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The $(2,3)$-fuzzy set and its application in BCK-algebras and BCI-algebras

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Abstract

It is well-known that an intuitionistic fuzzy set is a generalization of a fuzzy set. Intuitionistic fuzzy sets deal with two types of fuzzy sets, namely membership function and non-membership function, under the condition that the sum of the membership degree and non-membership degree is less than or equal to 1. If the sum of membership degree and non-membership degree is greater than or equal to 1, the intuitionistic fuzzy set feels limited in its role. The Pythagorean fuzzy set that emerged to overcome these limitations is the generalization of the intuitionistic fuzzy set. As another form of generalization of the intuitionistic fuzzy set, the concept of the $(2,3)$-fuzzy set is introduced in this article and several attributes are investigated. The semigroup structure is assigned to the collection of $(2,3)$-fuzzy sets. The concepts of $(2,3)$-fuzzy subalgebra for BCK/BCI-algebra and closed $(2,3)$-fuzzy subalgebra for BCI-algebra are introduced and their properties are investigated. The relationship between the $(2,3)$-fuzzy subalgebra and the degree function is discussed. A new $(2,3)$-fuzzy subalgebra is generated using the given $(2,3)$-fuzzy subalgebra. The union and intersection of $(2,3)$-fuzzy subalgebras are addressed, and the characterization of the $(2,3)$-fuzzy subalgebra using the $(2,3)$-cutty set is addressed. Conditions for closing the $(2,3)$-fuzzy subalgebra in the BCI-algebra are retrieved.

Keywords: $(2,3)$-fuzzy set, degree function, (closed) $(2,3)$-fuzzy subalgebra, $(2,3)$-cutty set.

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1. Introduction

Inaccuracy is an important factor in all decision-making processes. Considering the inaccuracies of the decision-making, Zadeh [25] introduced the idea of fuzzy set. Intuitionistic fuzzy sets are sets whose elements have degrees of membership and non-membership. It has been introduced by Atanassov [1] as an extension of Zadeh's notion of fuzzy set, which itself extends the classical notion of a set. Intuitionistic fuzzy sets are widely used in all fields (see [3, 4, 15, 24] for applications in algebraic structures). Also, the ideas of intuitionistic fuzzy sets appear to be useful in modeling many real-life situations like career

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determination [7], medical diagnosis [5, 6, 8], and decision-making [9, 10, 13], etc. Intuitionistic fuzzy sets are addressed under the condition that the sum of the membership and non-membership degrees is less than or equal to one. Intuitionistic fuzzy sets cannot be applied to a problem that includes the condition the sum of the membership and non-membership degrees is greater than or equal to one. To overcome this limitation in IFS, Pythagorean fuzzy sets are introduced by Yager [21–23]. Pythagorean fuzzy sets are a new tool to deal with vagueness considering the membership grade (α) and non-membership grade (β) satisfying the conditions $\alpha + \beta \leqslant 1$ or $\alpha + \beta \geqslant 1$. So Pythagorean fuzzy sets can be interpreted as the generalization of intuitionistic fuzzy sets. Pythagorean fuzzy sets ae applied to groups (see [2]), UP-algebras (see [18]) and topological spaces (see [17]). Senapati et al. [19] introduced Fermatean fuzzy set and it is applied to groups (see [20]). Ibrahim et al. [14] introduced $(3, 2)$-fuzzy sets and applied it to topological spaces.

The purpose of this paper is to introduce another form of generalization of intuitionistic fuzzy sets. We introduce the notion of $(2, 3)$-fuzzy sets, and investigate their properties. We form a semigroup structure in the collection of $(2, 3)$-fuzzy sets. We apply the concept of $(2, 3)$-fuzzy sets to BCK-algebras and BCI-algebas. We introduce the notion of $(2, 3)$-fuzzy subalgebras in BCK/BCI-algebras, and closed $(2, 3)$-fuzzy subalgebras in BCI-algebras. We investigate relations between $(2, 3)$-fuzzy subalgebras and degree functions. Given a $(2, 3)$-fuzzy subalgebra we make a new $(2, 3)$-fuzzy subalgebra. We deal with the union and intersection of $(2, 3)$-fuzzy subalgebras. We establish a characterization of a $(2, 3)$-fuzzy subalgebra by using the $(2, 3)$-cutty set. We explore the conditions under which a $(2, 3)$-fuzzy subalgebra can be closed in BCI-algebras.

2. Preliminaries

If a set X has a special element "0" and a binary operation "$*$" satisfying the conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$;

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$;

(III) $(\forall x \in X)\ (x * x = 0)$;

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$,

then we say that X is a *BCI-algebra* (see [12, 16]). If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra* (see [12, 16]). A BCI-algebra X is said to be p-*semisimple* (see [12]) if $0 * (0 * x) = x$ for all $x \in X$.

The order relation "$\leqslant$" in a BCK/BCI-algebra X is defined as follows:

$$(\forall x, y \in X)(x \leqslant y\ \Leftrightarrow\ x * y = 0).$$

Every BCK/BCI-algebra X satisfies the following conditions (see [12, 16]):

$$\begin{aligned} &(\forall x \in X)\ (x * 0 = x)\,, \\ &(\forall x, y, z \in X)\ (x \leqslant y\ \Rightarrow\ x * z \leqslant y * z,\ z * y \leqslant z * x)\,, \\ &(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)\,. \end{aligned} \tag{2.1}$$

Every BCI-algebra X satisfies the following conditions (see [12]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y),$$
$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)).$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X (see [12, 16]) if $x * y \in S$ for all $x, y \in S$.

Every ideal A of a BCK/BCI-algebra X satisfies the next assertion (see [12, 16]).

$$(\forall x, y \in X)\,(x \leqslant y,\ y \in A\ \Rightarrow\ x \in A)\,.$$

Let X and Y be BCK/BCI-algebras. A mapping $\Psi : X \to Y$ is call a *homomorphism* (see [12, 16]) if it satisfies:

$$(\forall x, y \in X)(\Psi(x * y) = \Psi(x) * \Psi(y)).$$

Let $\zeta_X : X \to [0,1]$ and $\gamma_X : X \to [0,1]$ be fuzzy sets in a set X. The structure $\Omega := \{\langle x, \zeta_X(x), \gamma_X(x)\rangle \mid x \in X\}$ is called

- an *intuitionistic fuzzy set* in X (see [1]) if it satisfies:

$$(\forall x \in X)(0 \leqslant \zeta_X(x) + \gamma_X(x) \leqslant 1),$$

- a *Pythgorean fuzzy set* in X (see [21]) if it satisfies:

$$(\forall x \in X)(0 \leqslant (\zeta_X(x))^2 + (\gamma_X(x))^2 \leqslant 1),$$

- a *Fermatean fuzzy set* in X (see [19]) if it satisfies:

$$(\forall x \in X)(0 \leqslant (\zeta_X(x))^3 + (\gamma_X(x))^3 \leqslant 1).$$

Let X be a BCK-algebra or a BCI-algebra. A fuzzy set $\zeta_X : X \to [0,1]$ is called

- a *fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)(\zeta_X(x * y) \geqslant \min\{\zeta_X(x), \zeta_X(y)\}),$$

- an *anti fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X)(\zeta_X(x * y) \leqslant \max\{\zeta_X(x), \zeta_X(y)\}).$$

Let X be a BCK-algebra or a BCI-algebra. A pair (ξ, η) of two fuzzy sets ξ and η in X is called an *intuitionistic fuzzy subalgebra* of X (see [11]) if it is an intuitionistic fuzzy set in X, ξ is a fuzzy subalgebra of X and η is an anti fuzzy subalgebra of X.

3. (2,3)-fuzzy sets

Definition 3.1. Let $\xi : X \to [0,1]$ and $\eta : X \to [0,1]$ be fuzzy sets in a set X. The pair (ξ, η) of ξ and η is called a $(2,3)$-*fuzzy set* on X if it satisfies:

$$(\forall x \in X)((\xi(x))^2 + (\eta(x))^3 \leqslant 1).$$

In what follows, we use the notations $\xi^2(x)$ and $\eta^3(x)$ instead of $(\xi(x))^2$ and $(\eta(x))^3$, respectively, and the $(2,3)$-fuzzy set on X is denoted by $\mathcal{E} := (X, \xi, \eta)$ and it can be represented as follows:

$$\mathcal{E} := (X, \xi, \eta) : X \to [0,1] \times [0,1],\ x \mapsto (\xi(x), \eta(x))\,.$$

The collection of $(2,3)$-fuzzy sets on X is denoted by $\mathcal{F}_2^3(X)$. It is clear that the mappings $\mathcal{E}_{\widetilde{01}} := (X, \tilde{0}, \tilde{1})$ and $\mathcal{E}_{\widetilde{10}} := (X, \tilde{1}, \tilde{0})$ which are described as follows:

$$\mathcal{E}_{\widetilde{01}} := (X, \tilde{0}, \tilde{1}) : X \to [0,1] \times [0,1],\ x \mapsto (0,1),$$
$$\mathcal{E}_{\widetilde{10}} := (X, \tilde{1}, \tilde{0}) : X \to [0,1] \times [0,1],\ x \mapsto (1,0),$$

are $(2,3)$-fuzzy sets on X.

Given an $\mathcal{E} := (X, \xi, \eta) \in \mathcal{F}_2^3(X)$, consider functions below

$$\begin{aligned} d_{\mathcal{E}} &: X \to [0,1],\ x \mapsto \left(1 - \xi^2(x) - \eta^3(x)\right)^{\frac{1}{5}}, \\ d_{\xi} &: X \to [0,1],\ x \mapsto 1 - \xi^2(x), \\ d_{\eta} &: X \to [0,1],\ x \mapsto 1 - \eta^3(x). \end{aligned}$$

We say that $d_{\mathcal{E}}$ (resp., d_ξ and d_η) is the *degree function* (resp., ξ*-degree function* and η*-degree function*) of $\mathcal{E} := (X, \xi, \eta)$ and $d_{\mathcal{E}}(x)$ (resp., $d_\xi(x)$ and $d_\eta(x)$) is the *degree* (resp., ξ*-degree* and η*-degree*) of x in $\mathcal{E}$.

Proposition 3.2. *Let $\mathcal{E} := (X, \xi, \eta) \in \mathcal{F}_2^3(X)$. If $d^5(x) = 0$ for $x \in X$, then η is given as follows:*

$$\eta : X \to [0,1],\ x \mapsto ((1 + \xi(x))(1 - \xi(x)))^{\frac{1}{3}}.$$

Proof. Let $x \in X$. If $d_{\mathcal{E}}^5(x) = 0$, then $\xi^2(x) + \eta^3(x) = 1$ and so $\eta^3(x) = 1 - \xi^2(x) = (1 + \xi(x))(1 - \xi(x)$. Hence $\eta(x) = ((1 + \xi(x))(1 - \xi(x)))^{\frac{1}{3}}$ for all $x \in X$. □

Example 3.3. Let $X = \{0, a, b, c, d\}$ be a set and define a $(2,3)$-fuzzy set $\mathcal{E} := (X, \xi, \eta)$ on X as follows:

$$\mathcal{E} := (X, \xi, \eta) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.93, 0.18), & \text{if } x = 0, \\ (0.43, 0.74), & \text{if } x = a, \\ (0.86, 0.26), & \text{if } x = b, \\ (0.77, 0.54), & \text{if } x = c, \\ (0.68, 0.37), & \text{if } x = d. \end{cases}$$

Then the degree function $d_{\mathcal{E}}$ of $\mathcal{E} := (X, \xi, \eta)$ is given as follows:

$$d_{\mathcal{E}} : X \to [0,1],\ x \mapsto \begin{cases} 0.6641989824327921, & \text{if } x = 0, \\ 0.8366243447144909, & \text{if } x = a, \\ 0.7507212976971708, & \text{if } x = b, \\ 0.7576374662818331, & \text{if } x = c, \\ 0.8659570233410206, & \text{if } x = d. \end{cases}$$

Remark 3.4. It is certain that intuitionistic fuzzy set and Pythagorean fuzzy set are $(2,3)$-fuzzy set. The $(2,3)$-fuzzy set is neither an intuitionistic fuzzy set nor a Pythagorean fuzzy set.

Example 3.5. Consider the $(2,3)$-fuzzy set $\mathcal{E} := (X, \xi, \eta)$ in Example 3.3. It is both a Pythagorean fuzzy set and a Fermatean fuzzy set. But it is not an intuitionistic fuzzy set because of $\xi(a) + \eta(a) = 0.43 + 0.74 = 1.17 > 1$.

Example 3.6. Consider a mapping $\mathcal{E} := (X, \xi, \eta)$ on a set $X = \{0, a, b, c, d\}$ given as follows:

$$\mathcal{E} := (X, \xi, \eta) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.88, 0.54), & \text{if } x = 0, \\ (0.69, 0.80), & \text{if } x = a, \\ (0.97, 0.26), & \text{if } x = b, \\ (0.77, 0.54), & \text{if } x = c, \\ (0.68, 0.37), & \text{if } x = d. \end{cases}$$

Then $\mathcal{E} := (X, \xi, \eta)$ is a $(2,3)$-fuzzy set on X, but it is not a Pythagorean fuzzy set on X since $\xi^2(a) + \eta^2(a) = 0.69^2 + 0.80^2 = 1.1161 > 1$ or $\xi^2(b) + \eta^2(b) = 0.97^2 + 0.26^2 = 1.0085 > 1$.

Remark 3.7. It is certain that $(2,3)$-fuzzy set is Fermatean fuzzy set. But the converse is not valid.

Example 3.8. Let $X = \{0, a, b, c\}$ be a set and define a mapping $\mathcal{E} := (X, \xi, \eta)$ on X as follows:

$$\mathcal{E} := (X, \xi, \eta) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.17, 0.93), & \text{if } x = 0, \\ (0.43, 0.74), & \text{if } x = a, \\ (0.19, 0.82), & \text{if } x = b, \\ (0.67, 0.83), & \text{if } x = c. \end{cases}$$

It is routine to verify that $\mathcal{E} := (X, \xi, \eta)$ is a Fermatean fuzzy set on X. But it is not a $(2,3)$-fuzzy set on X since $\xi(c)^2 + \eta^3(c) = 0.67^2 + 0.83^3 = 1.020687 > 1$.

It can be observed that the $(2,3)$-fuzzy set is located between Pythagorean fuzzy set and Fermatean fuzzy set by Remarks 3.4 and 3.7.

Definition 3.9. We define a binary relation " $\precsim$ " and the equality " $=$ " in $\mathcal{F}_2^3(X)$ as follows:

$$\mathcal{E}_1 \precsim \mathcal{E}_2 \ \Leftrightarrow\ \xi_1 \leqslant \xi_2,\ \eta_1 \geqslant \eta_2,$$
$$\mathcal{E}_1 = \mathcal{E}_2 \ \Leftrightarrow\ \xi_1 = \xi_2,\ \eta_1 = \eta_2$$

for all $\mathcal{E}_1 := (X, \xi_1, \eta_1)$, $\mathcal{E}_2 := (X, \xi_2, \eta_2) \in \mathcal{F}_2^3(X)$.

The notation $\mathcal{E}_1 \precnsim \mathcal{E}_2$ means $\mathcal{E}_1 \precsim \mathcal{E}_2$ and $\mathcal{E}_1 \neq \mathcal{E}_2$. It is clear that $(\mathcal{F}_2^3(X), \precsim)$ is a poset.

Definition 3.10. For all $\mathcal{E}_1 := (X, \xi_1, \eta_1)$, $\mathcal{E}_2 := (X, \xi_2, \eta_2) \in \mathcal{F}_2^3(X)$, we define the *union* ($\Cup$) and the *intersection* ($\Cap$) as follows:

$$\Cup : \mathcal{F}_2^3(X) \times \mathcal{F}_2^3(X) \to \mathcal{F}_2^3(X),\ (\mathcal{E}_1, \mathcal{E}_2) \mapsto \mathcal{E}_1 \Cup \mathcal{E}_2,$$
$$\Cap : \mathcal{F}_2^3(X) \times \mathcal{F}_2^3(X) \to \mathcal{F}_2^3(X),\ (\mathcal{E}_1, \mathcal{E}_2) \mapsto \mathcal{E}_1 \Cap \mathcal{E}_2,$$

where $\mathcal{E}_1 \Cup \mathcal{E}_2 = (X, \xi_1 \vee \xi_2, \eta_1 \wedge \eta_2)$ and $\mathcal{E}_1 \Cap \mathcal{E}_2 = (X, \xi_1 \wedge \xi_2, \eta_1 \vee \eta_2)$ with

$$(\xi_1 \vee \xi_2)(x) = \max\{\xi_1(x), \xi_2(x)\},$$
$$(\xi_1 \wedge \xi_2)(x) = \min\{\xi_1(x), \xi_2(x)\},$$
$$(\eta_1 \vee \eta_2)(x) = \max\{\eta_1(x), \eta_2(x)\},$$
$$(\eta_1 \wedge \eta_2)(x) = \min\{\eta_1(x), \eta_2(x)\}.$$

It is clear that the union ($\Cup$) and the intersection ($\Cap$) are associative binary operations in $\mathcal{F}_2^3(X)$.

Example 3.11. Let $X = \{0, a, b, c\}$ be a set and define $(2,3)$-fuzzy sets $\mathcal{E}_1 := (X, \xi_1, \eta_1)$ and $\mathcal{E}_2 := (X, \xi_2, \eta_2)$ on X as follows:

$$\mathcal{E}_1 := (X, \xi_1, \eta_1) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.87, 0.23), & \text{if } x = 0, \\ (0.43, 0.64), & \text{if } x = a, \\ (0.69, 0.42), & \text{if } x = b, \\ (0.78, 0.53), & \text{if } x = c, \end{cases}$$

and

$$\mathcal{E}_2 := (X, \xi_2, \eta_2) : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.77, 0.33), & \text{if } x = 0, \\ (0.53, 0.64), & \text{if } x = a, \\ (0.68, 0.45), & \text{if } x = b, \\ (0.82, 0.53), & \text{if } x = c, \end{cases}$$

respectively. Then the union $\mathcal{E}_1 \Cup \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is given as follows:

$$\mathcal{E}_1 \Cup \mathcal{E}_2 = (X, \xi_1 \vee \xi_2, \eta_1 \wedge \eta_2) : X \to [0,1] \times [0,1],\ \ x \mapsto \begin{cases} (0.87, 0.23), & \text{if } x = 0, \\ (0.53, 0.64), & \text{if } x = a, \\ (0.69, 0.42), & \text{if } x = b, \\ (0.82, 0.53), & \text{if } x = c. \end{cases}$$

Also, the intersection $\mathcal{E}_1 \Cap \mathcal{E}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is given as

$$\mathcal{E}_1 \Cap \mathcal{E}_2 = (X, \xi_1 \wedge \xi_2, \eta_1 \vee \eta_2) : X \to [0,1] \times [0,1], \quad x \mapsto \begin{cases} (0.77, 0.33), & \text{if } x = 0, \\ (0.43, 0.64), & \text{if } x = a, \\ (0.68, 0.45), & \text{if } x = b, \\ (0.78, 0.53), & \text{if } x = c. \end{cases}$$

Proposition 3.12. *Given* $\mathcal{E}_1 := (X, \xi_1, \eta_1)$, $\mathcal{E}_2 := (X, \xi_2, \eta_2) \in \mathcal{F}_2^3(X)$, *we have*

$$\begin{aligned} \mathcal{E}_1 \Cap \mathcal{E}_2 &= \mathcal{E}_2 \Cap \mathcal{E}_1, \\ \mathcal{E}_1 \Cup \mathcal{E}_2 &= \mathcal{E}_2 \Cup \mathcal{E}_1, \\ (\mathcal{E}_1 \Cap \mathcal{E}_2) \Cup \mathcal{E}_2 &= \mathcal{E}_2, \\ (\mathcal{E}_1 \Cup \mathcal{E}_2) \Cap \mathcal{E}_2 &= \mathcal{E}_2. \end{aligned}$$

Proof. Straightforward. □

Proposition 3.13. *Every element of* $\mathcal{F}_2^3(X)$ *is idempotent under the binary operators "* $\Cap$ *" and "* $\Cup$ *".*

Proof. Straightforward. □

Let $\mathcal{E} := (X, \xi, \eta) \in \mathcal{F}_2^3(X)$. Since $(\xi \vee \tilde{0})(x) = \max\{\xi(x), \tilde{0}(x)\} = \xi(x)$ and $(\eta \wedge \tilde{1})(x) = \min\{\eta(x), \tilde{1}(x)\} = \eta(x)$, we get $\mathcal{E} \Cup \mathcal{E}_{\widetilde{01}} = \mathcal{E}_{\widetilde{01}} \Cup \mathcal{E} = \mathcal{E}$. Also, we have $\mathcal{E} \Cap \mathcal{E}_{\widetilde{10}} = \mathcal{E}_{\widetilde{01}} \Cap \mathcal{E} = \mathcal{E}$ since $(\xi \wedge \tilde{1})(x) = \min\{\xi(x), \tilde{1}(x)\} = \xi(x)$ and $(\eta \vee \tilde{0})(x) = \max\{\eta(x), \tilde{0}(x)\} = \eta(x)$. Based on the above observation, we obtain the following conclusions.

Theorem 3.14. *Given a set* X, $\left(\mathcal{F}_2^3(X), \Cup, \mathcal{E}_{\widetilde{01}}\right)$ *and* $\left(\mathcal{F}_2^3(X), \Cap, \mathcal{E}_{\widetilde{10}}\right)$ *are idempotent commutative monoids.*

4. (2,3)-fuzzy subalgebras of BCK/BCI-algebras

In what follows, let X represent the BCK-algebra or BCI-algebra unless otherwise specified.

Given two fuzzy sets ξ and η in X, the pair (ξ, η) is said to be a *fuzzy couple* on X and it is denoted by $(\xi, \eta)_X$. It can be represented as follows:

$$(\xi, \eta)_X : X \to [0,1] \times [0,1], \; x \mapsto (\xi(x), \eta(x)).$$

Definition 4.1. A fuzzy couple $(\xi, \eta)_X$ is called a $(2,3)$-*fuzzy subalgebra* of X if it satisfies:

$$(\forall x, y \in X) \begin{pmatrix} \xi^2(x*y) \geqslant \xi^2(x) \text{ or } \xi^2(x*y) \geqslant \xi^2(y) \\ \eta^3(x*y) \leqslant \eta^3(x) \text{ or } \eta^3(x*y) \leqslant \eta^3(y) \end{pmatrix}. \tag{4.1}$$

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with a binary operation " $*$ " in the table below:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Then X is a BCK-algebra (see [16]). Define a fuzzy couple $(\xi, \eta)_X$ as follows:

$$(\xi, \eta)_X : X \to [0,1] \times [0,1], \; x \mapsto \begin{cases} (0.83, 0.13), & \text{if } x = 0, \\ (0.61, 0.41), & \text{if } x = a, \\ (0.61, 0.41), & \text{if } x = b, \\ (0.74, 0.56), & \text{if } x = c. \end{cases}$$

It is routine to verify that $(\xi, \eta)_X$ is a $(2,3)$-fuzzy subalgebra of X.

Lemma 4.3. *Every* $(2,3)$*-fuzzy subalgebra* $(\xi,\eta)_X$ *of* X *satisfies:*

$$(\forall x \in X)(\xi^2(0) \geqslant \xi^2(x),\ \eta^3(0) \leqslant \eta^3(x)).$$

Proof. It is straightforward by the combination of (III) and (4.1). □

Theorem 4.4. *If* $(\xi,\eta)_X$ *is a* $(2,3)$*-fuzzy subalgebra of* X*, then the set*

$$X_0 := \{x \in X \mid \xi^2(x) = \xi^2(0),\ \eta^3(x) = \eta^3(0)\}$$

is a subalgebra of X*.*

Proof. If $x, y \in X_0$, then $\xi^2(x) = \xi^2(0)$, $\xi^2(y) = \xi^2(0)$, $\eta^3(x) = \eta^3(0)$ and $\eta^3(y) = \eta^3(0)$. It follows from (4.1) that

$$\xi^2(x*y) \geqslant \xi^2(x) = \xi^2(0) \text{ or } \xi^2(x*y) \geqslant \xi^2(y) = \xi^2(0)$$

and

$$\eta^3(x*y) \leqslant \eta^3(x) = \eta^3(0) \text{ or } \eta^3(x*y) \leqslant \eta^3(y) = \eta^3(0).$$

Combining these and Lemma 4.3 induces $\xi^2(x*y) = \xi^2(0)$ and $\eta^3(x*y) = \eta^3(0)$, and so $x*y \in X_0$. Hence X_0 is a subalgebra of X. □

Theorem 4.5. *If* $(\xi,\eta)_X$ *is a* $(2,3)$*-fuzzy subalgebra of* X*, then the* ξ*-degree function* d_ξ *is an anti fuzzy subalgebra of* X *and the* η*-degree function* d_η *is a fuzy subalgebra of* X*.*

Proof. Assume that $(\xi,\eta)_X$ is a $(2,3)$-fuzzy subalgebra of X. Then

$$\xi^2(x*y) \geqslant \xi^2(x) \text{ or } \xi^2(x*y) \geqslant \xi^2(y)$$

and

$$\eta^3(x*y) \leqslant \eta^3(x) \text{ or } \eta^3(x*y) \leqslant \eta^3(y).$$

It follows that

$$d_\xi(x*y) = 1 - \xi^2(x*y) \leqslant 1 - \xi^2(x) = d_\xi(x)$$

or

$$d_\xi(x*y) = 1 - \xi^2(x*y) \leqslant 1 - \xi^2(y) = d_\xi(y).$$

Hence $d_\xi(x*y) \leqslant \max\{d_\xi(x), d_\xi(y)\}$, and therefore d_ξ is an anti fuzzy subalgebra of X. Also, we have

$$d_\eta(x*y) = 1 - \eta^3(x*y) \geqslant 1 - \eta^3(x) = d_\eta(x)$$

or

$$d_\eta(x*y) = 1 - \eta^3(x*y) \geqslant 1 - \eta^3(y) = d_\eta(y).$$

It follows that $d_\eta(x*y) \geqslant \min\{d_\eta(x), d_\eta(y)\}$. Hence d_η is a fuzzy subalgebra of X. □

We present the following question and try to find the answer.

Question 4.6. *If* $(\xi,\eta)_X$ *is a* $(2,3)$*-fuzzy subalgebra of* X*, then is the pair* (d_η, d_ξ) *of* η*-degree function* d_η *and* ξ*-degree function* d_ξ *of* $(\xi,\eta)_X$ *an intuitionistic fuzzy subalgebra of* X*?*

The following example presents a negative answer to Question 4.6.

Example 4.7. Consider the $(2,3)$-fuzzy subalgebra $(\xi,\eta)_X$ of X which is described in Example 4.2. Then the η-degree function d_η and the ξ-degree function d_ξ of $(\xi,\eta)_X$ are given as follows.

$$d_\eta : X \to [0,1],\ x \mapsto \begin{cases} 0.3111, & \text{if } x = 0,\\ 0.6279, & \text{if } x \in \{a,b\},\\ 0.4524, & \text{if } x = c \end{cases}$$

and

$$d_\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.997803, & \text{if } x = 0,\\ 0.931079, & \text{if } x \in \{a,b\},\\ 0.824384, & \text{if } x = c. \end{cases}$$

It is routine to verify that d_ξ is an anti fuzzy subalgebra of X and d_η is a fuzzy subalgebra of X. But the pair (d_η, d_ξ) is not an intuitionistic fuzzy subalgebra of X because of $d_\eta(a) + d_\xi(a) = 0.6279 + 0.931079 = 1.558979 > 1$.

Given a fuzzy couple $(\xi,\eta)_X$, we define a new fuzzy couple $(\tilde{\xi},\tilde{\eta})_X$ on X as follows:

$$\tilde{\xi} : X \to [0,1],\ x \mapsto \frac{\xi(x)}{\sup\{\xi(x) \mid x \in X\}}, \quad \tilde{\eta} : X \to [0,1],\ x \mapsto \frac{\eta(x)}{\inf\{\eta(x) \mid x \in X\}},$$

where $\inf\{\eta(x) \mid x \in X\} \neq 0$.

Theorem 4.8. *If $(\xi,\eta)_X$ is a $(2,3)$-fuzzy subalgebra of X with $\eta(0) \neq 0$, then $(\tilde{\xi},\tilde{\eta})_X$ is a $(2,3)$-fuzzy subalgebra of X.*

Proof. Suppose that $(\xi,\eta)_X$ is a $(2,3)$-fuzzy subalgebra of X with $\eta(0) \neq 0$. Then $\sup\{\xi(x) \mid x \in X\} = \xi(0)$ and $\inf\{\eta(x) \mid x \in X\} = \eta(0) \neq 0$. Hence

$$\tilde{\xi}^2(x*y) = \left(\frac{\xi(x*y)}{\sup\{\xi(x*y) \mid x*y \in X\}}\right)^2 = \left(\frac{\xi(x*y)}{\xi(0)}\right)^2 = \frac{\xi^2(x*y)}{\xi^2(0)} \geqslant \frac{\xi^2(x)}{\xi^2(0)} = \left(\frac{\xi(x)}{\xi(0)}\right)^2 = \tilde{\xi}^2(x)$$

or

$$\tilde{\xi}^2(x*y) = \left(\frac{\xi(x*y)}{\sup\{\xi(x*y) \mid x*y \in X\}}\right)^2 = \left(\frac{\xi(x*y)}{\xi(0)}\right)^2 = \frac{\xi^2(x*y)}{\xi^2(0)} \geqslant \frac{\xi^2(y)}{\xi^2(0)} = \left(\frac{\xi(y)}{\xi(0)}\right)^2 = \tilde{\xi}^2(y).$$

Also, we have

$$\tilde{\eta}^3(x*y) = \left(\frac{\eta(x*y)}{\inf\{\eta(x*y) \mid x*y \in X\}}\right)^3 = \left(\frac{\eta(x*y)}{\eta(0)}\right)^3 = \frac{\eta^3(x*y)}{\eta^3(0)} \leqslant \frac{\eta^3(x)}{\eta^3(0)} = \left(\frac{\eta(x)}{\eta(0)}\right)^3 = \tilde{\eta}^3(x)$$

or

$$\tilde{\eta}^3(x*y) = \left(\frac{\eta(x*y)}{\inf\{\eta(x*y) \mid x*y \in X\}}\right)^3 = \left(\frac{\eta(x*y)}{\eta(0)}\right)^3 = \frac{\eta^3(x*y)}{\eta^3(0)} \leqslant \frac{\eta^3(y)}{\eta^3(0)} = \left(\frac{\eta(y)}{\eta(0)}\right)^3 = \tilde{\eta}^3(y).$$

Therefore $(\tilde{\xi},\tilde{\eta})_X$ is a $(2,3)$-fuzzy subalgebra of X. □

Theorem 4.9. *If $(\xi_1,\eta_1)_X$ and $(\xi_2,\eta_2)_X$ are $(2,3)$-fuzzy subalgebras of X, then their intersection $(\xi_1,\eta_1)_X \Cap (\xi_2,\eta_2)_X = (\xi_1 \wedge \xi_2, \eta_1 \vee \eta_2)_X$ is also a $(2,3)$-fuzzy subalgebra of X.*

Proof. Let $x, y \in X$. Then

$$(\xi_1 \wedge \xi_2)^2(x*y) = \min\{\xi_1^2(x*y), \xi_2^2(x*y)\} \geqslant \min\{\xi_1^2(x), \xi_2^2(x)\} = (\xi_1 \wedge \xi_2)^2(x)$$

or

$$(\xi_1 \wedge \xi_2)^2(x*y) = \min\{\xi_1^2(x*y), \xi_2^2(x*y)\} \geqslant \min\{\xi_1^2(y), \xi_2^2(y)\} = (\xi_1 \wedge \xi_2)^2(y).$$

Also, we have

$$(\eta_1 \vee \eta_2)^3(x*y) = \max\{\eta_1^3(x*y), \eta_2^3(x*y)\} \leqslant \max\{\eta_1^3(x), \eta_2^3(x)\} = (\eta_1 \vee \eta_2)^3(x)$$

or

$$(\eta_1 \vee \eta_2)^3(x*y) = \max\{\eta_1^3(x*y), \eta_2^3(x*y)\} \leqslant \max\{\eta_1^3(y), \eta_2^3(y)\} = (\eta_1 \vee \eta_2)^3(y).$$

Hence $(\xi_1,\eta_1)_X \Cap (\xi_2,\eta_2)_X = (\xi_1 \wedge \xi_2, \eta_1 \vee \eta_2)_X$ is a $(2,3)$-fuzzy subalgebra of X. □

The following example shows that the union of two $(2,3)$-fuzzy subalgebras may not be a $(2,3)$-fuzzy subalgebra.

Example 4.10. Let $X = \{0, a, b, c\}$ be a set with a binary operation "$*$" in the table below:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then X is a BCI-algebra (see [12, 16]). Define two fuzzy couples $(\xi_1, \eta_1)_X$ and $(\xi_2, \eta_2)_X$ by the table as below, respectively.

$$(\xi_1, \eta_1)_X : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.78, 0.24), & \text{if } x = 0, \\ (0.63, 0.58), & \text{if } x = a, \\ (0.47, 0.33), & \text{if } x = b, \\ (0.47, 0.58), & \text{if } x = c, \end{cases}$$

and

$$(\xi_2, \eta_2)_X : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.78, 0.24), & \text{if } x = 0, \\ (0.53, 0.56), & \text{if } x = a, \\ (0.53, 0.63), & \text{if } x = b, \\ (0.69, 0.63), & \text{if } x = c. \end{cases}$$

Then the union $(\xi_1, \eta_1)_X \Cup (\xi_2, \eta_2)_X = (\xi_1 \vee \xi_2, \eta_1 \wedge \eta_2)_X$ is calculated as follows:

$$\xi_1 \vee \xi_2 : X \to [0,1],\ x \mapsto \begin{cases} 0.78 & \text{if } x = 0, \\ 0.63 & \text{if } x = a, \\ 0.53 & \text{if } x = b, \\ 0.69 & \text{if } x = c, \end{cases}$$

and

$$\eta_1 \wedge \eta_2 : X \to [0,1],\ x \mapsto \begin{cases} 0.24 & \text{if } x = 0, \\ 0.56 & \text{if } x = a, \\ 0.33 & \text{if } x = b, \\ 0.58 & \text{if } x = c. \end{cases}$$

Since $(\xi_1 \vee \xi_2)^2(a * c) = (\xi_1 \vee \xi_2)^2(b) = 0.53^2 \ngeqslant 0.63^2 = (\xi_1 \vee \xi_2)^2(a)$ and

$$(\xi_1 \vee \xi_2)^2(a * c) = (\xi_1 \vee \xi_2)^2(b) = 0.53^2 \ngeqslant 0.69^2 = (\xi_1 \vee \xi_2)^2(c),$$

or since $(\eta_1 \wedge \eta_2)^3(a * b) = (\eta_1 \wedge \eta_2)^3(c) = 0.58^3 \nleqslant 0.56^3 = (\eta_1 \wedge \eta_2)^3(a)$ and

$$(\eta_1 \wedge \eta_2)^3(a * b) = (\eta_1 \wedge \eta_2)^3(c) = 0.33^3 \nleqslant 0.56^3 = (\eta_1 \wedge \eta_2)^3(b),$$

we know that $(\xi_1, \eta_1)_X \Cup (\xi_2, \eta_2)_X = (\xi_1 \vee \xi_2, \eta_1 \wedge \eta_2)_X$ is not a $(2,3)$-fuzzy subalgebra of X.

Let $(\xi, \eta)_X$ be a fuzzy couple. For every $(s, t) \in [0,1] \times [0,1]$ with $0 \leqslant s^2 + t^3 \leqslant 1$, we consider the sets

$$X(\xi, s) := \{x \in X \mid \xi^2(x) \geqslant s\} \text{ and } X(\eta, t) := \{x \in X \mid \eta^3(x) \leqslant t\}.$$

The set $X((\xi, \eta), (s, t)) := X(\xi, s) \cap X(\eta, t)$ is called the $(2,3)$-*cutty set* of $(\xi, \eta)_X$.

Proposition 4.11. *Given fuzzy couples* $(\xi,\eta)_X$ *and* $(\alpha,\beta)_X$*, if* $(\xi,\eta)_X \precsim (\alpha,\beta)_X$*, then* $X(\xi,s) \subseteq X(\alpha,s)$*,* $X(\eta,t) \subseteq X(\beta,t)$ *and* $X((\xi,\eta),(s,t)) \subseteq X((\alpha,\beta),(s,t))$.

Proof. Assume that $(\xi,\eta)_X \precsim (\alpha,\beta)_X$. Then $\xi \leqslant \alpha$ and $\eta \geqslant \beta$, that is, $\xi(x) \leqslant \alpha(x)$ and $\eta(x) \geqslant \beta(x)$ for all $x \in X$. If $x \in X(\xi,s)$, then $s \leqslant \xi^2(x) \leqslant \alpha^2(x)$ and so $x \in X(\alpha,s)$. If $x \in X(\eta,t)$, then $t \geqslant \eta^3(x) \geqslant \beta^3(x)$ and so $x \in X(\beta,t)$. Hence $X(\xi,s) \subseteq X(\alpha,s)$ and $X(\eta,t) \subseteq X(\beta,t)$. It follows that $X((\xi,\eta),(s,t)) = X(\xi,s) \cap X(\eta,t) \subseteq X(\alpha,s) \cap X(\beta,t) = X((\alpha,\beta),(s,t))$. □

Theorem 4.12. *A fuzzy couple* $(\xi,\eta)_X$ *is a* $(2,3)$*-fuzzy subalgebra of* X *if and only if* $X(\xi,s)$ *and* $X(\eta,t)$ *are subalgebras of* X *for all* $(s,t) \in [0,1] \times [0,1]$ *with* $0 \leqslant s^2 + t^3 \leqslant 1$.

Proof. Assume that $(\xi,\eta)_X$ is a $(2,3)$-fuzzy subalgebra of X and let $(s,t) \in [0,1] \times [0,1]$ with $0 \leqslant s^2 + t^3 \leqslant 1$. If $x,y \in X(\xi,s)$ and $a,b \in X(\eta,t)$, then $\xi^2(x) \geqslant s$, $\xi^2(y) \geqslant s$, $\eta^3(a) \leqslant t$ and $\eta^3(b) \leqslant t$. It follows from (4.1) that $\xi^2(x*y) \geqslant s$ and $\eta^3(a*b) \leqslant t$. So $x*y \in X(\xi,s)$ and $a*b \in X(\eta,t)$, which shows that $X(\xi,s)$ and $X(\eta,t)$ are subalgebras of X.

Conversely, suppose that $X(\xi,s)$ and $X(\eta,t)$ are subalgebras of X for all $(s,t) \in [0,1] \times [0,1]$ with $0 \leqslant s^2 + t^3 \leqslant 1$. For evey $x,y \in X$, let $s := \min\{\xi^2(x),\xi^2(y)\}$ and $t := \max\{\eta^3(x),\eta^3(y)\}$. Then $x,y \in X(\xi,s)$ and $x,y \in X(\eta,t)$, which imply that $x*y \in X(\xi,s)$ and $x*y \in X(\eta,t)$. It follows that $\xi^2(x*y) \geqslant s := \min\{\xi^2(x),\xi^2(y)\}$ and $\eta^3(x*y) \leqslant t := \max\{\eta^3(x),\eta^3(y)\}$. Hence $(\xi,\eta)_X$ satisfies the condition (4.1), and therefore it is a $(2,3)$-fuzzy subalgebra of X. □

Corollary 4.13. *If a fuzzy couple* $(\xi,\eta)_X$ *is a* $(2,3)$*-fuzzy subalgebra of* X*, then the* $(2,3)$*-cutty set* $X((\xi,\eta),(s,t))$ *of* $(\xi,\eta)_X$ *is a subalgebra of* X *for all* $(s,t) \in [0,1] \times [0,1]$ *with* $0 \leqslant s^2 + t^3 \leqslant 1$.

Theorem 4.14. *If a fuzzy couple* $(\xi,\eta)_X$ *is a* $(2,3)$*-fuzzy subalgebra of* X*, then the following assertions are equivalent:*

$$(\forall x,y \in X)(\xi^2(x*y) \geqslant \xi^2(y), \quad \eta^3(x*y) \leqslant \eta^3(y)), \tag{4.2}$$

$$(\forall x \in X)(\xi^2(x) = \xi^2(0), \quad \eta^3(x) = \eta^3(0)). \tag{4.3}$$

Proof. If we put $y = 0$ in (4.2) and use (2.1), then $\xi^2(x) = \xi^2(x*0) \geqslant \xi^2(0)$ and $\eta^3(x) = \eta^3(x*0) \leqslant \eta^3(0)$. The combination of this and Lemma 4.3 induces the condition (4.3).

Conversely, if (4.3) is valid, then

$$\xi^2(x*y) \geqslant \min\{\xi^2(x),\xi^2(y)\} = \min\{\xi^2(0),\xi^2(y)\} = \xi^2(y)$$

and

$$\eta^3(x*y) \leqslant \max\{\eta^3(x),\eta^3(y)\} = \max\{\eta^3(0),\eta^3(y)\} = \eta^3(y)$$

by Lemma 4.3. □

Definition 4.15. Let X be a BCI-algebra. A $(2,3)$-fuzzy subalgebra $(\xi,\eta)_X$ of X is said to be *closed* if it satisfies:

$$(\forall x \in X)(\xi^2(0*x) = \xi^2(x), \eta^3(0*x) = \eta^3(x)). \tag{4.4}$$

Example 4.16.

(1) Given a BCI-algebra X, define a fuzzy couple $(\xi,\eta)_X$ as follows.

$$(\xi,\eta)_X : X \to [0,1] \times [0,1], \; x \mapsto \begin{cases} (0.83, 0.13), & \text{if } x \in \{y \in X \mid 0 \leqslant y\}, \\ (0.61, 0.41), & \text{otherwise.} \end{cases}$$

It is routine to verify that $(\xi,\eta)_X$ is a closed $(2,3)$-fuzzy subalgebra of X.

(2) Let $X = \{0, a_1, a_2, a_3, a_4\}$ be a set with the binary operation "$*$" which is given in the Cayley table below:

$*$	0	a_1	a_2	a_3	a_4
0	0	0	0	a_3	a_3
a_1	a_1	0	a_1	a_4	a_3
a_2	a_2	a_2	0	a_3	a_3
a_3	a_3	a_3	a_3	0	0
a_4	a_4	a_3	a_4	a_1	0

Then X is a BCI-algebra (see [12]). Define a fuzzy couple $(\xi, \eta)_X$ as follows:

$$(\xi, \eta)_X : X \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (0.71, 0.23), & \text{if } x = 0, \\ (0.63, 0.41), & \text{if } x = a_1, \\ (0.54, 0.51), & \text{if } x = a_2, \\ (0.49, 0.61), & \text{if } x = a_3, \\ (0.49, 0.61), & \text{if } x = a_4. \end{cases}$$

It is routine to verify that $(\xi, \eta)_X$ is a $(2,3)$-fuzzy subalgebra of X. But it is not closed since $\xi^2(0 * a_2) = 0.71^2 \neq 0.54^2 = \xi^2(a_2)$ and/or $\eta^3(0 * a_2) = 0.23^3 \neq 0.51^3 = \eta^3(a_2)$.

Proposition 4.17. *Let X be a BCI-algebra. Then every closed $(2,3)$-fuzzy subalgebra $(\xi, \eta)_X$ of X satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} \xi^2(x * (0 * y)) \geqslant \xi^2(x) \text{ or } \xi^2(x * (0 * y)) \geqslant \xi^2(y) \\ \eta^3(x * (0 * y)) \leqslant \eta^3(x) \text{ or } \eta^3(x * (0 * y)) \leqslant \eta^3(y) \end{pmatrix}. \tag{4.5}$$

Proof. Let $(\xi, \eta)_X$ be a closed $(2,3)$-fuzzy subalgebra of X and let $x, y \in X$. Then

$$\xi^2(x * (0 * y)) \geqslant \xi^2(x) \text{ or } \xi^2(x * (0 * y)) \geqslant \xi^2(0 * y) = \xi^2(y)$$

and

$$\eta^3(x * (0 * y)) \leqslant \eta^3(x) \text{ or } \eta^3(x * (0 * y)) \leqslant \eta^3(0 * y) = \eta^3(y).$$

Thus (4.5) is valid. □

Note that the $(2,3)$-fuzzy subalgebra of a BCI-algebra is generally not closed as shown in Example 4.16 (2). So, we suggest conditions that can be closed.

Lemma 4.18. *In a BCI-algebra X, every $(2,3)$-fuzzy subalgebra $(\xi, \eta)_X$ of X satisfies:*

$$(\forall x \in X)(\xi^2(0 * x) \geqslant \xi^2(x),\ \eta^3(0 * x) \leqslant \eta^3(x)).$$

Proof. Straightforward. □

Theorem 4.19. *In a p-semisimple BCI-algebra, every $(2,3)$-fuzzy subalgebra is closed.*

Proof. Let $(\xi, \eta)_X$ be a $(2,3)$-fuzzy subalgebra of a p-semisimple BCI-algebra X. Since X is p-semisimple, it follows from Lemma 4.18 that

$$\xi^2(0 * x) \leqslant \xi^2(0 * (0 * x)) = \xi^2(x) \text{ and } \eta^3(0 * x) \geqslant \eta^3(0 * (0 * x)) = \eta^3(x)$$

for all $x \in X$. Hence (4.4) is valid, and therefore $(\xi, \eta)_X$ is a closed $(2,3)$-fuzzy subalgebra of X. □

Corollary 4.20. *If a BCI-algebra* X *satisfies any one of the following conditions:*

$$\begin{aligned}&\text{every element } x \text{ of } X \text{ is minimal},\\ &X = \{0 * x \mid x \in X\},\\ &(\forall x, y \in X)(x * (0 * y) = y * (0 * x)),\\ &(\forall x \in X)(0 * x = 0 \ \Rightarrow \ x = 0),\end{aligned}$$

then every $(2, 3)$*-fuzzy subalgebra is closed.*

Lemma 4.21 ([12])**.** *If* X *is a BCI-algebra, then its BCK-part* X_+ *is given by* $X_+ = \{x * (0 * (0 * x)) \mid x \in X\}$*.*

Theorem 4.22. *Let* X *be a BCI-algebra in which its BCK-part* X_+ *is* $\{0\}$*. Then every* $(2, 3)$*-fuzzy subalgebra is closed.*

Proof. Assume that $X_+ = \{0\}$ in a BCI-algebra X and let $(\xi, \eta)_X$ be a $(2, 3)$-fuzzy subalgebra of X. Then $x * (0 * (0 * x)) = 0$, i.e., $x \leqslant 0 * (0 * x)$, for all $x \in X$ by Lemma 4.21. Since $0 * (0 * x)$ is a minimal element, it follows that $x = 0 * (0 * x)$ for all $x \in X$. Hence X is a p-semisimple BCI-algebra, and therefore $(\xi, \eta)_X$ is a closed $(2, 3)$-fuzzy subalgebra of X by Theorem 4.19. □

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