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Ideal theory of semigroups based on $(3, 2)$-fuzzy sets

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Abstract

In this paper, the notions of $(3, 2)$-fuzzy ideal, $(3, 2)$-fuzzy bi-ideal, $(3, 2)$-fuzzy interior ideal and $(3, 2)$-fuzzy $(1, 2)$-ideal of a semigroup are introduced and investigated their properties. The relation between ($(3, 2)$-fuzzy) ideal, bi-ideal, interior ideal and $(1, 2)$-ideal are given. We characterized $(3, 2)$-fuzzy $(1, 2)$-ideal in terms of f^3-level α-cut and g^2-level α-cut. A necessary and sufficient condition for a subset of a semigroup to be $(1, 2)$-ideal in terms of $(3, 2)$-fuzzy $(1, 2)$-ideal of a semigroup is given.

Keywords: $(3, 2)$-fuzzy set, $(3, 2)$-fuzzy subalgebra, $(3, 2)$-fuzzy ideal.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [12]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 3, 4]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision making [5–7]. Yager [11] offered a new fuzzy set called a Pythgorean fuzzy set, which is the generalization of intuitionistic fuzzy sets. Fermatean fuzzy sets were introduced by Senapati and Yager [10], and they also defined basic operations over the Fermatean fuzzy sets. The concept of $(3, 2)$-fuzzy sets are introduced and studied in [8]. In this paper, the notions of $(3, 2)$-fuzzy ideal, $(3, 2)$-fuzzy bi-ideal, $(3, 2)$-fuzzy interior ideal and $(3, 2)$-fuzzy $(1, 2)$-ideal of a semigroup are introduced and investigated their properties. The relation between ($(3, 2)$-fuzzy) ideal, bi-ideal, interior ideal and $(1, 2)$-ideal are given. We characterized $(3, 2)$-fuzzy $(1, 2)$-ideal in terms of f^3-level α-cut and g^2-level α-cut. A necessary and sufficient condition for a subset of a semigroup to be $(1, 2)$-ideal in terms of $(3, 2)$-fuzzy $(1, 2)$-ideal of a semigroup is given.

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2. Preliminaries

Let S be a semigroup. By a subsemigroup [9] of S, we mean a non-empty subset A of S such that $A^2 \subset A$, and by a left (right) ideal [9] of S we mean a non-empty subset A of S such that $SA \subset A$ ($AS \subset A$). By two-sided ideal [9] or simply ideal, we mean a non-empty subset of S which is both a left and a right ideal of S. A subsemigroup A of a semigroup S is called a bi-ideal [9] of S if $ASA \subset A$. A subsemigroup A of S is called a $(1,2)$-ideal [9] of S if $ASA^2 \subset A$. A semigroup S is said to be $(2,2)$-regular [9] if $x \in x^2Sx^2$ for any $x \in S$. A semigroup S is said to be regular [9] if for each $x \in S$, there exists $y \in S$ such that $x = xyx$. A semigroup S is said to be intraregular [9] if for each $a \in S$, there exists $x, y \in S$ such that $a = xa^2y$. A semigroup S is said to be completely regular [9] if for each $x \in S$, there exists $y \in S$ such that $x = xyx$ and $xy = yx$. For a semigroup S, note that S is completely regular if and only if S is a union of groups if and only if S is $(2,2)$-regular. A semigroup S is said to be left (resp. right) duo [9] if every left (resp. right) ideal of S is a two-sided ideal of S.

Definition 2.1 ([8])**.** Let X be a nonempty set. The $(3,2)$-fuzzy set on X is defined to be a structure

$$\mathcal{C}_X := \{\langle x, f(x), g(x)\rangle \mid x \in X\}, \tag{2.1}$$

where $f : X \to [0,1]$ is the degree of membership of x to $\mathcal{C}$ and $g : X \to [0,1]$ is the degree of non-membership of x to $\mathcal{C}$ such that $0 \leqslant (f(x))^3 + (g(x))^2 \leqslant 1$.

In what follows, we use the notations $f^3(x)$ and $g^2(x)$ instead of $(f(x))^3$ and $(g(x))^2$, respectively, and the $(3,2)$-fuzzy set in (2.1) is simply denoted by $\mathcal{C}_X := (X, f, g)$.

3. $(3,2)$-fuzzy ideals in semigroup

In this section, let S denote a semigroup unless otherwise specified.

Definition 3.1. A $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ in S is called a $(3,2)$-fuzzy sub-semigroup of S, if

$$(\forall x, y \in S)\left(\begin{array}{c} f^3(xy) \geqslant \min\{f^3(x), f^3(y)\} \\ g^2(xy) \leqslant \max\{g^2(x), g^2(y)\} \end{array}\right). \tag{3.1}$$

Definition 3.2. A $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ in S is called a $(3,2)$-fuzzy left ideal of S, if

$$(\forall x, y \in S)\left(\begin{array}{c} f^3(xy) \geqslant f^3(y) \\ g^2(xy) \leqslant g^2(y) \end{array}\right). \tag{3.2}$$

Definition 3.3. A $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ in S is called a $(3,2)$-fuzzy right ideal of S, if

$$(\forall x, y \in S)\left(\begin{array}{c} f^3(xy) \geqslant f^3(x) \\ g^2(xy) \leqslant g^2(x) \end{array}\right). \tag{3.3}$$

A $(3,2)$-fuzzy set $\mathcal{C}_X := (X, f, g)$ in S is called a $(3,2)$-fuzzy ideal of S, if it is both a $(3,2)$-fuzzy left and $(3,2)$-fuzzy right ideal of S.

Example 3.4. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table.

	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	b	b	b	b

We define a $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ in S as follows.

S	a	b	c	d
f	0.91	0.62	0.43	0.65
g	0.31	0.52	0.73	0.64

Thus $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy sub-semigroup of S.

Example 3.5. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table.

	a	b	c	d
a	b	b	d	b
b	b	b	b	b
c	a	b	c	d
d	b	b	d	b

We define a $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ in S as follows.

S	a	b	c	d
f	0.90	0.18	0.16	0.15
g	0.21	0.5	0.17	0.64

Thus $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy left ideal of S. But $\mathcal{C} = (f, g)$ is not a $(3,2)$-fuzzy right ideal of S, since

$$f^3(ab) = f^3(b) = 0.00583 \ngeq 0.729 = f^3(a)$$

and

$$g^2(ab) = 0.25 \nleq 0.0441 = g^2(a).$$

Example 3.6. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table.

	a	b	c	d
a	b	c	c	c
b	b	c	c	c
c	c	c	c	c
d	b	c	c	c

We define a $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ in S as follows.

S	a	b	c	d
f	0.6	0.7	0.8	0.45
g	0.7	0.6	0.25	0.8

Thus $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy ideal of S.

It is clear that any $(3,2)$-fuzzy left (resp. right) ideal of S is a $(3,2)$-fuzzy sub-semigroup of S. But converse, need not be true which we can see in the following example.

Example 3.7. Let S be a semigroup in Example 3.6. Define a $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ on S by the table below.

S	a	b	c	d
f	0.2	0.4	0.5	0.7
g	0.9	0.7	0.6	0.8

Thus $\mathcal{C} = (f, g)$ is a $(3, 2)$-fuzzy subsemigroup of S. But $\mathcal{C} = (f, g)$ is not a $(3, 2)$-fuzzy ideal of S, since

$$f^3(ad) = f^3(c) = 0.125 \ngeq 0.343 = f^3(d)$$

and

$$f^3(da) = f^3(b) = 0.064 \ngeq 0.343 = f^3(d).$$

Definition 3.8. A $(3, 2)$-fuzzy sub-semigroup $\mathcal{C} = (f, g)$ of S is called a $(3, 2)$-fuzzy bi-ideal of S if

$$(\forall x, y, z \in S) \left(\begin{array}{l} f^3(xzy) \geqslant \min\{f^3(x), f^3(y)\} \\ g^2(xzy) \leqslant \max\{g^2(x), g^2(y)\} \end{array} \right).$$

Example 3.9. Let S be a semigroup in Example 3.4. Define a $(3, 2)$-fuzzy set $\mathcal{C} = (f, g)$ on S by the table below.

S	a	b	c	d
f	0.81	0.42	0.33	0.54
g	0.31	0.52	0.73	0.45

Thus $\mathcal{C} = (f, g)$ is a $(3, 2)$-fuzzy bi-ideal of S.

Theorem 3.10. *If $\{\mathcal{C}_i = (f_i, g_i)\}_{i\in I}$ is a family of $(3, 2)$-fuzzy bi-ideal of S, then $\cap\mathcal{C}_i$ is a $(3, 2)$-fuzzy bi-ideal of S, where $\cap\mathcal{C}_i = (\wedge f_i, \vee g_i)$ and $\wedge f_i = \inf\{f_i(x) : i \in I, x \in S\}, \vee g_i = \sup\{g_i(x) : i \in I, x \in S\}$.*

Proof. Let $x, y \in S$. Then we have

$$\wedge f_i^3(xy) \geqslant \inf\{\min\{f_i^3(x), f_i^3(y)\}\} = \inf\{\min\{f_i^3(x), f_i^3(y)\}\} = \inf\{\min\{f_i^3(x)\}, \min\{f_i^3(y)\}\} = \inf\{\wedge f_i^3(x), \wedge f_i^3(y)\},$$
$$\begin{aligned} \vee g_i^2(xy) &\leqslant \sup\{\max\{g_i^2(x), g_i^2(y)\}\} \\ &= \sup\{\max\{g_i^2(x), g_i^2(y)\}\} = \sup\{\max\{g_i^2(x)\}, \max\{g_i^2(y)\}\} = \sup\{\vee g_i^2(x), \vee g_i^2(y)\}. \end{aligned}$$

Hence $\cap\mathcal{C}_i$ is a $(3, 2)$-fuzzy sub-semigroup of S. Next for $u, x, y \in S$, we obtain

$$\begin{aligned} \wedge f_i^3(xuy) &\geqslant \sup\{\min\{f_i^3(x), f_i^3(y)\}\} \\ &= \inf\{\min\{f_i^3(x), f_i^3(y)\}\} = \inf\{\min\{f_i^3(x)\}, \min\{f_i^3(y)\}\} = \inf\{\wedge f_i^3(x), \wedge f_i^3(y)\}, \\ \vee g_i^2(xuy) &\leqslant \sup\{\max\{g_i^2(x), g_i^2(y)\}\} \\ &= \sup\{\max\{g_i^2(x), g_i^2(y)\}\} = \sup\{\max\{g_i^2(x)\}, \max\{g_i^2(y)\}\} = \sup\{\vee g_i^2(x), \vee g_i^2(y)\}. \end{aligned}$$

Hence $\cap\mathcal{C}_i$ is a $(3, 2)$-fuzzy bi-ideal of S. □

Theorem 3.11. *Every $(3, 2)$-fuzzy left (right) ideal of S is a $(3, 2)$-fuzzy bi-ideal of S.*

Proof. Let $\mathcal{C} = (f, g)$ be a $(3, 2)$-fuzzy left ideal of S and $u, x, y \in S$. Then

$$f^3(xuy) \geqslant f^3(y) \geqslant \min\{f^3(x), f^3(y)\} \quad \text{and} \quad g^2(xuy) \leqslant g^2(y) \leqslant \max\{g^2(x), g^2(y)\}.$$

Thus $\mathcal{C} = (f, g)$ is a $(3, 2)$-fuzzy bi-ideal of S. The right case is proved in a similar manner. □

Definition 3.12. A $(3, 2)$-fuzzy sub-semigroup $\mathcal{C} = (f, g)$ of S is called a $(3, 2)$-fuzzy $(1, 2)$-ideal of S if

$$(\forall x, y, z, u \in X) \left(\begin{array}{l} f^3(xu(yz)) \geqslant \min\{f^3(x), f^3(y), f^3(z)\} \\ g^2(xu(yz)) \leqslant \max\{g^2(x), g^2(y), g^2(z)\} \end{array} \right).$$

Example 3.13. Consider a semigroup $S = \{a, b, c, d\}$ with the following Cayley table.

	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	c	c	b
d	a	b	d	d

We define a $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ as follows.

S	a	b	c	d
f	0.91	0.72	0.53	0.44
g	0	0.11	0.32	0.43

Thus $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy $(1,2)$-fuzzy ideal of S.

Theorem 3.14. *Every $(3,2)$-fuzzy bi-ideal of S is a $(3,2)$-fuzzy $(1,2)$-ideal of S.*

Proof. Let $\mathcal{C} = (f, g)$ be a $(3,2)$-fuzzy bi-ideal of S and let $u, x, y, z \in S$. Then

$$f^3(xu(yz)) = f^3((xuy)z) \geqslant \min\{f^3(xuy), f^3(z)\} \geqslant \min\{\min\{f^3(x), f^3(y)\}, f^3(z)\} = \min\{f^3(x), f^3(y), f^3(z)\},$$
$$g^2(xu(yz)) = g^2((xuy)z) \leqslant \max\{g^2(xuy), g^2(z)\} \leqslant \max\{\max\{g^2(x), g^2(y)\}, g^2(z)\} = \max\{g^2(x), g^2(y), g^2(z)\}.$$

Hence $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy $(1,2)$-ideal of S. □

Corollary 3.15. *Every $(3,2)$-fuzzy left (right) of S is a $(3,2)$-fuzzy $(1,2)$-ideal of S.*

Theorem 3.16. *If S is a regular semigroup, then every $(3,2)$-fuzzy $(1,2)$-ideal of S is a $(3,2)$-fuzzy bi-ideal of S.*

Proof. Assume that a semigroup S is regular and let $\mathcal{C} = (f, g)$ be a $(3,2)$-fuzzy $(1,2)$-ideal of S. Let $u, x, y, z \in S$. Since S is regular, we have $xu \in (xSx)S \subset xSx$, which implies that $xu = xsx$ for some $s \in S$. Thus

$$f^3(xuy) = f^3((xsx)y) = f^3(xs(xy)) \geqslant \min\{f^3(x), f^3(x), f^3(y)\} = \min\{f^3(x), f^3(y)\},$$
$$g^2(xuy) = g^2((xsx)y) = g^2(xs(xy)) \leqslant \max\{g^2(x), g^2(x), g^2(y)\} = \max\{g^2(x), g^2(y)\}.$$

Therefore $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy bi-ideal of S. □

Theorem 3.17. *If $\mathcal{C}_1 = (f_1, g_1)$ and $\mathcal{C}_2 = (f_2, g_2)$ are $(3,2)$-fuzzy $(1,2)$-ideals of a semigroup S, then*

$$\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2 = (f = f_1 \wedge f_2, g = g_1 \vee g_2)$$

is a $(3,2)$-fuzzy $(1,2)$-ideal of a semigroup S.

Proof. If $\mathcal{C}_1 = (f_1, g_1)$ and $\mathcal{C}_2 = (f_2, g_2)$ are $(3,2)$-fuzzy $(1,2)$-ideals of a semigroup S, then the following hold

$$(\forall x, y \in X) \begin{pmatrix} f_1^3(xy) \geqslant \min\{f_1^3(x), f_1^3(y)\} \\ g_1^2(xy) \leqslant \max\{g_1^2(x), g_1^2(y)\} \\ f_2^3(xy) \geqslant \min\{f_2^3(x), f_2^3(y)\} \\ g_2^2(xy) \leqslant \max\{g_2^2(x), g_2^2(y)\} \end{pmatrix}, \tag{3.4}$$

$$(\forall x, y, z, u \in X) \begin{pmatrix} f_1^3(xu(yz)) \geqslant \min\{f_1^3(x), f_1^3(y), f_1^3(z)\} \\ g_1^2(xu(yz)) \leqslant \max\{g_1^2(x), g_1^2(y), g_1^2(z)\} \\ f_2^3(xu(yz)) \geqslant \min\{f_2^3(x), f_2^3(y), f_2^3(z)\} \\ g_2^2(xu(yz)) \leqslant \max\{g_2^2(x), g_2^2(y), g_2^2(z)\}. \end{pmatrix}. \tag{3.5}$$

Let $x, y \in X$. It follows from (3.4) that

$$\begin{aligned} f^3(xy) &= \min\{f_1^3(xy), f_2^3(xy)\} \\ &\geqslant \min\{\min\{f_1^3(x), f_1^3(y)\}, \min\{f_2^3(x), f_2^3(y)\}\} \\ &= \min\{\min\{f_1^3(x), f_2^3(x)\}, \min\{f_1^3(y), f_2^3(y)\}\} = \min\{f^3(x), f^3(y)\}, \\ g^2(xy) &= \max\{g_1^2(xy), g_2^2(xy)\} \\ &\leqslant \max\{\max\{g_1^2(x), g_1^2(y)\}, \max\{g_2^2(x), g_2^2(y)\}\} \\ &= \max\{\max\{g_1^2(x), g_2^2(x)\}, \max\{g_1^2(y), g_2^2(y)\}\} = \max\{g^2(x), g^2(y)\}. \end{aligned}$$

Let $x, y, z, u \in X$. It follows from (3.5) that

$$\begin{aligned} f^3(xu(yz)) &= \min\{f_1^3(xu(yz)), f_2^3(xu(yz))\} \\ &\geqslant \min\{\min\{f_1^3(x), f_1^3(y), f_1^3(z)\}, \min\{f_2^3(x), f_2^3(y), f_2^3(z)\}\} \\ &= \min\{\min\{f_1^3(x), f_2^3(x)\}, \min\{f_1^3(y), f_2^3(y)\}, \min\{f_1^3(z), f_2^3(z)\}\} = \min\{f^3(x), f^3(y), f^3(z)\}, \\ g^2(xu(yz)) &= \max\{g_1^2(xu(yz)), g_2^2(xu(yz))\} \\ &\leqslant \max\{\min\{g_1^2(x), g_1^2(y), g_1^2(z)\}, \max\{g_2^2(x), g_2^2(y), g_2^2(z)\}\} \\ &= \max\{\max\{g_1^2(x), g_2^2(x)\}, \max\{g_1^2(y), g_2^2(y)\}, \max\{g_1^2(z), g_2^2(z)\}\} = \max\{g^2(x), g^2(y), g^2(z)\}. \end{aligned}$$

Hence $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2 = (f, g)$ is a $(3, 2)$-fuzzy $(1, 2)$-ideal of a semigroup S. □

Theorem 3.18. *If $\mathcal{C}_i = (f_i, g_i)_{i\in\Delta}$ are $(3, 2)$-fuzzy $(1, 2)$-ideals of a semigroup S, then $\mathcal{C} = \bigcap_{i\in\Delta} \mathcal{C}_i = (f, g)$ is a $(3, 2)$-fuzzy $(1, 2)$-ideal of a semigroup S.*

Proof. If $\mathcal{C}_i = (f_i, g_i)_{i\in\Delta}$ are $(3, 2)$-fuzzy $(1, 2)$-ideals of a semigroup S, then the following hold

$$(\forall x, y \in X) \begin{pmatrix} f_i^3(xy) \geqslant \min\{f_i^3(x), f_i^3(y)\} \\ g_i^2(xy) \leqslant \max\{g_i^2(x), g_i^2(y)\} \end{pmatrix}, \tag{3.6}$$

$$(\forall x, y, z, u \in X) \begin{pmatrix} f_i^3(xu(yz)) \geqslant \min\{f_i^3(x), f_i^3(y), f_i^3(z)\} \\ g_i^2(xu(yz)) \leqslant \max\{g_i^2(x), g_i^2(y), g_i^2(z)\}. \end{pmatrix}. \tag{3.7}$$

Let $x, y \in X$. It follows from (3.6) that

$$\begin{aligned} f^3(xy) &= \inf\{f_i^3(xy)\} \geqslant \inf\{\min\{f_i^3(x), f_i^3(y)\}\}\} = \min\{\inf\{f_i^3(x)\}, \inf\{f_i^3(y)\}\} = \min\{f^3(x), f^3(y)\}, \\ g^2(xy) &= \sup\{g_i^2(xy)\} \leqslant \sup\{\max\{g_i^2(x), g_i^2(y)\}\}\} = \max\{\sup\{g_i^2(x)\}, \sup\{g_i^2(y)\}\} = \max\{g^2(x), g^2(y)\}. \end{aligned}$$

Let $x, y, z, u \in X$. It follows from (3.7) that

$$\begin{aligned} f_i^3(xu(yz)) &= \inf\{f_i^3(xu(yz))\} \geqslant \inf\{\min\{f_i^3(x), f_i^3(y), f_i^3(z)\}\} \\ &= \min\{\inf\{f_i^3(x)\}, \inf\{f_i^3(y)\}, \inf\{f_i^3(z)\}\} = \min\{f^3(x), f^3(y), f^3(z)\}, \\ g_i^2(xu(yz)) &= \sup\{g_i^2(xu(yz))\} \geqslant \sup\{\max\{g_i^2(x), g_i^2(y), g_i^2(z)\}\} \\ &= \max\{\sup\{g_i^2(x)\}, \sup\{g_i^2(y)\}, \sup\{g_i^2(z)\}\} = \max\{g^2(x), g^2(y), g^2(z)\}. \end{aligned}$$

Hence $\mathcal{C} = \bigcap_{i\in\Delta} \mathcal{C}_i = (f, g)$ is a $(3, 2)$-fuzzy $(1, 2)$-ideal of a semigroup S. □

Definition 3.19. A $(3, 2)$-fuzzy set $\mathcal{C} = (f, g)$ in S is called a $(3, 2)$-fuzzy interior ideal of S if it satisfies the following

$$(\forall u, x, y \in S) \begin{pmatrix} f^3(xuy) \geqslant f^3(u) \\ g^2(xuy) \leqslant g^2(u) \end{pmatrix}.$$

Example 3.20. Let $S = \{0, a, b, c, d\}$ be a set with the following Cayley table.

	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	c	0
b	0	0	b	0	d
c	0	c	0	0	a
d	0	0	d	b	0

Define a $(3, 2)$-fuzzy set $\mathcal{C} = (f, g)$ in S as follows.

S	0	a	b	c	d
f	1	1	1	0	0
g	0	0	0	1	1

Thus $\mathcal{C} = (f, g)$ is a $(3, 2)$-fuzzy interior ideal of S.

Theorem 3.21. *If $\{\mathcal{C}_i = (f_i, g_i)\}_{i\in I}$ is a family of $(3, 2)$-fuzzy interior ideal of S, then $\cap\mathcal{C}_i$ is a $(3, 2)$-fuzzy interior ideal of S, where $\cap\mathcal{C}_i = (\wedge f_i, \vee g_i)$ and $\wedge f_i(x)$ and $\vee g_i(x)$ are defined as follows:*

$$\wedge f_i(x) = \inf\{f_i(x) : i \in I, x \in S\}, \qquad \vee g_i(x) = \sup\{g_i(x) : i \in I, x \in S\}.$$

Proof. Let $u, x, y \in S$. Then

$$\wedge f_i^3(xy) \geqslant \inf\{\min\{f_i^3(x), f_i^3(y)\}\} = (\wedge f_i^3(x)) \wedge (\wedge f_i^3(y)),$$
$$\vee g_i^2(xy) \leqslant \sup\{\max\{g_i^2(x), g_i^2(y)\}\} = (\vee g_i^2(x)) \vee (\vee g_i^2(y))$$

Hence $\wedge f_i^3(xy) \geqslant \wedge f_i^3(u)$ and $\vee g_i^2(xy) \leqslant \vee g_i^2(u)$. Hence $\cap\mathcal{C}_i$ is a $(3, 2)$-fuzzy interior ideal of S. □

Definition 3.22. Let $\mathcal{C} = (f, g)$ be a $(3, 2)$-fuzzy set of S and $\alpha \in [0, 1]$. Then the sets $U(f, \alpha) = \{x \in S : f^3(x) \geqslant \alpha\}$ and $L(g, \alpha) = \{x \in S : g^2(x) \leqslant \alpha\}$ are called a f^3-level α-cut and g^2-level α-cut of $\mathcal{C}$, respectively.

Theorem 3.23. *If a $(3, 2)$-fuzzy set $\mathcal{C} = (f, g)$ in S is a $(3, 2)$-fuzzy interior ideal of S, then the sets $U(f, \alpha)$ and $L(g, \alpha)$ of $\mathcal{C}$ are interior ideals of S for every $\alpha \in Im(f) \cap Im(g) \subset [0, 1]$.*

Proof. Let $\alpha \in Im(f) \cap Im(g) \subset [0, 1]$ and $x, y \in U(f, \alpha)$. Then $f^3(x) \geqslant \alpha$ and $f^3(y) \geqslant \alpha$. It follows from (3.1) that $f^3(xy) \geqslant \min\{f^3(x), f^3(y)\} \geqslant \alpha$ so that $xy \in U(f, \alpha)$. If $x, y \in L(g, \alpha)$, then $g^2(x) \leqslant \alpha$ and $g^2(y) \leqslant \alpha$ and so $g^2(xy) \leqslant \max\{g^2(x), g^2(y)\} \leqslant \alpha$, that is, $xy \in L(g, \alpha)$. Hence $U(f, \alpha)$ and $L(g, \alpha)$ are sub-semigroup of S. Now let $xy \in S$ and $u \in U(f, \alpha)$. Then $f^3(xuy) \geqslant f^3(u) \geqslant \alpha$ and so $xuy \in U(f, \alpha)$. If $\alpha \in L(g, \alpha)$, then $g^2(xuy) \leqslant g^2(u) \leqslant \alpha$. Thus $xuy \in L(g, \alpha)$. Therefore $U(f, \alpha)$ and $L(g, \alpha)$ are interior ideal of S. □

Theorem 3.24. *Let $\mathcal{C} = (f, g)$ be a $(3, 2)$-fuzzy set S such that the nonempty sets $U(f, \alpha)$ and $L(g, \alpha)$ are interior ideals of S for all $\alpha \in [0, 1]$. Then $\mathcal{C} = (f, g)$ is a $(3, 2)$-fuzzy interior ideal of S.*

Proof. Let $\alpha \in [0, 1]$ and suppose that $U(f, \alpha)$ $(\neq \emptyset)$ and $L(g, \alpha)$ $(\neq \emptyset)$ are interior ideals of S. We must show that $\mathcal{C} = (f, g)$ satisfies conditions (3.1) and (3.3). If condition (3.1) is false, then there exist $x_0, y_0 \in S$ such that $f^3(x_0y_0) < \min\{f^3(x_0), f^3(y_0)\}$. Taking $\alpha_0 = \frac{1}{2}(f^3(x_0y_0) + \min\{f^3(x_0), f^3(y_0)\})$, we have $f^3(x_0y_0) < \alpha_0 < \min\{f^3(x_0), f^3(y_0)\}$. It follows that $x_0, y_0 \in U(f, \alpha)$ and $x_0y_0 \notin U(f, \alpha)$, which is a contradiction. Hence condition (3.1) is true. The proof of other conditions are similar to the case (3.1), we omit the proof. □

Theorem 3.25. *Let M be an interior ideal of S and let $\mathcal{C} = (f, g)$ be a $(3, 2)$-fuzzy in S defined by*

$$f(x) = \begin{cases} \alpha_0, & \text{if } x \in M, \\ \alpha_1, & \text{otherwise,} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \beta_0, & \text{if } x \in M, \\ \beta_1, & \text{otherwise,} \end{cases}$$

for all $x \in S$ and $\alpha_i, \beta_i \in [0, 1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$, and $\alpha_i^3 + \beta_i^2 \leqslant 1$ for $i = 0, 1$. Then $\mathcal{C} = (f, g)$ is a $(3, 2)$-fuzzy interior ideal of S and $U(f, \alpha_0) = M = L(g, \beta_0)$.

Proof. Let $x, y \in S$. If anyone of x and y does not belong to M, then $f^3(xy) \geqslant \alpha_1 = \min\{f^3(x), f^3(y)\}$, $g^2(xy) \leqslant \beta_1 = \max\{g^2(x), g^2(y)\}$. Other cases are trivial, and we omit the proof. Hence $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy subsemigroup of S. Now let $x, y, a \in S$. If $a \notin M$, then $f^3(xay) \geqslant \alpha_1 = f^3(a)$ and $g^2(xay) \leqslant \beta_1 = g^2(a)$. Assume that $a \in M$. Since M is a $(3,2)$-fuzzy interior ideal of S, $xay \in M$. Hence $f^3(xay) = \alpha_0 = f^3(a)$ and $g^2(xay) = \beta_0 = g^2(a)$. Hence $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy interior ideal of S. Obviously $U(f, \alpha_0) = M = L(g, \beta_0)$. □

Theorem 3.26. *If a $(3,2)$-fuzzy set $\mathcal{C} = (f, g)$ is a $(3,2)$-fuzzy interior ideal of S, then $f(x) = \sup\{\alpha \in [0,1] : x \in U(f, \alpha)\}$, $g(x) = \inf\{\alpha \in [0,1] : x \in L(g, \alpha)\}$ for all $x \in S$.*

Proof. Let $\delta = \sup\{\alpha \in [0,1] : x \in U(f, \alpha)\}$ and $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(f, \alpha)$. It follows that $\delta - \varepsilon < f^3(x)$ so that $\delta \leqslant f^3(x)$ since ε is arbitrary. We now show that $f^3(x) \leqslant \delta$. Let $f^3(x) = \beta$. Then $x \in U(f, \beta)$ and so $\beta = \{\alpha \in [0,1] : x \in U(f, \alpha_0)\}$. Hence $f^3(x) = \beta \leqslant \sup\{\alpha \in [0,1] : x \in U(f, \alpha)\} = \delta$. Therefore $f^3(x) = \delta = \sup\{\alpha \in [0,1] : x \in U(f, \alpha)\}$. Now, let $\eta = \inf\{\beta \in [0,1] : x \in L(g, \beta)\}$. Then $\inf\{\beta \in [0,1] : x \in L(g, \beta)\} < \eta + \varepsilon$ for any $\varepsilon < 0$, and so $\beta < \eta + \varepsilon$ for some $\beta \in [0,1]$ with $x \in L(g, \alpha)$. Since $g^2(x) \leqslant \beta$ and ε is arbitrary, it follows that $g^2(x) \leqslant \eta$. To prove $g^2(x) \geqslant \eta$, let $g^2(x) = \zeta$. Then $x \in L(g, \zeta)$ and thus $\zeta \in \{\beta \in [0,1] : x \in L(g, \beta)\}$. Hence $\inf\{\beta \in [0,1] : x \in L(g, \beta)\} \leqslant \zeta$, that is, $\eta \leqslant \zeta = g^2(x)$. Consequently, $g^2(x) = \eta = \inf\{\beta \in [0,1] : x \in L(g, \beta)\}$. □

Theorem 3.27. *Let $\{I_t : t \in \Delta \subset [0,1]\}$ be a collection of interior ideals of S such that*

1. $S = \bigcup_{t \in \Delta} I_t$;
2. $s > t$ *if and only if* $I_s \subset I_t$ *for all* $s, t \in \Delta$.

Then a $(3,2)$-fuzzy $\mathcal{C} = (f, g)$ in X defined by $f^3(x) = \sup\{t \in \Delta : x \in I_t\}$ and $g^2(x) = \inf\{t \in \Delta : x \in I_t\}$ for all $x \in S$ is a $(3,2)$-fuzzy interior ideal of S.

Proof. According to Theorem 3.24, it is sufficient to show that the non-empty sets $U(f, t)$ and $L(g, t)$ are ideals of S for every $t \in [0, f(0)]$ and $s \in [g(0), 1]$. In order to prove that $U(f, t)$ is an ideal of S, we divide the proof into the following two cases:

(1) $t = \sup\{q \in \Delta : q < t\}$;

(2) $t \neq \sup\{q \in \Delta : q < t\}$.

The case (1) implies that $x \in U(f, t) \Leftrightarrow x \in I_q, \forall q < t \Leftrightarrow x \in \bigcap_{q<t} I_q$, so that $U(f, t) = \bigcap_{q<t} I_q$, which is an ideal of X. For the case (2), we claim that $U(f, t) = \bigcup_{q \geqslant t} I_q$. If $x \in \bigcup_{q \geqslant t} I_q$, then $x \in I_q$ for some $q \geqslant t$. It follows that $f^3(x) \geqslant q \geqslant t$, so that $x \in U(f, t)$. This shows that $\bigcup_{q \geqslant t} I_q \subset U(f, t)$. Now assume that $x \notin \bigcup_{q \geqslant t} I_q$. Then $x \notin I_q$ for all $q \geqslant t$. Since $t = \sup\{q \in \Delta : q < t\}$, there exists $\epsilon > 0$ such that $(t - \epsilon, t) \cap \Delta = \emptyset$. Hence $x \notin I_q$ for all $q > t - \epsilon$, which means that $x \in I_q$, then $q \leqslant t - \epsilon$. Thus $f^3(x) \leqslant t - \epsilon < t$ and so $x \notin U(f, t)$. Therefore $U(f, t) \subset \bigcup_{q \geqslant t} I_q$ and thus $U(f, t) = \bigcup_{q \geqslant t} I_q$, which is an ideal of S. Next we prove that $L(g, t)$ is an ideal of S. We consider the following two cases:

(3) $s = \inf\{r \in \Delta : s < r\}$;

(4) $s \neq \inf\{r \in \Delta : s < r\}$.

For the case (3), we have $x \in L(g, s) \Leftrightarrow x \in I_r, \forall s < r \Leftrightarrow x \in \bigcap_{s<r} I_r$ and hence $L(g, s) = \bigcap_{s<r} I_r$, which is an ideal of S. For the case (4), there exists $\epsilon > 0$ such that $(s, s + \epsilon) \cap \Delta = \emptyset$. We will show that

$L(g,s)=\bigcup_{s\geqslant r} I_r$. If $x\in\bigcup_{s\geqslant r} I_r$, then $x\in I_r$ for some $r\leqslant s$. It follows that $g^2(x)\leqslant r\leqslant s$, so that $x\in L(g,s)$. Hence $\bigcup_{s\geqslant r} I_r\subset L(g,s)$. Conversely, if $x\notin\bigcup_{s\geqslant r} I_r$, then $x\notin I_r$ for all $r\leqslant s$, which implies that $x\notin I_r$ for all $r<s+\epsilon$, that is, if $x\in I_r$, then $r\geqslant s+\epsilon$. Thus $g^2(x)\geqslant s+\epsilon>s$, that is, $x\notin g_s$. Therefore $L(g,s)\subset\bigcup_{s\geqslant r} I_r$ and consequently $L(g,s)=\bigcup_{s\geqslant r} I_r$, which is an ideal of S. □

Theorem 3.28. *Let $\mathcal{C}=(f,g)$ be a $(3,2)$-fuzzy bi-ideal of S. If S is a completely regular, then $f^3(a)=f^3(a^2)$ and $g^2(a)=g^2(a^2)$ for all $a\in S$.*

Proof. Let $a\in S$. Then there exists $x\in S$ such that $a=a^2xa^2$. Hence

$$f^3(a)=f^3(a^2xa^2)\geqslant\min\{f^3(a^2),f^3(a^2)\}=f^3(a^2)\geqslant\min\{f^3(a),f^3(a)\}=f^3(a),$$
$$g^2(a)=g^2(a^2xa^2)\leqslant\max\{g^2(a^2),g^2(a^2)\}=g^2(a^2)\leqslant\max\{g^2(a),g^2(a)\}=g^2(a).$$

It follows that $f^3(a)=f^3(a^2)$ and $g^2(a)=g^2(a^2)$. □

Theorem 3.29. *Let $\mathcal{C}=(f,g)$ be a $(3,2)$-fuzzy bi-ideal of S. If S is an intraregular, then*

1. *$f^3(a)=f^3(a^2)$ and $g^2(a)=g^2(a^2)$ for all $a\in S$;*
2. *$f^3(ab)=f^3(ba)$ and $g^2(ab)=g^2(ba)$ for all $a,b\in S$.*

Proof.
(1). Let a be any element of S. Then since S is intraregular, there exist x and y in S such that $a=xa^2y$. Hence since $\mathcal{C}=(f,g)$ is a $(3,2)$-fuzzy ideal,

$$f^3(a)=f^3(xa^2y)\geqslant f^3(xa^2)\geqslant f^3(a^2)\geqslant\min\{f^3(a),f^3(a)\}=f^3(a)$$

and

$$g^2(a)=g^2(xa^2y)\leqslant g^2(xa^2)\leqslant g^2(a^2)\leqslant\max\{g^2(a),g^2(a)\}=g^2(a).$$

Hence $f^3(a)=f^3(a^2)$ and $g^2(a)=g^2(a^2)$.
(2). Let $a,b\in S$. It follows from Theorem 3.28 that

$$f^3(ab)=f^3((ab)^2)\geqslant f^3(a(ba)b)\geqslant f^3(ba)=f^3((ba)^2)\geqslant f^3(b(ab)a)\geqslant f^3(ab)$$

and

$$g^2(ab)=g^2((ab)^2)=g^2(a(ba)b)\leqslant g^2(ba)=g^2((ba)^2)=g^2(b(ab)a)\leqslant g^2(ab).$$

So we have $f^3(ab)=f^3(ba)$ and $g^2(ab)=g^2(ba)$. □

4. Conclusions

In this paper, we have introduced $(3,2)$-fuzzy ideal, $(3,2)$-fuzzy bi-ideal, $(3,2)$-fuzzy interior ideal, and $(3,2)$-fuzzy $(1,2)$-ideal of a semigroup. The relations between ($(3,2)$-fuzzy) ideal, bi-ideal, interior ideal, and $(1,2)$-ideal are studied. We characterized $(3,2)$-fuzzy $(1,2)$-ideal in terms of f^3-level α-cut and g^2-level α-cut. A necessary and sufficient condition for a subset of a semigroup to be $(1,2)$-ideal in terms of $(3,2)$-fuzzy $(1,2)$-ideal of a semigroup is proved.

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