Subclasses of analytic and bi-univalent functions involving a generalized Mittag-Leffler function based on quasi-subordination

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Abstract

Two quasi-subordination subclasses \(Q_{\Sigma}^{\gamma,k}(\theta,p;\phi)\) and \(M_{\Sigma}^{\gamma,k}(\tau,\rho;\phi)\) of the class \(\Sigma\) of analytic and bi-univalent functions associated with the convolution operator involving Mittag-Leffler function are introduced and investigated. Then, the corresponding bound estimates of the coefficients \(a_2\) and \(a_3\) are provided. Meanwhile, Fekete-Szegő functional inequalities for these classes are proved. Besides, some consequences and connections to all the theorems would be interpreted, which generalize and improve earlier known results.

Keywords: Fekete-Szegő inequality, bi-univalent function, Mittag-Leffler function, quasi-subordination.

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1. Introduction

Denote by \(A\) the class of analytic function \(f\) given by Taylor-Maclaurin’s series

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

in the open unit disk \(U = \{z \in \mathbb{C} : |z| < 1\}\) and normalized as \(f(0) = 0 = f'(0) - 1\). Also, let \(S\) be the subclass of \(A\) which are univalent functions in \(U\).

For \(f_1, f_2 \in A\), as given by \(f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n (j = 1, 2)\), the Hadamard product or convolution \(f_1 \ast f_2\) is given by

\[
(f_1 \ast f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n, \quad (z \in U).
\]

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For two analytic functions $F$ and $G$, if there exist two analytic functions $\varphi$ and $h$ with $|\varphi(z)| \leq 1$, $h(0) = 0$ and $|h(z)| < 1$ for $z \in \mathbb{U}$ so that $F(z) = \varphi(z)G(h(z))$, then $F$ is quasi-subordinate to $G$, i.e., $F \prec_{\text{quasi}} G$. Note that if $\varphi \equiv 1$, then $F$ is subordinate to $G$ in $\mathbb{U}$, i.e., $F \prec G$. Further, if $G \in \mathcal{H}$, then there exists the next equivalent relation ([13, 37]):

$$F \prec G \iff F(\mathbb{U}) \subset G(\mathbb{U}) \text{ and } F(0) = G(0).$$

In addition, if $h(z) = z$, then $F$ is majorized by $G$ in $\mathbb{U}$, i.e., $F \preceq G$. For the related work on quasi-subordination relation, refer to Robertson [50]. From now on, let $\varphi$ and $\phi$ be denoted by

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} E_n z^n, \quad (E_1 > 0, z \in \mathbb{U})$$

and

$$\phi(z) = B_0 + B_1 z + B_2 z^2 + B_3 z^3, \ldots, \quad (|\varphi(z)| \leq 1, z \in \mathbb{U}).$$

1.1. Mittag-Leffler type function

Recently, there has been awesome attention to the study of Mittag-Leffler functions. The Mittag-Leffler type functions have widely applied in fractional differential equations, stochastic systems, dynamical systems, statistical distributions, and chaotic systems, and some aspects of the applications of these functions can also be seen in physics, biology, chemistry, engineering and some other applied sciences (for more applications, we refer to [51] pp. 269-296). The Mittag-Leffler function arises naturally in the solutions of fractional differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and the study of complex systems. Since Wiman [66, 67] firstly defined and investigated the special function named as Mittag-Leffler function is the one and two-parametric Mittag-Leffler functions are fractional generalizations of the basic functions. Observe that the function $E_{\alpha,\beta}$ takes many well-known functions as its special case, for example, $E_{1,1}(z) = e^z$, $E_{1,2}(z) = e^{z^2}/z$, $E_{2,1}(z^2) = \cosh z$, $E_{2,1}(-z^2) = \cos z$, $E_{2,2}(z^2) = \sinh z$, $E_{2,2}(-z^2) = \sin z$, $E_4(z) = \frac{1}{2}(\cos z^{1/4} + \cos z^{1/4})$ and $E_3(z) = \frac{1}{2}(e^{z/3} + 2e^{-\frac{1}{2}z^{1/3}} \cos(\frac{\sqrt{3}}{2}z^{1/3}))$; see
[20, 28, 30, 70] and references therein. We note that Mittag-Leffler function $E_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is expected to cogitate the following normalization of Mittag-Leffler functions:

$$E_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} z^n,$$

which is due to Bansal and Prajapat [10] and also holds for positive real parameters $\alpha, \beta$ and $z \in \mathbb{C}$. In this paper, we will restrict our focuses to the case of real-valued $\alpha, \beta$ and $z \in \mathbb{U}$.

The generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,k}(z)$ [60] is denoted by

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma(\alpha n + \beta) n!}, \quad (z \in \mathbb{C})$$

for the complex parameters $\beta, \gamma, \Re(\alpha) > \max\{0, \Re(k) - 1\}$ or $\Re(k) > 0$ and $\Re(\alpha) = 0$ when $\Re(k) = 1$ with $\beta \neq 0$, where $(\ell)_m$ is the Pochhammer symbol or shifted factorial by

$$(\ell)_m = \begin{cases} 1, & \text{if } m = 0 \text{ and } \ell \in \mathbb{C} \setminus \{0\}, \\ \ell(\ell + 1)(\ell + 2) \ldots (\ell + m - 1), & \text{if } m \in \mathbb{N} = \{1, 2, 3, \ldots\} \text{ and } \ell \in \mathbb{C}. \end{cases}$$

Set

$$E_{\alpha,\beta}^{\gamma,k}(z) = z\Gamma(\beta)E_{\alpha,\beta}^{\gamma,k}(z).$$

Following the methods of Aouf and Seoudy [7], by Hadamard product or convolution we define the linear operator $\mathcal{J}_{\alpha,\beta}^{\gamma,k}$ as below:

$$\mathcal{J}_{\alpha,\beta}^{\gamma,k} f(z) = E_{\alpha,\beta}^{\gamma,k}(z) * f(z) = z + \sum_{n=2}^{\infty} \Lambda_n a_n z^n,$$

where

$$\Lambda_n := \Lambda_n(\alpha, \beta, \gamma, k) = \frac{(\gamma)_{n-1} \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)(n-1)!}.$$ 

In particular

$$\Lambda_2 := \Lambda_2(\alpha, \beta, \gamma, k) = \frac{(\gamma)_{2k} \Gamma(\beta)}{\Gamma(2\alpha + \beta)(2k)!} \quad \text{and} \quad \Lambda_3 := \Lambda_3(\alpha, \beta, \gamma, k) = \frac{(\gamma)_{2k} \Gamma(\beta)}{\Gamma(2\alpha + \beta)(2k)!}. \quad (1.3)$$

Furthermore, note that

$$\mathcal{J}_{0,\beta}^{1,1} f(z) = f(z) \text{ and } \mathcal{J}_{0,\beta}^{2,1} f(z) = zf'(z).$$

Here, we remark that if $\gamma = 1$ and $k = 1$, $E_{\alpha,\beta}^{1,1} f(z) \equiv E_{\alpha,\beta} f(z)$ is a two-parameter Mittag-Leffler function by Wiman [66, 67]. Further, if $\beta = 1$, then $E_{\alpha,1} f(z) \equiv E_{\alpha} f(z)$ is the classical (or one-parameter) Mittag-Leffler function [40, 41].

1.2. Bi-univalent function class $\Sigma$

According to the Koebe’s one-quarter theorem [17], every $f \in \mathbb{S}$ has the compositional inverse $f^{-1}$ satisfying

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w, \quad (w \in \mathbb{U}_\rho),$$

where $\rho \geq \frac{1}{4}$ is the radius of the image $f(\mathbb{U})$. As is well known that $f^{-1}(w)$ has the normalized Taylor-Maclaurin’s series

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n, \quad (w \in \mathbb{U}_\rho), \quad (1.4)$$
where

\[ b_n = \frac{(-1)^{n+1}}{n!} |A_{ij}|, \]

and \(|A_{ij}|\) is the \((n - 1)^{th}\) order determinant whose entries are denoted by (see [41])

\[ |A_{ij}| = \begin{cases} (i-j+1)n+j-1a_{i-j+2}, & \text{if } i + 1 \geq j, \\ 0, & \text{if } i + 1 < j. \end{cases} \]

Then

\[ f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + 4)a_4 w^4 + \cdots. \]  

(1.5)

If \(f \in \mathcal{A}\) and its inverse \(f^{-1}\) both are univalent in \(\mathbb{U}\), then \(f\) is said to be bi-univalent and let \(\Sigma\) denote the class of all bi-univalent functions in \(\mathbb{U}\). Lewin [34] introduced the analytic and bi-univalent function and proved that \(|a_2| < 1.51\), Brannan and Clunie [11] conjectured that \(|a_2| \leq \sqrt{2}\), and Netanyahu [43] obtained that \(\max_{f \in \Sigma} |a_2| = 1.51\). However, Styer and Wright [61] showed that there exists \(f(z)\) so that \(|a_2| > \frac{4}{3}\).

In fact, up to now the upper bound estimate \(|a_2| < 1.485\) of coefficient for functions in \(\Sigma\) by Tan [63] is best. But, as far the coefficient estimate problem for every Taylor-Maclaurin coefficient \(|a_n|\) \(n \in \mathbb{N} \setminus \{1, 2\}\) is probably still an open problem. Since the works of Brannan and Taha [12] and Srivastava et al. [57], many subclasses of the class \(\Sigma\) of analytic and bi-univalent functions were introduced and investigated, and the non-sharp estimates of first two Taylor-Maclaurin coefficients \(|a_2|\) and \(|a_3|\) were provided; see [16, 19, 24, 29, 35, 38, 39, 47, 49, 54–56, 58, 59, 68, 69] and the references cited therein. In addition, for the subclasses of analytic and bi-univalent functions associated with quasi-subordination we mainly refer to [15, 22, 48]. To study the determinations of the sharp upper bounds for the subclass of \(\mathcal{S}\), Fekete-Szegö [18] proposed the functional problem named by himself, which was considered in many classes of functions; refer to Abdel-Gawad [1] for class of quasi-convex functions, Koepf [31] for class of close-to-convex functions, Orhan and Răducanu [46] for class of starlike functions, Panigrahi and Raina [48] for class of quasi-subordination functions, Orhan et al. [45] for the classes of bi-convex and bi-starlike type functions, and Tang et al. [65] for classes of m-mold symmetric bi-univalent functions.

Lately, the mathematicians paid more attention to the behaviors of the Mittag-Leffler function and extended their related results to the more complex domain. Until now, many properties of the emphasized special function have been derived in the literature [3, 4, 9, 27, 36, 42, 52, 53] for some other perspective. In this article inspired by the above works, we define and explore two new subclasses of the function class \(\Sigma\) of analytic and bi-univalent functions concomitant with the quasi-subordination and convolution operator concerning Mittag-Leffler function, and study the analogous estimates of the coefficients \(a_2\) and \(a_3\). Instantaneously, we also obtain the corresponding Fekete-Szegö functional inequalities. In addition, the significances and influences to some earlier known results are established.

We now introduce the following general subclasses of \(\Sigma\) associated with generalized Mittag-Leffler function based on quasi-subordination.

**Definition 1.1.** A function \(f \in \Sigma\) as assumed in (1.1), belongs to the class \(\Omega_{\Sigma; \alpha, \beta}^{\gamma, k}(\theta, \rho; \phi)\) if the following quasi-subordinations

\[
(1 - \theta) \left[ \frac{T_{\alpha, \beta}^{\gamma, k} f(z)}{z} \right]^\rho + \theta (T_{\alpha, \beta}^{\gamma, k} f)'(z) \left[ \frac{T_{\alpha, \beta}^{\gamma, k} f(z)}{z} \right]^{\rho - 1} - 1 \prec_{\text{quasi}} \phi(z) - 1
\]

and

\[
(1 - \theta) \left[ \frac{T_{\alpha, \beta}^{\gamma, k} g(w)}{w} \right]^\rho + \theta (T_{\alpha, \beta}^{\gamma, k} g)'(w) \left[ \frac{T_{\alpha, \beta}^{\gamma, k} g(w)}{w} \right]^{\rho - 1} - 1 \prec_{\text{quasi}} \phi(w) - 1
\]

are satisfied for \(z, w \in \mathbb{U}\), where \(\theta \geq 0, \rho \geq 0\) and \(g = f^{-1}\) is given by (1.4) or (1.5).
From the Definition 1.1, by specializing the parameters one can define the following subclasses based on Mittag-Leffler function which are new and not yet studied so far:

- $\Omega \Sigma_{\alpha,\beta}^{\gamma,k}(1, \rho; \varphi) = \mathcal{B} \Sigma_{\alpha,\beta}^{\gamma,k}(\rho; \varphi)$, the class of bi-Bazlevic functions associated with Mittag-Leffler functions;
- $\Omega \Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta, 0; \varphi) = \mathcal{S} \Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta; \varphi)$, the class of bi-starlike functions associated with Mittag-Leffler functions;
- $\Omega \Sigma_{\alpha,\beta}^{\gamma,k}(0, 1; \varphi) = \mathcal{T} \Sigma_{\alpha,\beta}^{\gamma,k}(\varphi)$, the class of bi-univalent functions associated with Mittag-Leffler functions.

**Definition 1.2.** A function $f(z) \in \Sigma$ is assumed as in (1.1), then $f \in M \Sigma_{\alpha,\beta}^{\gamma,k}(\tau, \vartheta, \rho; \varphi)$ if the following two quasi-subordinations

$$1 \left[ z \left( \frac{w(\gamma_{\alpha,\beta}^{\gamma,k} g)''(w) + \theta \rho w (\gamma_{\alpha,\beta}^{\gamma,k} g)'''(w)}{(1 - \vartheta + \rho)(\gamma_{\alpha,\beta}^{\gamma,k} f)(z) + (\rho - \vartheta)z(\gamma_{\alpha,\beta}^{\gamma,k} f)'(z) + \theta \rho z^2 (\gamma_{\alpha,\beta}^{\gamma,k} f)''(z)} \right) - 1 \right] \prec \text{ quasi } (\varphi(w) - 1)$$

and

$$1 \left[ z \left( \frac{w(\gamma_{\alpha,\beta}^{\gamma,k} f)''(z) + (2\vartheta + \rho - \varphi)w^2 (\gamma_{\alpha,\beta}^{\gamma,k} f)'''(w) + \theta \rho w^2 (\gamma_{\alpha,\beta}^{\gamma,k} f)''(z)}{(1 - \vartheta + \rho)(\gamma_{\alpha,\beta}^{\gamma,k} f)(z) + (\rho - \vartheta)z(\gamma_{\alpha,\beta}^{\gamma,k} f)'(z) + \theta \rho z^2 (\gamma_{\alpha,\beta}^{\gamma,k} f)''(z)} \right) - 1 \right] \prec \text{ quasi } (\varphi(f(z)) - 1)$$

are satisfied for $z, w \in \mathbb{U}$, where $\tau \in \mathbb{C}^s = \mathbb{C} \setminus \{0\}$, $\vartheta, \rho \geq 0$ and $g = f^{-1}$ is given by (1.4) or (1.5).

2. Functional estimates for $\Omega \Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta, \rho; \varphi)$

In the section, making use of the following lemma we study the estimates for the class $\Omega \Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta, \rho; \varphi)$.

**Lemma 2.1 ([17, 21]).** Let $\mathcal{P}$ be the class of all analytic functions $h(z)$ of the following form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{U}$$

satisfying $\Re h(z) > 0$ and $h(0) = 1$. Then the sharp estimates $|c_n| \leq 2(n \in \mathbb{N})$ are true. In particular, the equality holds for all $n$ about the next function

$$h(z) = \frac{1 + z}{1 - z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

For the sake of brevity we assume that $\Lambda_2$ and $\Lambda_3$ are assumed as in (1.3), $\alpha$ and $\beta$ are positive and real, $\vartheta \geq 0, \rho \geq 0,$ and $g = f^{-1}$ is given by (1.4) or (1.5), $z, w \in \mathbb{U}$, unless otherwise stated.

**Theorem 2.2.** Let $f(z)$ be given by (1.1). If $f \in \Omega \Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta, \rho; \varphi)$, then

$$|a_2| \leq \min \left\{ \frac{E_1}{\vartheta + \rho}, \sqrt{\frac{2(|E_2 - E_1| + E_1)}{(\rho + 1)(2\vartheta + \rho)|B_0|}}, \frac{E_1}{\sqrt{2E_1}} \right\} |B_0| \frac{1}{|A_2|} \tag{2.1}$$

and

$$|a_3| \leq \left| \frac{|B_0| + |B_1|}{(2\vartheta + \rho)|A_3|} \right| + \min \left\{ \left| \frac{|B_0|E_1^2 - 2(|E_2 - E_1| + E_1)}{(2\vartheta + \rho)^2}, \frac{2(|E_2 - E_1| + E_1)}{(\rho + 1)(2\vartheta + \rho)} \right| \right\} |B_0| \frac{1}{|A_3|} \tag{2.2}$$

where

$$\Pi(\vartheta, \rho, B_0, E_1, E_2) = (\rho + 1)(2\vartheta + \rho)B_0E_1^2 + 2(\vartheta + \rho)^2(E_1 - E_2). \tag{2.3}$$
Proof. If $f(z) \in \bigcup_{\alpha, \beta} \Omega^{\gamma, k}_{\alpha, \beta}(\theta, \rho; \phi)$, then by Lemma 2.1 there exist two analytic functions $u(z)$ and $v(w) \in \mathcal{P}$ so that

\[
(1 - \theta) \left[ \frac{\gamma_{\alpha, \beta} f(z)}{z} \right] - 1 = \varphi(z)[\phi(u(z)) - 1] \tag{2.4}
\]

and

\[
(1 - \theta) \left[ \frac{\gamma_{\alpha, \beta} g(w)}{w} \right] - 1 = \varphi(w)[\phi(v(w)) - 1]. \tag{2.5}
\]

Expanding the left hand sides of (2.4) and (2.5), we know that

\[
(1 - \theta) \left[ \frac{\gamma_{\alpha, \beta} f(z)}{z} \right] + \theta(\gamma f)'(z) \left[ \frac{\gamma_{\alpha, \beta} f(z)}{z} \right] - 1 = (\theta + \rho) \Lambda_2 a_2 z + \left(2 \theta + \rho \right) \Lambda_3 a_3 + \frac{(\rho - 1)(2 \theta + \rho)}{2} \Lambda^2 = z^2 + \ldots \tag{2.6}
\]

and

\[
(1 - \theta) \left[ \frac{\gamma_{\alpha, \beta} g(w)}{w} \right] + \theta(\gamma g)'(w) \left[ \frac{\gamma_{\alpha, \beta} g(w)}{w} \right] - 1 = -(\theta + \rho) \Lambda_2 a_2 w - \left(2 \theta + \rho \right) \Lambda_3 a_3 - \frac{(\rho + 3)(2 \theta + \rho)}{2} \Lambda^2 w^2 + \ldots \tag{2.7}
\]

Denote the functions $p, q \in \mathcal{P}$ by

\[
p(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \text{and} \quad q(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + \sum_{n=1}^{\infty} d_n w^n, \quad (z, w \in \mathcal{U}). \tag{2.8}
\]

Equivalently, from (2.8) we know that

\[
u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1}{2} z + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \ldots, \quad (z \in \mathcal{U}), \tag{2.9}
\]

and

\[
v(w) = \frac{q(w) - 1}{q(w) + 1} = \frac{d_1}{2} w + \frac{1}{2} \left( d_2 - \frac{d_1^2}{2} \right) w^2 + \ldots, \quad (w \in \mathcal{U}). \tag{2.10}
\]

Then, by (2.9) and (1.2) we derive that

\[
\phi(u(z)) = 1 + \frac{1}{2} E_1 c_1 z + \left[ \frac{1}{2} E_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} E_2 c_1^2 \right] z^2 + \ldots, \quad (z \in \mathcal{U}),
\]

and

\[
\phi(v(w)) = 1 + \frac{1}{2} E_1 d_1 w + \left[ \frac{1}{2} E_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} E_2 d_1^2 \right] w^2 + \ldots, \quad (w \in \mathcal{U}).
\]

Furthermore, we see that

\[
\varphi(z)[\phi(u(z)) - 1] = \frac{1}{2} B_0 E_1 c_1 z + \left[ \frac{1}{2} B_1 E_1 c_1 + \frac{1}{2} B_0 E_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_0 E_2 c_1^2 \right] z^2 + \ldots \tag{2.11}
\]

and

\[
\varphi(w)[\phi(v(w)) - 1] = \frac{1}{2} B_0 E_1 d_1 w + \left[ \frac{1}{2} B_1 E_1 d_1 + \frac{1}{2} B_0 E_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} B_0 E_2 d_1^2 \right] w^2 + \ldots. \tag{2.12}
\]
Therefore, from (2.4)-(2.7) and (2.11)-(2.12) we have that

$$ \frac{1}{2} B_0 E_1 c_1, $$

and

$$ \frac{1}{2} B_1 E_1 c_1 + \frac{1}{2} B_0 E_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_0 E_2 c_1^2, $$

and

$$ \frac{1}{2} B_0 E_1 d_1, $$

and

$$ \frac{1}{2} B_1 E_1 d_1 + \frac{1}{2} B_0 E_1 \left( d_2 - \frac{d_1^2}{2} \right) + \frac{1}{4} B_0 E_2 d_1^2. $$

In view of (2.13) and (2.15), we express that

$$ a_2 = \frac{B_0 E_1 c_1}{2(\theta + \rho) \Lambda_2} = - \frac{B_0 E_1 d_1}{2(\theta + \rho) \Lambda_2} $$

such that

$$ c_1 = -d_1 $$

and

$$ B_0^2 E_1^2 (c_1^2 + d_1^2) = 8(\theta + \rho)^2 \Lambda_2^2 a_2^2. $$

By (2.14) and (2.16), we obtain that

$$ \frac{1}{4} B_0 (E_2 - E_1) (c_1^2 + d_1^2) + \frac{1}{2} B_0 E_1 (c_2 + d_2) = (p + 1)(2\theta + \rho) \Lambda_2^2 a_2^2. $$

Then, together with (2.19)-(2.20) we get that

$$ a_2^2 = \frac{B_0^2 E_1^3 (c_2 + d_2)}{2[(p + 1)(2\theta + \rho)B_0 E_1^2 + 2(E_1 - E_2)(\theta + \rho)^2] \Lambda_2}. $$

Moreover, it infers from Lemma 2.1 and (2.19)-(2.21) that

$$ |a_2| \leq \frac{|B_0| E_1}{(\theta + \rho) \Lambda_2}, $$

and

$$ |a_2| \leq \frac{\sqrt{2|B_0||(E_2 - E_1) + E_1}}{(p + 1)(2\theta + \rho) \Lambda_2}, $$

and

$$ |a_2| \leq \frac{|B_0| E_1 \sqrt{2E_1}}{\sqrt{(p + 1)(2\theta + \rho)B_0 E_1^2 + 2(E_1 - E_2)(\theta + \rho)^2} \Lambda_2}, $$

then (2.1) holds. Similarly, from (2.14), (2.16), and (2.18), it also implies that

$$ B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2) = 4(2\theta + \rho) |\Lambda_2 a_3 - \Lambda_2^2 a_2^2|. $$

Hence, from (2.19) and (2.22) we derive that

$$ a_3 = \frac{B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2)}{4(2\theta + \rho) \Lambda_3} + \frac{B_0^2 E_1^2 (c_1^2 + d_1^2)}{8(\theta + \rho)^2 \Lambda_3}. $$

Further, from Lemma 2.1 we remark that

$$ |a_3| \leq \frac{(|B_0| + |B_1|) E_1}{(2\theta + \rho) \Lambda_3} + \frac{B_0^2 E_1^2}{(\theta + \rho)^2 \Lambda_3}. $$
On the other hand, by (2.20) and (2.22) we deduce that
\[
\alpha_3 = \frac{B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2)}{4(2\theta + \rho)\Lambda_3} + \frac{B_0 (E_2 - E_1)(c_1^2 + d_1^2) + 2B_0 E_1 (c_2 + d_2)}{4(\rho + 1)(2\theta + \rho)\Lambda_3}.
\]
Therefore, from Lemma 2.1 we see that
\[
|\alpha_3| \leq \frac{(|B_0| + |B_1|)E_1}{(2\theta + \rho)|\Lambda_3|} + \frac{2|B_0(\rho + 1)(|E_2 - E_1| + |E_1|)}{(\rho + 1)|\Lambda_3|}.
\]
(2.24)
Then, from (2.23) and (2.24) we know (2.2) is true.

If we let \( \theta = 1 \) or \( \rho = 0 \) in Theorem 2.2, respectively, then we can state the following corollaries.

**Corollary 2.3.** Let \( f(z) \) be given by (1.1). If \( f \in \mathcal{B}_{\alpha_0, \beta_0}(\rho; \phi) \), then
\[
|\alpha_2| \leq \min \left\{ \frac{E_1}{\rho + 1}, \sqrt{2|E_2 - E_1| + E_1} \right\} \frac{E_1 \sqrt{2E_1}}{|\Lambda_2|} \left| \frac{B_0}{|\Lambda_2|} \right|
\]
and
\[
|\alpha_3| \leq \frac{(|B_0| + |B_1|)E_1}{(\rho + 1)|\Lambda_3|} + \frac{\left\{ \frac{|B_0|E_1^2}{\rho + 1}, \frac{2|E_2 - E_1| + E_1}{\rho + 2} \right\} |B_0|}{(\rho + 1)|\Lambda_3|},
\]
where
\[
\Pi(\rho, B_0, E_1, E_2) = (\rho + 1)((\rho + 1)B_0E_1^2 + 2(\rho + 1)(E_1 - E_2)).
\]

**Corollary 2.4.** Let \( f(z) \) be given by (1.1) and \( \theta > 0 \), and if \( f \in \mathcal{S}_{\alpha_0, \beta_0}(\theta; \phi) \), then
\[
|\alpha_2| \leq \min \left\{ \frac{E_1}{\theta}, \sqrt{\frac{|E_2 - E_1| + E_1}{\theta|B_0|}} \right\} \frac{E_1 \sqrt{E_1}}{|\Lambda_2|} \left| \frac{B_0}{|\Lambda_2|} \right|
\]
and
\[
|\alpha_3| \leq \frac{(|B_0| + |B_1|)E_1}{2\theta|\Lambda_3|} + \frac{\left\{ \frac{|B_0|E_1^2}{\theta}, |E_2 - E_1| + E_1 \right\} |B_0|}{\theta|\Lambda_3|},
\]

**Remark 2.5.** By fixing \( \varphi(z) \equiv 1 \) the coefficient estimates for this class are partially due to Tang et al. [64].
Furthermore, suitably fixing the parameters \( \theta, \rho, \) and \( \phi \) in Theorem 2.2, we provide the better estimates for the functions classes noted below:

- \( \Omega_{\alpha, \beta}(1, 1; \phi) = \mathcal{H}_\Sigma(\phi) \), refer to Ali et al. [6];
- \( \Omega_{\alpha, \beta}(\theta, 1; \phi) = \mathcal{H}_\Sigma(\phi) \) and \( \Omega_{\alpha, \beta}(1, \rho; \phi) = \mathcal{H}_\Sigma^\rho(\phi) \), refer to Kumar et al. [32].

In addition, if we take
\[
\phi(z) = \left( \frac{1 + z}{1 - z} \right)^{\xi} = \psi_{\xi}(z) \text{ or } \phi(z) = \frac{1 + (1 - 2\xi)z}{1 - z} = \chi_{\xi}(z),
\]
then we have another reduced classes as follows:

- \( \Omega_{\alpha, \beta}(\theta, \rho; \psi_{\xi}) = \mathcal{H}_\Sigma^\xi(\theta, \xi) \) or \( \Omega_{\alpha, \beta}(\theta, \rho; \chi_{\xi}) = \mathcal{H}_\Sigma^\rho(\theta, \xi) \), refer to Caglar et al. [14];
- \( \Omega_{\alpha, \beta}(1, 1; \psi_{\xi}) = \mathcal{H}_\Sigma(\theta, \xi) \) or \( \Omega_{\alpha, \beta}(1, 1; \chi_{\xi}) = \mathcal{H}_\Sigma(\theta, \xi) \), refer to Frasin and Aouf [19];
- \( \Omega_{\alpha, \beta}(1, 1; \psi_{\xi}) = \mathcal{H}_\Sigma^{\rho}(\xi) \) or \( \Omega_{\alpha, \beta}(1, 1; \chi_{\xi}) = \mathcal{H}_\Sigma^{\rho}(\xi) \), refer to Srivastava et al. [57].
Now we determine to study the Fekete-Szegő functional inequality for \( f \in \Omega \Sigma_{\alpha,\beta}^{\gamma,k}(\theta, \rho; \phi) \).

**Theorem 2.6.** Let \( f(z) \) be given by (1.1). If \( f \in \Omega \Sigma_{\alpha,\beta}^{\gamma,k}(\theta, \rho; \phi) \) and \( \delta \in \mathbb{R} \), then

\[
|a_3 - \delta a_2^2| \leq \begin{cases} \frac{|B_0| + |B_1|}{|2 \theta + \rho| |A_3|}, & \text{if } 2(2 \theta + \rho)|B_0|E_1^2 |A_2^2 - \delta A_3| \leq |\Pi \Lambda_2^3|, \\ \frac{|B_1| |E_1|}{|2 \theta + \rho| |A_3|} + \frac{2|B_0|^2 E_1^2 |A_2^2 - \delta A_3|}{|\Pi \Lambda_2^3|}, & \text{if } 2(2 \theta + \rho)|B_0|E_1^2 |A_2^2 - \delta A_3| \geq |\Pi \Lambda_2^3|, \end{cases}
\]

where \( \Pi = \Pi(\theta, \rho, B_0, E_1, E_2) \) is fixed as in (2.3).

**Proof.** From (2.22), we have

\[
a_3 - \frac{\Lambda_2^3 a_2^2}{3} = \frac{B_1 E_1 (c_1 - d_1) + B_0 E_1 (c_2 - d_2)}{4(2 \theta + \rho) \Lambda_3}.
\]

By (2.21) we easily obtain that

\[
a_3 - \delta a_2^2 = \frac{B_1 E_1 (c_1 - d_1)}{4(2 \theta + \rho) \Lambda_3} + \frac{B_0 E_1 [2B_0 E_1^2 (2 \theta + \rho) (A_2^2 - \delta A_3) + \Pi \Lambda_2^3] c_2}{4(2 \theta + \rho) \Pi \Lambda_2^3 \Lambda_3} + \frac{B_0 E_1 [2B_0 E_1^2 (2 \theta + \rho) (A_2^2 - \delta A_3) - \Pi \Lambda_2^3] d_2}{4(2 \theta + \rho) \Pi \Lambda_2^3 \Lambda_3}.
\]

Hence, from Lemma 2.1 we imply that

\[
|a_3 - \delta a_2^2| \leq \frac{|B_0| + |B_1|}{(2 \theta + \rho) |A_3|},
\]

if \( 2(2 \theta + \rho)|B_0|E_1^2 |A_2^2 - \delta A_3| \leq |\Pi \Lambda_2^3| \), and

\[
|a_3 - \delta a_2^2| \leq \frac{|B_1| E_1}{(2 \theta + \rho) |A_3|} + \frac{2|B_0|^2 E_1^2 |A_2^2 - \delta A_3|}{\Pi \Lambda_2^3 |A_3|},
\]

if \( 2(2 \theta + \rho)|B_0|E_1^2 |A_2^2 - \delta A_3| \geq |\Pi \Lambda_2^3| \).

By fixing \( \delta = 0 \), we may formulate the following corollary.

**Corollary 2.7.** Let \( f(z) \) be given by (1.1). If \( f \in \Omega \Sigma_{\alpha,\beta}^{\gamma,k}(\theta, \rho; \phi) \), then

\[
|a_3| \leq \begin{cases} \frac{|B_0| + |B_1|}{|2 \theta + \rho| |A_3|}, & \text{if } 2(2 \theta + \rho)|B_0|E_1^2 \leq |\Pi|, \\ \frac{|B_1| E_1}{|2 \theta + \rho| |A_3|} + \frac{2|B_0|^2 E_1^2}{|\Pi \Lambda_3|}, & \text{if } 2(2 \theta + \rho)|B_0|E_1^2 \geq |\Pi|, \end{cases}
\]

where \( \Pi = \Pi(\theta, \rho, B_0, E_1, E_2) \) is given in (2.3).

### 3. Functional estimates for \( f \in \mathcal{M} \Sigma_{\alpha,\beta}^{\gamma,k}(\tau, \theta, \rho; \phi) \)

Now we consider the coefficients for the class \( \mathcal{M} \Sigma_{\alpha,\beta}^{\gamma,k}(\tau, \theta, \rho; \phi) \) and give the next theorem for coefficient bounds.

**Theorem 3.1.** If \( f(z) \) is assumed as in (1.1) and \( f \in \mathcal{M} \Sigma_{\alpha,\beta}^{\gamma,k}(\tau, \theta, \rho; \phi) \), then

\[
|a_2| \leq \min\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}
\]
for
\[J_1 = \frac{|B_0\tau|E_1}{|2\theta \rho + \theta - \rho + 1|\Lambda_2}, \quad J_2 = \sqrt{\frac{|B_0\tau|(|E_2 - E_1| + E_1)}{|\Omega(\theta, \rho)|\Lambda_3^2}},\]
and
\[J_3 = \frac{|B_0\tau|E_1\sqrt{E_1}}{\sqrt{|B_0E_1^2\tau\Omega(\theta, \rho) + (E_1 - E_2)(2\theta \rho + \theta - \rho + 1)^2|\Lambda_2}|},\]
and
\[|a_3| \leq \min\{J_1, J_2\}\]
for
\[G_1 = \frac{E_1|\tau|(|B_0| + |B_1|)}{2|6\theta \rho + 2\theta - 2\rho + 1|\Lambda_3|} + \frac{|B_0\tau|^2E_1^2}{|2\theta \rho + \theta - \rho + 1|^2\Lambda_3|},\]
and
\[G_2 = \frac{E_1|\tau|(|B_0| + |B_1|)}{2|6\theta \rho + 2\theta - 2\rho + 1|\Lambda_3|} + \frac{|\tau B_0|(|E_2 - E_1| + E_1)}{|\Omega(\theta, \rho)|\Lambda_3^2|},\]
where
\[\Omega(\theta, \rho) = 2(6\theta \rho + 2\theta - 2\rho + 1) - (2\theta \rho + \theta - \rho + 1)^2. \tag{3.1}\]

Proof. Since \(f(z) \in M_{\Sigma^\gamma, \alpha, \beta}(\tau, \theta, \rho; \phi)\), there exist two analytic functions \(u(z), v(z) : \mathbb{U} \to \mathbb{U}\) with \(u(0) = 0\) and \(v(0) = 0\) such that
\[
1 \frac{z[(\mathcal{T}_{\gamma, \alpha, \beta})'](z) + (2\theta \rho + \theta - \rho)z^2(\mathcal{T}_{\gamma, \alpha, \beta})''(z) + \theta \rho z^3(\mathcal{T}_{\gamma, \alpha, \beta})'''(z)}{(1 - \theta + \rho)(\mathcal{T}_{\gamma, \alpha, \beta})(z) + (\theta - \rho)z(\mathcal{T}_{\gamma, \alpha, \beta})(z) + \theta \rho z^2(\mathcal{T}_{\gamma, \alpha, \beta})''(z) - 1} = \phi(z)[\phi(u(z)) - 1] \tag{3.2}
\]
and
\[
1 \frac{w[(\mathcal{T}_{\gamma, \alpha, \beta})'](z) + (2\theta \rho + \theta - \rho)w^2(\mathcal{T}_{\gamma, \alpha, \beta})''(w) + \theta \rho w^3(\mathcal{T}_{\gamma, \alpha, \beta})'''(w)}{(1 - \theta + \rho)(\mathcal{T}_{\gamma, \alpha, \beta})(w) + (\theta - \rho)w(\mathcal{T}_{\gamma, \alpha, \beta})(w) + \theta \rho w^2(\mathcal{T}_{\gamma, \alpha, \beta})''(w) - 1} = \varphi(w)[\varphi(v(w)) - 1]. \tag{3.3}
\]
Expanding the left hand sides of (3.2) and (3.3), we obtain that
\[
1 \frac{z[(\mathcal{T}_{\gamma, \alpha, \beta})'](z) + (2\theta \rho + \theta - \rho)z^2(\mathcal{T}_{\gamma, \alpha, \beta})''(z) + \theta \rho z^3(\mathcal{T}_{\gamma, \alpha, \beta})'''(z)}{(1 - \theta + \rho)(\mathcal{T}_{\gamma, \alpha, \beta})(z) + (\theta - \rho)z(\mathcal{T}_{\gamma, \alpha, \beta})(z) + \theta \rho z^2(\mathcal{T}_{\gamma, \alpha, \beta})''(z) - 1} = \frac{2\theta \rho + \theta - \rho + 1}{\tau}\Lambda_2 a_2 \Lambda_3 a_3 + \frac{2(6\theta \rho + 2\theta - 2\rho + 1)}{\tau}\Lambda_3 a_3 - \frac{(2\theta \rho + \theta - \rho + 1)^2}{\tau}\Lambda_2 a_2^2 z^2 + \cdots \tag{3.4}
\]
and
\[
1 \frac{w[(\mathcal{T}_{\gamma, \alpha, \beta})'](z) + (2\theta \rho + \theta - \rho)w^2(\mathcal{T}_{\gamma, \alpha, \beta})''(w) + \theta \rho w^3(\mathcal{T}_{\gamma, \alpha, \beta})'''(w)}{(1 - \theta + \rho)(\mathcal{T}_{\gamma, \alpha, \beta})(w) + (\theta - \rho)w(\mathcal{T}_{\gamma, \alpha, \beta})(w) + \theta \rho w^2(\mathcal{T}_{\gamma, \alpha, \beta})''(w) - 1} = \frac{-2\theta \rho + \theta - \rho + 1}{\tau}\Lambda_2 a_2 w
\]
\[
+ \frac{-2(6\theta \rho + 2\theta - 2\rho + 1)}{\tau}\Lambda_3 a_3 + \frac{4(6\theta \rho + 2\theta - 2\rho + 1) - (2\theta \rho + \theta - \rho + 1)^2}{\tau}\Lambda_2 a_2^2 w^2 + \cdots \tag{3.5}
\]
Furthermore, from (2.11)-(2.12) and (3.2)-(3.5), by the coefficient comparison method we know that
\[
\frac{2\theta \rho + \theta - \rho + 1}{\tau}\Lambda_2 a_2 = \frac{1}{2} B_0 E_1 c_1, \tag{3.6}
\]
Therefore, by (3.10)-(3.11) we obtain that
\[
\frac{2(6\theta \rho + 2\theta - 2\rho + 1)}{\tau} \Lambda_3 a_3 - \frac{(2\theta \rho + \theta - \rho + 1)^2}{\tau} \Lambda_2^2 a_2^2 = \frac{1}{2} B_1 E_1 c_1 + \frac{1}{2} B_0 E_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_0 E_2 c_1^2, \tag{3.7}
\]
and
\[
-\frac{2(6\theta \rho + 2\theta - 2\rho + 1)}{\tau} \Lambda_2 a_2 = \frac{1}{2} B_0 E_1 d_1, \tag{3.8}
\]
Now proceeding on similar lines of Theorem 2.2 and using (3.6) and (3.8), we get
\[
a_2 = \frac{B_0 E_1 c_1 \tau}{2(2\theta \rho + \theta - \rho + 1) \Lambda_2} = -\frac{B_0 E_1 d_1 \tau}{2(2\theta \rho + \theta - \rho + 1) \Lambda_2}.
\]
Then, it infers that
\[
c_1 = -d_1
\]
and
\[
8(2\theta \rho + \theta - \rho + 1)^2 \Lambda_2^2 a_2^2 = B_0^2 E_1^2 \tau^2 (c_1^2 + d_1^2). \tag{3.10}
\]
By (3.7) and (3.9), we have that
\[
\frac{2\Lambda_2^2 a_2^2}{\tau} \Omega(\theta, \rho) = \frac{1}{4} B_0 (c_1^2 + d_1^2) (E_2 - E_1) + \frac{1}{2} B_0 E_1 (c_2 + d_2), \tag{3.11}
\]
where
\[
\Omega(\theta, \rho) = 2(6\theta \rho + 2\theta - 2\rho + 1) - (2\theta \rho + \theta - \rho + 1)^2.
\]
Therefore, by (3.10)-(3.11) we obtain that
\[
a_2^2 = \frac{\frac{1}{4} B_0^2 E_1^2 \tau^2 (c_2 + d_2)}{B_0^2 E_1^2 \tau \Omega(\theta, \rho) + (E_1 - E_2)(2\theta \rho + \theta - \rho + 1)^2 \Lambda_2^2}. \tag{3.12}
\]
Hence, from (3.10)-(3.12) and Lemma 2.1 we know that
\[
|a_2| \leq \frac{|B_0 \tau| E_1}{|2\theta \rho + \theta - \rho + 1| \Lambda_2}, \tag{3.13}
\]
Correspondingly, by detracting (3.9) from (3.7), it infers that
\[
\frac{1}{2} B_1 E_1 (c_1 - d_1) + \frac{1}{2} B_0 E_1 (c_2 - d_2) = \frac{4(6\theta \rho + 2\theta - 2\rho + 1)(\Lambda_3 a_3 - \Lambda_2^2 a_2^2)}{\tau}. \tag{3.14}
\]
Hence, by (3.10) and (3.13), it trails that
\[
a_3 = \frac{E_1 \tau [B_1 (c_1 - d_1) + B_0 (c_2 - d_2)]}{8(6\theta \rho + 2\theta - 2\rho + 1) \Lambda_3} + \frac{B_0^2 E_1^2 \tau^2 (c_1^2 + d_1^2)}{8(2\theta \rho + \theta - \rho + 1)^2 \Lambda_3}.
\]
Thus, from Lemma 2.1 we obtain that

\[ |a_3| \leq \frac{E_1|\tau|(|B_0| + |B_1|)}{2(\theta^4 + \theta - 2\theta^2 + 1)||\Lambda_1||} + \frac{|B_0\tau|E_1^2}{(2\theta - \theta + 1)^2||\Lambda_1||}. \]

On the other hand, by (3.11) and (3.13) we infer that

\[ a_3 = \frac{E_1\tau|B_0(c_1 - d_1) + B_0(c_2 - d_2)|}{8(6\theta^2 + 2\theta - 2\theta^2 + 1)||\Lambda_3||} + \frac{\tau B_0([E_2 - E_1](c_1^2 + d_1^2) + 2E_1(c_2 + d_2))}{8\Omega(\theta, \rho)||\Lambda_3||}. \]

Thus, from Lemma 2.1 we obtain that

\[ |a_3| \leq \frac{E_1|\tau|(|B_0| + |B_1|)}{2(\theta^4 + \theta - 2\theta^2 + 1)||\Lambda_1||} + \frac{|\tau B_0|(|E_2 - E_1| + 1)}{||\Lambda_3||}. \]

Then, we complete the proof of Theorem 3.1. \(\square\)

If we choose \(\theta = 1\) and \(\rho = 0\), respectively, we let \(M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, 1, \rho; \phi) = M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, \rho; \phi), M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, 0, \rho; \phi) = M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, 0; \phi)\) and have two corollaries below.

**Corollary 3.2.** If \(f(z)\) is as assumed in (1.1) and \(f \in M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, \rho; \phi)\), then

\[ |a_2| \leq \min \left\{ \frac{|B_0\tau|E_1}{(\rho + 2)||\Lambda_1||}, \sqrt{\frac{|B_0\tau|(|E_2 - E_1| + 1)}{|\Omega(\rho)||\Lambda_1||}}, \frac{|B_0\tau|E_1}{|\Lambda_1||}\right\}, \]

and

\[ |a_3| \leq \frac{E_1|\tau|(|B_0| + |B_1|)}{2(4\rho + 3)||\Lambda_1||} \cdot \begin{cases} \frac{|B_0\tau|E_1^2}{(\rho + 2)||\Lambda_1||}, & |B_0\tau|/|\Omega(\rho)||\Lambda_1||, \\ \frac{|\tau B_0|(|E_2 - E_1| + 1)}{||\Lambda_1||}, & \end{cases} \]

where

\[ \Omega(\rho) = 6 - (\rho - 2)^2. \]

**Corollary 3.3.** If \(f(z)\) is as assumed in (1.1) and \(f \in M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, \theta; \phi)\), then

\[ |a_2| \leq \min \left\{ \frac{|B_0\tau|E_1}{(\theta + 1)||\Lambda_1||}, \sqrt{\frac{|B_0\tau|(|E_2 - E_1| + 1)}{|\Omega(\theta)||\Lambda_1||}}, \right\}, \]

for

\[ \mathcal{F} = \frac{|B_0\tau|E_1}{\sqrt{|B_0E_1^2\tau\Omega(\theta) + (E_2 - E_1)(\theta + 1)^2||\Lambda_1||}}, \]

and

\[ |a_3| \leq \frac{E_1|\tau|(|B_0| + |B_1|)}{2(2\theta + 1)||\Lambda_1||} \cdot \begin{cases} \frac{|B_0\tau|E_1^2}{(\theta + 1)^2||\Lambda_1||}, & |B_0\tau|/|\Omega(\theta)||\Lambda_1||, \\ \frac{|\tau B_0|(|E_2 - E_1| + 1)}{||\Lambda_1||}, & \end{cases} \]

where

\[ \Omega(\theta) = 2 - (\theta - 1)^2. \]

**Remark 3.4.** If the quasi-subordination reduces to the subordination (i.e., fixing \(\varphi(z) \equiv 1\)), then our result in Theorem 3.1, partly belongs to Tang et al. [64]. Furthermore, suitably choosing the parameter \(\theta, \rho, \) and \(\phi,\) we afford the subsequent condensed forms for \(M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, \theta; \phi)\) and analogues results as in Theorem 3.1:

1. \(M_{\Sigma_\alpha,\beta}^{\gamma,k}(\tau, \theta, 0; \phi) = N_{\Sigma_\alpha}^{\theta}(\phi) (\theta \geq 0, \tau \in \mathbb{C}^\ast),\) refer to Kumar et al. [32];
2. $M_{\Sigma_{\alpha,\beta}}^{\gamma,k}(1,0,0; \psi_{\ell}) = S_{\Sigma_{a}}^{X}(\xi)$ (see Remark 2.5 for $\psi_{\ell}$), refer to Taha [62];

3. $M_{\Sigma_{\alpha,\beta}}^{\gamma,k}(1,1,0; \phi) = H_{\Sigma}(\phi)$, refer to Lashin [33], furthermore, $M_{\Sigma_{\alpha,\beta}}^{\gamma,k}(1,1,0; \chi_{\ell}) = \mathcal{K}_{\Sigma}(\zeta)$ and $M_{\Sigma_{\alpha,\beta}}^{\gamma,k}(1,0,0; \chi_{\ell}) = \delta_{\Sigma}^{X}(\zeta)$ (0 $\leq \zeta < 1$) (see Remark 2.5 for $\chi_{\ell}$), refer to Brannan and Taha [12].

Next, we primarily cogitate Fejér-Szegő problems for $M_{\Sigma_{\alpha,\beta}}^{\gamma,k}(\tau, \delta, \rho; \phi)$.

**Theorem 3.5.** Let $f(z)$ be fixed as in (1.1) and $f \in M_{\Sigma_{\alpha,\beta}}^{\gamma,k}(\tau, \delta, \rho; \phi)$ and $\delta \in \mathbb{R}$. Then

$$|\alpha_{3} - \delta\alpha_{2}^{2}| \leq \frac{|B_{1}|E_{1}}{2|6\delta\rho + 2\delta - 2\rho + 1||\Lambda_{3}|} + \frac{|B_{0}\tau^{2}|E_{1}^{3}|\Lambda_{2}^{2} - \delta\Lambda_{3}|}{2|6\delta\rho + 2\delta - 2\rho + 1||\Lambda_{3}|}$$

if

$$2|B_{0}\tau|E_{1}^{2}|(6\delta\rho + 2\delta - 2\rho + 1)(\Lambda_{2}^{2} - \delta\Lambda_{3})| \geq |B_{0}\tau^{2}|E_{1}^{2}\tau\Omega(\delta, \rho) + (E_{1} - E_{2})(2\delta\rho + \theta - \rho + 1)^{2}\|\Lambda_{2}\|^{2},$$

or

$$|\alpha_{3} - \delta\alpha_{2}^{2}| \leq \frac{|(B_{0} + |B_{1}|)\tau|E_{1}}{2|6\delta\rho + 2\delta - 2\rho + 1||\Lambda_{3}|}$$

if

$$2|B_{0}\tau|E_{1}^{2}|(6\delta\rho + 2\delta - 2\rho + 1)(\Lambda_{2}^{2} - \delta\Lambda_{3})| \leq |B_{0}\tau^{2}|E_{1}^{2}\tau\Omega(\delta, \rho) + (E_{1} - E_{2})(2\delta\rho + \theta - \rho + 1)^{2}\|\Lambda_{2}\|^{2},$$

where $\Omega(\delta, \rho)$ is given by (3.1).

**Proof.** From (3.13), we have

$$\alpha_{3} - \frac{\Lambda_{2}^{2}}{\Lambda_{3} \alpha_{2}^{2}} = \frac{E_{1}\tau[B_{1}(c_{1} - d_{1}) + B_{0}(c_{2} - d_{2})]}{8(6\delta\rho + 2\delta - 2\rho + 1)\Lambda_{3}}.$$

By (3.12) we obtain that

$$\alpha_{3} - \delta\alpha_{2}^{2} = \frac{B_{1}E_{1}\tau(c_{1} - d_{1})}{8(6\delta\rho + 2\delta - 2\rho + 1)\Lambda_{3}} + \frac{B_{0}E_{1}\tau(2B_{0}E_{1}^{2}\tau(6\delta\rho + 2\delta - 2\rho + 1)(\Lambda_{2}^{2} - \delta\Lambda_{3}) + |B_{0}\tau^{2}|E_{1}^{2}\tau\Omega(\delta, \rho) + (E_{1} - E_{2})(2\delta\rho + \theta - \rho + 1)^{2}\|\Lambda_{2}\|^{2})c_{2}}{8(6\delta\rho + 2\delta - 2\rho + 1)\Lambda_{3}} + \frac{B_{0}E_{1}\tau(2B_{0}E_{1}^{2}\tau(6\delta\rho + 2\delta - 2\rho + 1)(\Lambda_{2}^{2} - \delta\Lambda_{3}) - |B_{0}\tau^{2}|E_{1}^{2}\tau\Omega(\delta, \rho) + (E_{1} - E_{2})(2\delta\rho + \theta - \rho + 1)^{2}\|\Lambda_{2}\|^{2})d_{2}}{8(6\delta\rho + 2\delta - 2\rho + 1)\Lambda_{3}}.$$

Hence, from Lemma 2.1 we imply that

$$|\alpha_{3} - \delta\alpha_{2}^{2}| \leq \frac{|B_{1}|E_{1}}{2|6\delta\rho + 2\delta - 2\rho + 1||\Lambda_{3}|} + \frac{|B_{0}\tau^{2}|E_{1}^{3}|\Lambda_{2}^{2} - \delta\Lambda_{3}|}{2|6\delta\rho + 2\delta - 2\rho + 1||\Lambda_{3}|}$$

when

$$2|B_{0}\tau|E_{1}^{2}|(6\delta\rho + 2\delta - 2\rho + 1)(\Lambda_{2}^{2} - \delta\Lambda_{3})| \geq |B_{0}\tau^{2}|E_{1}^{2}\tau\Omega(\delta, \rho) + (E_{1} - E_{2})(2\delta\rho + \theta - \rho + 1)^{2}\|\Lambda_{2}\|^{2},$$

or

$$|\alpha_{3} - \delta\alpha_{2}^{2}| \leq \frac{|(B_{0} + |B_{1}|)\tau|E_{1}}{2|6\delta\rho + 2\delta - 2\rho + 1||\Lambda_{3}|}$$

when

$$2|B_{0}\tau|E_{1}^{2}|(6\delta\rho + 2\delta - 2\rho + 1)(\Lambda_{2}^{2} - \delta\Lambda_{3})| \leq |B_{0}\tau^{2}|E_{1}^{2}\tau\Omega(\delta, \rho) + (E_{1} - E_{2})(2\delta\rho + \theta - \rho + 1)^{2}\|\Lambda_{2}\|^{2}. $$

□
By taking $\delta = 0$, we provide the following corollary.

**Corollary 3.6.** Let $f(z)$ given by (1.1) belongs to the class $M_{\alpha, \beta}^{\gamma,k}(\tau, \vartheta, \rho; \phi)$. Then

$$|a_3| \leq \frac{|B_1\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|} + \frac{|B_0E_1^2\tau\Omega(\vartheta, \rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_3|}{|B_0E_1^2\tau\Omega(\vartheta, \rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_3|}$$

if

$$2|B_0\tau|E_1^2|6\vartheta\rho + 2\vartheta - 2\rho + 1| \geq |B_0E_1^2\tau\Omega(\vartheta, \rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2|,$$

or

$$|a_3| \leq \frac{(|B_0| + |B_1|)|\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|}$$

if

$$2|B_0\tau|E_1^2|6\vartheta\rho + 2\vartheta - 2\rho + 1| \leq |B_0E_1^2\tau\Omega(\vartheta, \rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2|,$$

where $\Omega(\vartheta, \rho)$ is assumed by (3.1).

4. **Conclusions**

By fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$E_{z,1}(z) = e^{z^2} \cdot \text{erfc}(-z) = e^{z^2} \left(1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n + 1)} z^{2n+1}\right).$$

We can define and explore new subclasses of the function class $\Sigma$ of analytic and bi-univalent functions concomitant with the quasi-subordination and convolution operator concerning error function, and study the analogous estimates of the coefficients $a_2$ and $a_3$ as given for $f \in M_{\alpha, \beta}^{\gamma,k}(\tau, \vartheta, \rho; \phi)$ or $f \in \Omega_{\alpha, \beta}^{\gamma,k}(\vartheta, \rho; \phi)$. Instantaneously, one can obtain the corresponding Fekete-Szegő functional inequalities. Besides, the consequences and connections to some earlier known results would be expounded which are new and has not been studied so far, we left this as an exercise to interested readers.

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**References**

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