{\psi(\sigma(t))}\omega(\sigma(t)) - \psi(t)Q(t) - \frac{\alpha\psi(t)}{(\psi(\sigma(t)))^{\frac{\alpha+1}{\alpha}}(r(\sigma(t)))^{\frac{1}{\alpha}}}(\omega(\sigma(t)))^{\frac{\alpha+1}{\alpha}}.
\end{aligned}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\alpha+1}{\alpha}} \leqslant \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},$$

with $B = \frac{\psi_+^{\Delta}(t)}{\psi(\sigma(t))}$ and $A = \frac{\alpha\psi(t)}{(\psi(\sigma(t)))^{\frac{\alpha+1}{\alpha}}(r(\sigma(t)))^{\frac{1}{\alpha}}}$, we get

$$\omega^{\Delta}(t) \leqslant -\psi(t)Q(t) + \frac{(\psi_+^{\Delta}(t))^{\alpha+1}r(\sigma(t))}{(\alpha+1)^{\alpha+1}(\psi(t))^{\alpha}}. \tag{2.24}$$

Integrating (2.24) from $t_2 \in [t_1,\infty)_{\mathbb{T}}$ to t, we have

$$\int_{t_2}^{t} \left(\psi(s)Q(s) - \frac{(\psi_+^{\Delta}(s))^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}(\psi(s))^{\alpha}} \right)\Delta s \leqslant \omega(t_2),$$

which contradicts (2.23). This completes the proof. □

3. Examples

Example 3.1. Let $\mathbb{T}=\mathbb{R}$, consider the differential equation

$$\left(x(t)+\frac{9}{10}x(t/4)\right)''+\frac{\lambda}{t^2}x(t/5)=0,\quad t\geqslant 1, \tag{3.1}$$

where $\lambda>0$, $r(t)=1$, $\alpha=1$, $p(t)=\frac{9}{10}$, $q(t)=\frac{\lambda}{t^2}$, and $\delta(t)=t/5$. Since $\mu(t)=0$, $Q(t)=\frac{\lambda}{10t^2}$, $R(t)=t-1$, and $\tilde{R}(t)=t-1+\frac{\lambda\left(t^2+4t-6t\log(t)-5\right)}{50t}$.

Applying Corollary 2.7, Eq. (2.22) takes the form

$$\begin{aligned}
&\limsup_{t\to\infty}\int_{t_2}^{t}\left(R^{\alpha}(\delta(s))Q(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{r(\sigma(s))\delta'(s)}{f_n(\rho)R(\delta(s))r^{1/\alpha}(\delta(s))r(s)}\right)ds\\
&=\limsup_{t\to\infty}\int_{1}^{t}\left(R(\delta(s))Q(s)-\left(\frac{1}{2}\right)^{2}\frac{\delta'(s)}{f_n(\rho)R(\delta(s))}\right)ds\\
&=\limsup_{t\to\infty}\int_{1}^{t}\left(\frac{(s-5)\lambda}{50s^2}+\frac{1}{20e-4es}\right)ds.
\end{aligned}$$

Then by Corollary 2.7, every solution of (3.1) is oscillatory if $\lambda>4.59849$.

Remark 3.2. Applying [34, Theorem 3.1] to Example 3.1 we find that (3.1) is oscillatory if $\lambda>\frac{23}{4}$. Thus for Example 3.1 Corollary 2.7 $\lambda>4.59849$. So our result is more strong for testing oscillations.

Example 3.3. Consider Euler dynamic equation

$$x^{\Delta\Delta}(t)+\frac{\mu}{t^2}x(t)=0,\quad t\geqslant 1, \tag{3.2}$$

where $r(t)=1$, $\alpha=1$, $p(t)=0$, $q(t)=\frac{\mu}{t^2}$, and $\delta(t)=t$.

Special case, for $\mathbb{T}=\mathbb{R}$, (3.2) gives Euler differential equation

$$x''(t)+\frac{\mu}{t^2}x(t)=0,\quad t\geqslant 1. \tag{3.3}$$

Hence for $\psi(t)=t$, we can apply Theorem 2.8 condition, then Eq. (2.23) takes the form

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\psi(s)Q(s)-\frac{(\psi'_+(s))^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}(\psi(s))^{\alpha}}\right)ds=\limsup_{t\to\infty}\int_{1}^{t}\left(\frac{\mu}{s}-\frac{1}{4s}\right)ds=(4\mu-1)\infty.$$

So it is clear that (3.3) satisfies Theorem 2.8 condition for $\mu>\frac{1}{4}$, then (3.3) is oscillatory when $\mu>\frac{1}{4}$.

Now, for $\mathbb{T}=\mathbb{Z}$, (3.2) gives the discrete Euler equation

$$\Delta^2 x_n+\frac{\mu}{n^2}x_n=0,\quad n\geqslant 1. \tag{3.4}$$

Hence for $\psi(t)=t$, we can apply Theorem 2.8 condition, then (2.23) takes the form

$$\limsup_{n\to\infty}\sum_{s=1}^{n}\left(\psi(s)Q(s)-\frac{(\Delta\psi_+(s))^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}(\psi(s))^{\alpha}}\right)=\limsup_{n\to\infty}\sum_{s=1}^{n}\left(\frac{\mu}{s}-\frac{1}{4s}\right).$$

So it is clear that (3.4) satisfies Theorem 2.8 condition for $\mu>\frac{1}{4}$, then (3.4) is oscillatory when $\mu>\frac{1}{4}$.

Remark 3.4. The obtained result in Example 3.3 is consistent with the results in [29, 36].

Example 3.5. Let $\mathbb{T} = \mathbb{R}$. Consider the second order neutral differential equation

$$((x'(t))^{1/3})' + \frac{q_0}{t^{4/3}} x^{1/3}(0.9t) = 0, \quad t \geqslant 1, \tag{3.5}$$

where $r(t) = 1$, $\alpha = 1/3$, $p(t) = 0$, $q(t) = \frac{q_0}{t^{4/3}}$, and $\delta(t) = 0.9t$. Hence for $\psi(t) = t$, we can apply Theorem 2.8 condition, then (2.23) takes the form

$$\limsup_{t\to\infty} \int_{t_2}^{t} \left(\psi(s)Q(s) - \frac{(\psi'_{+}(s))^{\alpha+1} r(\sigma(s))}{(\alpha+1)^{\alpha+1}(\psi(s))^{\alpha}} \right) ds = \limsup_{t\to\infty} \int_{1}^{t} \left(\frac{q_0}{s^{1/3}} - \frac{1}{\left(\frac{4}{3}\right)^{4/3} s^{1/3}} \right) ds.$$

So it is clear that (3.5) satisfies Theorem 2.8 condition for $q_0 > 0.68142$, then (3.5) is oscillatory for $q_0 > 0.68142$.

The known related criteria for oscillation of this equation are as follows.

Using [4, Theorem 2]	$q_0 > 3.61643$
Using [17, Theorem 3]	$q_0 > 1.92916$
Using Theorem 2.8	$q_0 > 0.68142$

This means that Theorem 2.8 presents more effective oscillation criteria for (3.5).

Remark 3.6. It would be interesting to extend the obtained results in the non-canonical case, when $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s < \infty$.

Acknowledgments

The authors express their sincere gratitude to the anonymous referee for careful reading of the original manuscript and useful comments that helped to improve presentation of results and accentuate important details.

References

[1] R. P. Agarwal, M. Bohner, W.-T. Li, *Nonoscillation and oscillation theory for functional differential equations*, Marcel Dekker, New York, (2004). 1

[2] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations*, Kluwer Academic Publishers, Dordrecht, (2002). 1

[3] R. P. Agarwal, D. O'Regan, S. H. Saker, *Philos-type oscillation criteria for second order half-linear dynamic equations on time scales*, Rocky Mountain J. Math., **37** (2007), 1085–1104. 1

[4] B. Baculíková, J. Džurina, *Oscillation theorems for second-order nonlinear neutral differential equations*, Comput. Math. Appl., **62** (2011), 4472–4478. 3.5

[5] B. Baculíková, J. Džurina, *Comparison theorems for higher-order neutral delay differential equations*, J. Appl. Math. Comput., **49** (2015), 107–118. 1

[6] M. Bohner, S. Grace, I. Jadlovská, *Oscillation criteria for second-order neutral delay differential equations*, Electron. J. Qual. Theory Differ. Equ., **2017** (2017), 12 pages. 1

[7] M. Bohner, T. S. Hassan, T. X. Li, *Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equation with deviating arguments*, Indag. Math. (N.S.), **29** (2018), 548–560.

[8] M. Bohner, T. X. Li, *Oscillation of second-order p-Laplace dynamic equations with a nonpositive neutral coefficient*, Appl. Math. Lett., **37** (2014), 72–76.

[9] M. Bohner, T. X. Li, *Kamenev-type criteria for nonlinear damped dynamic equations*, Sci. China Math., **58** (2015), 1445–1452. 1

[10] M. Bohner, A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Springer Science & Business Media, Berlin, (2012). 2.1

[11] G. E. Chatzarakis, I. Jadlovská, *Improved oscillation results for second-order half-linear delay differential equations*, Hacet. J. Math. Stat., **48** (2019), 170–179. 1

[12] K.-S. Chiu, T. X. Li, *Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments*, Math. Nachr., **292** (2019), 2153–2164.

[13] X.-H. Deng, Q.-R. Wang, Z. Zhou, *Oscillation criteria for second order nonlinear delay dynamic equations on time scales*, Appl. Math. Comput., **269** (2015), 834–840.

[14] J. Džurina, S. R. Grace, I. Jadlovská, T. X. Li, *Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term*, Math. Nachr., **293** (2020), 910–922.

[15] M. M. A. El-sheikh, M. H. Abdalla, A. M. Hassan, *Oscillatory behaviour of higher-order nonlinear neutral delay dynamic equations on time scales*, Filomat, **32** (2018), 2635–2649. 1

[16] L. H. Erbe, Q. K. Kong, B. G. Zhang, *Oscillation theory for functional-differential equations*, Marcel Dekker, New York, (1995). 1

[17] S. R. Grace, J. Džurina, I. Jadlovská, T.-X. Li, *An improved approach for studying oscillation of second-order neutral delay differential equations*, J. Inequal. Appl., **2018** (2018), 13 pages. 1, 1, 3.5

[18] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, (1991). 1

[19] B. Karpuz, Ö. Öcalan, *New oscillation tests and some refinements for first-order delay dynamic equations*, Turkish J. Math., **40** (2016), 850–863. 2

[20] T. X. Li, N. Pintus, G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys., **70** (2019), 18 pages. 1

[21] T.-X. Li, Y. V. Rogovchenko, *Oscillation of second-order neutral differential equations*, Math. Nachr., **288** (2015), 1150–1162.

[22] T.-X. Li, Y. V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Appl. Math. Lett., **61** (2016), 35–41.

[23] T.-X. Li, Y. V. Rogovchenko, *On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations*, Appl. Math. Lett., **67** (2017), 53–59.

[24] T. X. Li, Y. V. Rogovchenko, *Oscillation criteria for second–order superlinear Emden–Fowler neutral differential equations*, Monatsh. Math., **184** (2017), 489–500.

[25] T. X. Li, Y. V. Rogovchenko, *On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations*, Appl. Math. Lett., **105** (2020), 7 pages.

[26] T. Li, G. Viglialoro, *Boundedness for a nonlocal reaction chemotaxis model even in the attraction dominated regime*, Differ. Integ. Equ., **34** (2021), 315–336. 1

[27] O. Moaaz, *New criteria for oscillation of nonlinear neutral differential equations*, Adv. Difference Equ., **2019** (2019), 11 pages. 1

[28] D. O'Regan, S. H. Saker, A. M. Elshenhab, R. P. Agarwal, *Distributions of zeros of solutions to first order delay dynamic equations*, Adv. Difference Equ., **2017** (2017), 16 pages. 1, 1, 2

[29] S. H. Saker, *Oscillation theorems of nonlinear difference equations of second order*, Dedicated to the memory of Professor Revaz Chitashvili, Georgian Math. J., **10** (2003), 343–352. 3.4

[30] S. H. Saker, *Oscillation criteria of second-order half-linear dynamic equations on time scales*, J. Comput. Appl. Math., **177** (2005), 375–387. 1

[31] S. H. Saker, D. O'regan, R. P. Agarwal, *Oscillation theorems for second-order nonlinear neutral delay dynamic equations on time scales*, Acta Math. Sin. (Engl. Ser.), **24** (2008), 1409–1432. 1

[32] G. Viglialoro, T. E. Woolley, *Boundedness in a parabolic-elliptic chemotaxis system with nonlinear diffusion and sensitivity and logistic source*, Math. Methods Appl. Sci., **41** (2018), 1809–1824. 1

[33] J. J. Wang, M. M. A. El-Sheikh, R. A. Sallam, D. I. Elimy, T. X. Li, *Oscillation results for nonlinear second-order damped dynamic equations*, J. Nonlinear Sci. Appl., **8** (2015), 877–883. 1

[34] J. S. Yang, J. J. Wang, X. W. Qin, T. X. Li, *Oscillation of nonlinear second-order neutral delay differential equations*, J. Nonlinear Sci. Appl., **10** (2017), 2727–2734. 3.2

[35] C. H. Zhang, R. P. Agarwal, M. Bohner, T. X. Li, *Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators*, Bull. Malays. Math. Sci. Soc., **38** (2015), 761–778. 1

[36] G. Zhang, S. S. Cheng, *A necessary and sufficient oscillation condition for the discrete Euler equation*, PanAmer. Math. J., **9** (1999), 29–34. 3.4

[37] S.-Y. Zhang, Q.-R. Wang, *Oscillation of second-order nonlinear neutral dynamic equations on time scales*, Appl. Math. Comput., **216** (2010), 2837–2848. 1

[38] B. G. Zhang, Y. Zhou, *The distribution of zeros of solutions of differential equations with a variable delay*, J. Math. Anal. Appl., **256** (2001), 216–228. 1, 1