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Subclasses of analytic and bi-univalent functions involving a generalized Mittag-Leffler function based on quasi-subordination

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Abstract

Two quasi-subordination subclasses $\mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,\rho;\phi)$ and $\mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(\tau,\vartheta,\rho;\phi)$ of the class Σ of analytic and bi-univalent functions associated with the convolution operator involving Mittag-Leffler function are introduced and investigated. Then, the corresponding bound estimates of the coefficients a_2 and a_3 are provided. Meanwhile, Fekete-Szegö functional inequalities for these classes are proved. Besides, some consequences and connections to all the theorems would be interpreted, which generalize and improve earlier known results.

Keywords: Fekete-Szegö inequality, bi-univalent function, Mittag-Leffler function, quasi-subordination.

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1. Introduction

Denote by $\mathcal{A}$ the class of analytic function f given by Taylor-Maclaurin's series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized as $f(0) = 0 = f'(0) - 1$. Also, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent functions in $\mathbb{U}$.

For $f_1, f_2 \in \mathcal{A}$, as given by $f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n (j = 1, 2)$, the Hadamard product or convolution $f_1 * f_2$ is given by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n, \quad (z \in \mathbb{U}).$$

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For two analytic functions F and G, if there exist two analytic functions φ and h with $|\varphi(z)| \leqslant 1$, $h(0) = 0$ and $|h(z)| < 1$ for $z \in \mathbb{U}$ so that $F(z) = \varphi(z)G(h(z))$, then F is quasi-subordinate to G, i.e., $f \prec_{quasi} G$. Note that if $\varphi \equiv 1$, then F is subordinate to G in $\mathbb{U}$, i.e., $F \prec G$. Further, if $G \in \mathcal{S}$, then there exists the next equivalent relation ([13, 37]):

$$F \prec G \Leftrightarrow F(\mathbb{U}) \subset G(\mathbb{U}) \text{ and } F(0) = G(0).$$

In addition, if $h(z) = z$, then F is majorized by G in $\mathbb{U}$, i.e., $F \leqslant G$. For the related work on quasi-subordination relation, refer to Robertson [50]. From now on, let ϕ and φ be denoted by

$$\phi(z) = 1 + \sum_{n=1}^{\infty} E_n z^n, \quad (E_1 > 0, z \in \mathbb{U}) \tag{1.2}$$

and

$$\varphi(z) = B_0 + B_1 z + B_2 z^2 + B_3 z^3, \ldots, \quad (|\varphi(z)| \leqslant 1, z \in \mathbb{U}).$$

1.1. Mittag-Leffler type function

Recently, there has been awesome attention to the study of Mittag-Leffler functions. The Mittag-Lefferr type functions have widely applied in fractional differential equations, stochastic systems, dynamical systems, statistical distributions, and chaotic systems, and some aspects of the applications of these functions can also be seen in physics, biology, chemistry, engineering and some other applied sciences (for more applications, we refer to [51] pp. 269-296). The Mittag-Leffler function arises naturally in the solutions of fractional differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and the study of complex systems. Since Wiman [66, 67] firstly defined and investigated the special function named by Mittag-Leffler [40, 41] in term of power series with the complex parameter, the geometric properties as starlikeness, convexity and close-to-convexity of this kind of function have been examined on a large scale by a great deal of mathematicians. For example, Agarwal [2], Humbert [25] and Humbert and Agarwal [26] considered the two-parameter Mittag-Leffler function. For the more properties and applications of Mittag-Leffler functions and even generalized Mittag-Leffler functions, we may refer to [5, 8, 10, 20, 23, 27, 30, 44, 60] and their references therein.

Now we recall Mittag-Leffler function $\mathbf{E}_\alpha(z)$ [40, 41] denoted by

$$\mathbf{E}_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z \in \mathbb{C}, \mathrm{Re}(\alpha) > 0),$$

and it converges in the whole complex plane for all $\Re\{\alpha\} > 0$, while it diverges everywhere in $\mathbb{C} \setminus \{0\}$ for all $\Re\{\alpha\} < 0$. We also note that for $\Re\{\alpha\} = 0$, its radius of convergence is $R = e^{\pi/2|\mathrm{Im}(z)|}$. To generalize $\mathbf{E}_\alpha(z)$, Wiman [66, 67] defined a more general function $\mathbf{E}_{\alpha,\beta}$ given by

$$\mathbf{E}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (z, \alpha, \beta \in \mathbb{C}, \Re\{\alpha\} > 0,\ \Re\{\beta\} > 0).$$

For different values of parameters, the conditions of convergence vary. When α and β are positive and real, the above given series converges in the entire complex plane. Another important and interesting fact about Mittag-Leffler function is that the one and two-parametric Mittag-Leffler functions are fractional extensions of the basic functions. Observe that the function $\mathbf{E}_{\alpha,\beta}$ takes many well-known functions as its special case, for example, $\mathbf{E}_{1,1}(z) = e^z$, $\mathbf{E}_{1,2}(z) = \frac{e^z - 1}{z}$, $\mathbf{E}_{2,1}(z^2) = \cosh z$, $\mathbf{E}_{2,1}(-z^2) = \cos z$, $\mathbf{E}_{2,2}(z^2) = \frac{\sinh z}{z}$, $\mathbf{E}_{2,2}(-z^2) = \frac{\sin z}{z}$, $\mathbf{E}_4(z) = \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4})$ and $\mathbf{E}_3(z) = \frac{1}{2}(e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos(\frac{\sqrt{3}}{2} z^{1/3}))$; see

[20, 28, 30, 70] and references therein. We note that Mittag-Leffler function $\mathbf{E}_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is expected to cogitate the following normalization of Mittag-Leffler functions:

$$E_{\alpha,\beta}(z) = z\Gamma(\beta)\mathbf{E}_{\alpha,\beta}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} z^n,$$

which is due to Bansal and Prajapat [10] and also holds for positive real parameters α, β and $z \in \mathbb{C}$. In this paper, we will restrict our focuses to the case of real-valued α, β and $z \in \mathbb{U}$.

The generalized Mittag-Leffler function $\mathbb{E}^{\gamma,k}_{\alpha,\beta}(z)$ [60] is denoted by

$$\mathbf{E}^{\gamma,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \quad (z \in \mathbb{C})$$

for the complex parameters β, γ, $\Re\{\alpha\} > \max\{0, \Re\{k\} - 1\}$ or $\Re\{k\} > 0$ and $\Re\{\alpha\} = 0$ when $\Re\{k\} = 1$ with $\beta \neq 0$, where $(\ell)_m$ is the Pochhammer symbol or shifted factorial by

$$(\ell)_m = \frac{\Gamma(\ell+m)}{\Gamma(\ell)} = \begin{cases} 1, & \text{if } m = 0 \text{ and } \ell \in \mathbb{C} \setminus \{0\}, \\ \ell(\ell+1)(\ell+2)\ldots(\ell+m-1), & \text{if } m \in \mathbb{N} = \{1,2,3,\ldots\} \text{ and } \ell \in \mathbb{C}. \end{cases}$$

Set

$$E^{\gamma,k}_{\alpha,\beta}(z) = z\Gamma(\beta)\mathbf{E}^{\gamma,k}_{\alpha,\beta}(z).$$

Following the methods of Aouf and Seoudy [7], by Hadamard product or convolution we define the linear operator $\mathcal{T}^{\gamma,k}_{\alpha,\beta}$ as below:

$$\mathcal{T}^{\gamma,k}_{\alpha,\beta} f(z) = E^{\gamma,k}_{\alpha,\beta}(z) * f(z) = z + \sum_{n=2}^{\infty} \Lambda_n a_n z^n,$$

where

$$\Lambda_n := \Lambda_n(\alpha,\beta,\gamma,k) = \frac{(\gamma)_{(n-1)k}\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)(n-1)!}.$$

In particular

$$\Lambda_2 := \Lambda_2(\alpha,\beta,\gamma,k) = \frac{(\gamma)_k\Gamma(\beta)}{\Gamma(\alpha+\beta)} \text{ and } \Lambda_3 := \Lambda_3(\alpha,\beta,\gamma,k) = \frac{(\gamma)_{2k}\Gamma(\beta)}{\Gamma(2\alpha+\beta)(2)!}. \tag{1.3}$$

Furthermore, note that

$$\mathcal{T}^{1,1}_{0,\beta} f(z) = f(z) \text{ and } \mathcal{T}^{2,1}_{0,\beta} f(z) = zf'(z).$$

Here, we remark that if $\gamma = 1$ and $k = 1$, $\mathbf{E}^{1,1}_{\alpha,\beta} f(z) \equiv E_{\alpha,\beta} f(z)$ is a two-parameter Mittag-Leffler function by Wiman [66, 67]. Further, if $\beta = 1$, then $E_{\alpha,1} f(z) \equiv E_\alpha f(z)$ is the classical (or one-parameter) Mittag-Leffler function [40, 41].

1.2. Bi-univalent function class Σ

According to the Koebe's one-quarter theorem [17], every $f \in \mathcal{S}$ has the compositional inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w, \ \ (w \in \mathbb{U}_\rho),$$

where $\rho \geqslant \frac{1}{4}$ is the radius of the image $f(\mathbb{U})$. As is well known that $f^{-1}(w)$ has the normalized Taylor-Maclaurin's series

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} b_n w^n, \ \ (w \in \mathbb{U}_\rho), \tag{1.4}$$

where

$$b_n = \frac{(-1)^{n+1}}{n!}|A_{ij}|,$$

and $|A_{ij}|$ is the $(n-1)^{th}$ order determinant whose entries are denoted by (see [41])

$$|A_{ij}| = \begin{cases} [(i-j+1)n+j-1]a_{i-j+2}, & \text{if } i+1 \geqslant j, \\ 0, & \text{if } i+1 < j. \end{cases}$$

Then

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.5}$$

If $f \in \mathcal{A}$ and its inverse f^{-1} both are univalent in $\mathbb{U}$, then f is said to be bi-univalent and let Σ denote the class of all bi-univalent functions in $\mathbb{U}$. Lewin [34] introduced the analytic and bi-univalent function and proved that $|a_2| < 1.51$, Brannan and Clunie [11] conjectured that $|a_2| \leqslant \sqrt{2}$, and Netanyahu [43] obtained that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. However, Styer and Wright [61] showed that there exists $f(z)$ so that $|a_2| > \frac{4}{3}$. In fact, up to now the upper bound estimate $|a_2| < 1.485$ of coefficient for functions in Σ by Tan [63] is best. But, as far the coefficient estimate problem for every Taylor-Maclaurin coefficient $|a_n|(n \in \mathbb{N} \setminus \{1, 2\})$ is probably still an open problem. Since the works of Brannan and Taha [12] and Srivastava et al. [57], many subclasses of the class Σ of analytic and bi-univalent functions were introduced and investigated, and the non-sharp estimates of first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ were provided; see [16, 19, 24, 29, 35, 38, 39, 47, 49, 54–56, 58, 59, 68, 69] and the references cited therein. In addition, for the subclasses of analytic and bi-univalent functions associated with quasi-subordination we mainly refer to [15, 22, 48]. To study the determinations of the sharp upper bounds for the subclass of $\mathcal{S}$, Fekete-Szegö [18] proposed the functional problem named by himself, which was considered in many classes of functions; refer to Abdel-Gawad [1] for class of quasi-convex functions, Koepf [31] for class of close-to-convex functions, Orhan and Răducanu [46] for class of starlike functions, Panigrahi and Raina [48] for class of quasi-subordination functions, Orhan et al. [45] for the classes of bi-convex and bi-starlike type functions, and Tang et al. [65] for classes of m-mold symmetric bi-univalent functions.

Lately, the mathematicians paid more attention to the behaviors of the Mittag-Leffler function and extended their related results to the more complex domain. Until now, many properties of the emphasized special function have been derived in the literature [3, 4, 9, 27, 36, 42, 52, 53] for some other perspective. In this article inspired by the above works, we define and explore two new subclasses of the function class Σ of analytic and bi-univalent functions concomitant with the quasi-subordination and convolution operator concerning Mittag-Leffler function, and study the analogous estimates of the coefficients a_2 and a_3. Instantaneously, we also obtain the corresponding Fekete-Szegö functional inequalities. In addition, the significances and influences to some earlier known results are established.

We now introduce the following general subclasses of Σ associated with generalized Mittag-Leffler function based on quasi-subordination.

Definition 1.1. A function $f \in \Sigma$ as assumed in (1.1), belongs to the class $\mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta, \rho; \phi)$ if the following quasi-subordinations

$$(1-\vartheta)\left[\frac{\mathcal{T}^{\gamma,k}_{\alpha,\beta} f(z)}{z}\right]^{\rho} + \vartheta(\mathcal{T}^{\gamma,k}_{\alpha,\beta} f)'(z)\left[\frac{\mathcal{T}^{\gamma,k}_{\alpha,\beta} f(z)}{z}\right]^{\rho-1} - 1 \prec_{\text{quasi}} (\phi(z) - 1)$$

and

$$(1-\vartheta)\left[\frac{\mathcal{T}^{\gamma,k}_{\alpha,\beta} g(w)}{w}\right]^{\rho} + \vartheta(\mathcal{T}^{\gamma,k}_{\alpha,\beta} g)'(w)\left[\frac{\mathcal{T}^{\gamma,k}_{\alpha,\beta} g(w)}{w}\right]^{\rho-1} - 1 \prec_{\text{quasi}} (\phi(w) - 1)$$

are satisfied for $z, w \in \mathbb{U}$, where $\vartheta \geqslant 0, \rho \geqslant 0$ and $g = f^{-1}$ is given by (1.4) or (1.5).

From the Definition 1.1, by specializing the parameters one can define the following subclasses based on Mittag-Leffler function which are new and not yet studied so far:

- $\mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(1,\rho;\phi) \equiv \mathcal{B}\Sigma^{\gamma,k}_{\alpha,\beta}(\rho;\phi)$, the class of bi-Bazelvic functions associated with Mittag-Leffler functions;
- $\mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,0;\phi) \equiv \mathcal{S}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta;\phi)$, the class of bi-starlike functions associated with Mittag-Leffler functions;
- $\mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(0,1;\phi) \equiv \mathcal{F}\Sigma^{\gamma,k}_{\alpha,\beta}(\phi)$, the class of bi-univalent functions associated with Mittag-Leffler functions.

Definition 1.2. A function $f(z) \in \Sigma$ is assumed as in (1.1), then $f \in \mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(\tau,\vartheta,\rho;\phi)$ if the following two quasi-subordinations

$$\frac{1}{\tau}\left[\frac{z(\mathcal{T}^{\gamma,k}_{\alpha,\beta}f)'(z)+(2\vartheta\rho+\vartheta-\rho)z^2(\mathcal{T}^{\gamma,k}_{\alpha,\beta}f)''(z)+\vartheta\rho z^3(\mathcal{T}^{\gamma,k}_{\alpha,\beta}f)'''(z)}{(1-\vartheta+\rho)(\mathcal{T}^{\gamma,k}_{\alpha,\beta}f)(z)+(\vartheta-\rho)z(\mathcal{T}^{\gamma,k}_{\alpha,\beta}f)'(z)+\vartheta\rho z^2(\mathcal{T}^{\gamma,k}_{\alpha,\beta}f)''(z)}-1\right] \prec_{\text{quasi}} (\phi(z)-1)$$

and

$$\frac{1}{\tau}\left[\frac{w(\mathcal{T}^{\gamma,k}_{\alpha,\beta}g)'(z)+(2\vartheta\rho+\vartheta-\rho)w^2(\mathcal{T}^{\gamma,k}_{\alpha,\beta}g)''(w)+\vartheta\rho w^3(\mathcal{T}^{\gamma,k}_{\alpha,\beta}g)'''(w)}{(1-\vartheta+\rho)(\mathcal{T}^{\gamma,k}_{\alpha,\beta}g)(w)+(\vartheta-\rho)w(\mathcal{T}^{\gamma,k}_{\alpha,\beta}g)'(w)+\vartheta\rho w^2(\mathcal{T}^{\gamma,k}_{\alpha,\beta}g)''(w)}-1\right] \prec_{\text{quasi}} (\phi(w)-1)$$

are satisfied for $z,w \in \mathbb{U}$, where $\tau \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, $\vartheta,\rho \geqslant 0$ and $g = f^{-1}$ is given by (1.4) or (1.5).

2. Functional estimates for $\mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,\rho;\phi)$

In the section, making use of the following lemma we study the estimates for the class $\mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,\rho;\phi)$.

Lemma 2.1 ([17, 21])**.** *Let $\mathcal{P}$ be the class of all analytic functions $h(z)$ of the following form*

$$h(z) = 1+\sum_{n=1}^{\infty} c_n z^n, \quad (z\in\mathbb{U})$$

satisfying $\Re h(z) > 0$ and $h(0) = 1$. Then the sharp estimates $|c_n| \leqslant 2 (n \in \mathbb{N})$ are true. In Particular, the equality holds for all n about the next function

$$h(z) = \frac{1+z}{1-z} = 1+\sum_{n=1}^{\infty} 2z^n.$$

For the sake of brevity we assume that Λ_2 and Λ_3 are assumed as in (1.3), α and β are positive and real, $\vartheta \geqslant 0, \rho \geqslant 0$, and $g = f^{-1}$ is given by (1.4) or (1.5), $z,w \in \mathbb{U}$, unless otherwise stated.

Theorem 2.2. *Let $f(z)$ be given by (1.1). If $f \in \mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,\rho;\phi)$, then*

$$|a_2| \leqslant \min\left\{\frac{E_1}{\vartheta+\rho}, \sqrt{\frac{2(|E_2-E_1|+E_1)}{(\rho+1)(2\vartheta+\rho)|B_0|}}, \frac{E_1\sqrt{2E_1}}{\sqrt{|\Pi(\vartheta,\rho,B_0,E_1,E_2)|}}\right\}\frac{|B_0|}{|\Lambda_2|} \tag{2.1}$$

and

$$|a_3| \leqslant \frac{(|B_0|+|B_1|)E_1}{(2\vartheta+\rho)|\Lambda_3|} + \min\left\{\frac{|B_0|E_1^2}{(\vartheta+\rho)^2}, \frac{2(|E_2-E_1|+E_1)}{(\rho+1)(2\vartheta+\rho)}\right\}\frac{|B_0|}{|\Lambda_3|}, \tag{2.2}$$

where

$$\Pi(\vartheta,\rho,B_0,E_1,E_2) = (\rho+1)(2\vartheta+\rho)B_0E_1^2 + 2(\vartheta+\rho)^2(E_1-E_2). \tag{2.3}$$

Proof. If $f(z) \in \mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta,\rho;\phi)$, then by Lemma 2.1 there exist two analytic functions $u(z)$ and $v(w) \in \mathcal{P}$ so that

$$(1-\vartheta)\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}f(z)}{z}\right]^{\rho} + \vartheta(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'(z)\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}f(z)}{z}\right]^{\rho-1} - 1 = \varphi(z)[\phi(u(z))-1] \tag{2.4}$$

and

$$(1-\vartheta)\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}g(w)}{w}\right]^{\rho} + \vartheta(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'(w)\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}g(w)}{w}\right]^{\rho-1} - 1 = \varphi(w)[\phi(v(w))-1]. \tag{2.5}$$

Expanding the left hand sides of (2.4) and (2.5), we know that

$$\begin{aligned}(1-\vartheta)&\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}f(z)}{z}\right]^{\rho} + \vartheta(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'(z)\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}f(z)}{z}\right]^{\rho-1} - 1\\ &= (\vartheta+\rho)\Lambda_2 a_2 z + \left[(2\vartheta+\rho)\Lambda_3 a_3 + \frac{(\rho-1)(2\vartheta+\rho)}{2}\Lambda_2^2 a_2^2\right]z^2 + \cdots\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}(1-\vartheta)&\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}g(w)}{w}\right]^{\rho} + \vartheta(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'(w)\left[\frac{\mathcal{T}_{\alpha,\beta}^{\gamma,k}g(w)}{w}\right]^{\rho-1} - 1\\ &= -(\vartheta+\rho)\Lambda_2 a_2 w - \left[(2\vartheta+\rho)\Lambda_3 a_3 - \frac{(\rho+3)(2\vartheta+\rho)}{2}\Lambda_2^2 a_2^2\right]w^2 + \cdots.\end{aligned} \tag{2.7}$$

Denote the functions $p, q \in \mathcal{P}$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n \quad \text{and} \quad q(w) = \frac{1+v(w)}{1-v(w)} = 1 + \sum_{n=1}^{\infty} d_n w^n, \quad (z, w \in \mathbb{U}). \tag{2.8}$$

Equivalently, from (2.8) we know that

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots, \quad (z \in \mathbb{U}), \tag{2.9}$$

and

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{d_1}{2}w + \frac{1}{2}\left(d_2 - \frac{d_1^2}{2}\right)w^2 + \cdots, \quad (w \in \mathbb{U}). \tag{2.10}$$

Then, by (2.9)-(2.10) and (1.2) we derive that

$$\phi(u(z)) = 1 + \frac{1}{2}E_1 c_1 z + \left[\frac{1}{2}E_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}E_2 c_1^2\right]z^2 + \cdots, \quad (z \in \mathbb{U}),$$

and

$$\phi(v(w)) = 1 + \frac{1}{2}E_1 d_1 w + \left[\frac{1}{2}E_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{1}{4}E_2 d_1^2\right]w^2 + \cdots, \quad (w \in \mathbb{U}).$$

Furthermore, we see that

$$\varphi(z)[\phi(u(z))-1] = \frac{1}{2}B_0 E_1 c_1 z + \left[\frac{1}{2}B_1 E_1 c_1 + \frac{1}{2}B_0 E_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_0 E_2 c_1^2\right]z^2 + \cdots \tag{2.11}$$

and

$$\varphi(w)[\phi(v(w))-1] = \frac{1}{2}B_0 E_1 d_1 w + \left[\frac{1}{2}B_1 E_1 d_1 + \frac{1}{2}B_0 E_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{1}{4}B_0 E_2 d_1^2\right]w^2 + \cdots. \tag{2.12}$$

Therefore, from (2.4)-(2.7) and (2.11)-(2.12) we have that

$$(\vartheta+\rho)\Lambda_2 a_2 = \frac{1}{2}B_0E_1c_1, \tag{2.13}$$

$$(2\vartheta+\rho)\Lambda_3 a_3 + \frac{(\rho-1)(2\vartheta+\rho)}{2}\Lambda_2^2 a_2^2 = \frac{1}{2}B_1E_1c_1 + \frac{1}{2}B_0E_1\left(c_2-\frac{c_1^2}{2}\right) + \frac{1}{4}B_0E_2c_1^2, \tag{2.14}$$

$$-(\vartheta+\rho)\Lambda_2 a_2 = \frac{1}{2}B_0E_1d_1, \tag{2.15}$$

and

$$-(2\vartheta+\rho)\Lambda_3 a_3 + \frac{(\rho+3)(2\vartheta+\rho)}{2}\Lambda_2^2 a_2^2 = \frac{1}{2}B_1E_1d_1 + \frac{1}{2}B_0E_1\left(d_2-\frac{d_1^2}{2}\right) + \frac{1}{4}B_0E_2d_1^2. \tag{2.16}$$

In view of (2.13) and (2.15), we express that

$$a_2 = \frac{B_0E_1c_1}{2(\vartheta+\rho)\Lambda_2} = -\frac{B_0E_1d_1}{2(\vartheta+\rho)\Lambda_2} \tag{2.17}$$

such that

$$c_1 = -d_1 \tag{2.18}$$

and

$$B_0^2E_1^2(c_1^2+d_1^2) = 8(\vartheta+\rho)^2\Lambda_2^2 a_2^2. \tag{2.19}$$

By (2.14) and (2.16), we obtain that

$$\frac{1}{4}B_0(E_2-E_1)(c_1^2+d_1^2) + \frac{1}{2}B_0E_1(c_2+d_2) = (\rho+1)(2\vartheta+\rho)\Lambda_2^2 a_2^2. \tag{2.20}$$

Then, together with (2.19)-(2.20) we get that

$$a_2^2 = \frac{B_0^2E_1^3(c_2+d_2)}{2[(\rho+1)(2\vartheta+\rho)B_0E_1^2+2(E_1-E_2)(\vartheta+\rho)^2]\Lambda_2^2}. \tag{2.21}$$

Moreover, it infers from Lemma 2.1 and (2.19)-(2.21) that

$$|a_2| \leqslant \frac{|B_0|E_1}{(\vartheta+\rho)|\Lambda_2|},$$

$$|a_2| \leqslant \frac{\sqrt{2|B_0|(|E_2-E_1|+E_1)}}{\sqrt{(\rho+1)(2\vartheta+\rho)|\Lambda_2|}},$$

and

$$|a_2| \leqslant \frac{|B_0|E_1\sqrt{2E_1}}{\sqrt{|(\rho+1)(2\vartheta+\rho)B_0E_1^2+2(E_1-E_2)(\vartheta+\rho)^2||\Lambda_2|}},$$

then (2.1) holds. Similarly, from (2.14), (2.16), and (2.18), it also implies that

$$B_1E_1(c_1-d_1) + B_0E_1(c_2-d_2) = 4(2\vartheta+\rho)[\Lambda_3 a_3 - \Lambda_2^2 a_2^2]. \tag{2.22}$$

Hence, from (2.19) and (2.22) we derive that

$$a_3 = \frac{B_1E_1(c_1-d_1)+B_0E_1(c_2-d_2)}{4(2\vartheta+\rho)\Lambda_3} + \frac{B_0^2E_1^2(c_1^2+d_1^2)}{8(\vartheta+\rho)^2\Lambda_3}.$$

Further, from Lemma 2.1 we remark that

$$|a_3| \leqslant \frac{(|B_0|+|B_1|)E_1}{(2\vartheta+\rho)|\Lambda_3|} + \frac{B_0^2E_1^2}{(\vartheta+\rho)^2|\Lambda_3|}. \tag{2.23}$$

On the other hand, by (2.20) and (2.22) we deduce that

$$a_3 = \frac{B_1E_1(c_1-d_1)+B_0E_1(c_2-d_2)}{4(2\vartheta+\rho)\Lambda_3} + \frac{B_0(E_2-E_1)(c_1^2+d_1^2)+2B_0E_1(c_2+d_2)}{4(\rho+1)(2\vartheta+\rho)\Lambda_3}.$$

Therefore, from Lemma 2.1 we see that

$$|a_3| \leqslant \frac{(|B_0|+|B_1|)E_1}{(2\vartheta+\rho)|\Lambda_3|} + \frac{2|B_0|(|E_2-E_1|+E_1)}{(\rho+1)(2\vartheta+\rho)|\Lambda_3|}. \tag{2.24}$$

Then, from (2.23) and (2.24) we know (2.2) is true. □

If we let $\vartheta = 1$ or $\rho = 0$ in Theorem 2.2, respectively, then we can state the following corollaries.

Corollary 2.3. *Let* $f(z)$ *be given by* (1.1). *If* $f \in \mathcal{B}\Sigma_{\alpha,\beta}^{\gamma,k}(\rho;\phi)$, *then*

$$|a_2| \leqslant \min\left\{\frac{E_1}{\rho+1}, \sqrt{\frac{2(|E_2-E_1|+E_1)}{(\rho+1)(\rho+2)|B_0|}}, \frac{E_1\sqrt{2E_1}}{\sqrt{|\Pi(\rho,B_0,E_1,E_2)|}}\right\}\frac{|B_0|}{|\Lambda_2|}$$

and

$$|a_3| \leqslant \frac{(|B_0|+|B_1|)E_1}{(\rho+2)|\Lambda_3|} + \min\left\{\frac{|B_0|E_1^2}{\rho+1}, \frac{2(|E_2-E_1|+E_1)}{\rho+2}\right\}\frac{|B_0|}{(\rho+1)|\Lambda_3|},$$

where

$$\Pi(\rho,B_0,E_1,E_2) = (\rho+1)[(\rho+2)B_0E_1^2+2(\rho+1)(E_1-E_2)].$$

Corollary 2.4. *Let* $f(z)$ *be given by* (1.1) *and* $\vartheta > 0$, *and if* $f \in \mathcal{S}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta;\phi)$, *then*

$$|a_2| \leqslant \min\left\{\frac{E_1}{\vartheta}, \sqrt{\frac{|E_2-E_1|+E_1}{\vartheta|B_0|}}, \frac{E_1\sqrt{E_1}}{\sqrt{\vartheta|B_0E_1^2+\vartheta(E_1-E_2)|}}\right\}\frac{|B_0|}{|\Lambda_2|}$$

and

$$|a_3| \leqslant \frac{(|B_0|+|B_1|)E_1}{2\vartheta|\Lambda_3|} + \min\left\{\frac{|B_0|E_1^2}{\vartheta}, |E_2-E_1|+E_1\right\}\frac{|B_0|}{\vartheta|\Lambda_3|}.$$

Remark 2.5. By fixing $\varphi(z) \equiv 1$ the coefficient estimates for this class are partially due to Tang et al. [64]. Furthermore, suitably fixing the parameters ϑ, ρ, and ϕ in Theorem 2.2, we provide the better estimates for the functions classes noted below:

- $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(1,1;\phi) = \mathcal{H}_\Sigma(\phi)$, refer to Ali et al. [6];
- $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta,1;\phi) = \mathcal{H}_\Sigma(\vartheta,\phi)$ and $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(1,\rho;\phi) = \mathcal{H}_\Sigma^{\rho}(\phi)$, refer to Kumar et al. [32].

In addition, if we take

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\xi} = \psi_\xi(z) \text{ or } \phi(z) = \frac{1+(1-2\zeta)z}{1-z} = \chi_\zeta(z),$$

then we have another reduced classes as follows:

- $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta,\rho;\psi_\xi) = \mathcal{H}_\Sigma^{\rho}(\vartheta,\xi)$ or $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta,\rho;\chi_\zeta) = \mathcal{H}_\Sigma^{\rho}(\vartheta,\zeta)$, refer to Caglar et al. [14];
- $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta,1;\psi_\xi) = \mathcal{H}_\Sigma(\vartheta,\xi)$ or $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta,1;\chi_\zeta) = \mathcal{H}_\Sigma(\vartheta,\zeta)$, refer to Frasin and Aouf [19];
- $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(1,1;\psi_\xi) = \mathcal{H}_\Sigma^{\rho}(\xi)$ or $\mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(1,1;\chi_\zeta) = \mathcal{H}_\Sigma^{\rho}(\zeta)$, refer to Srivastava et al. [57].

Now we determine to study the Fekete-Szegö functional inequality for $f \in \mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,\rho;\phi)$.

Theorem 2.6. *Let* $f(z)$ *be given by* (1.1)*. If* $f \in \mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,\rho;\phi)$ *and* $\delta \in \mathbb{R}$*, then*

$$|a_3 - \delta a_2^2| \leqslant \begin{cases} \frac{(|B_0|+|B_1|)E_1}{(2\vartheta+\rho)|\Lambda_3|}, & \text{if } 2(2\vartheta+\rho)|B_0|E_1^2|\Lambda_2^2-\delta\Lambda_3| \leqslant |\Pi\Lambda_2^2|, \\ \frac{|B_1|E_1}{(2\vartheta+\rho)|\Lambda_3|} + \frac{2|B_0|^2E_1^3|\Lambda_2^2-\delta\Lambda_3|}{|\Pi\Lambda_2^2\Lambda_3|}, & \text{if } 2(2\vartheta+\rho)|B_0|E_1^2|\Lambda_2^2-\delta\Lambda_3| \geqslant |\Pi\Lambda_2^2|, \end{cases}$$

where $\Pi = \Pi(\vartheta,\rho,B_0,E_1,E_2)$ *is fixed as in* (2.3)*.*

Proof. From (2.22), we have

$$a_3 - \frac{\Lambda_2^2 a_2^2}{\Lambda_3} = \frac{B_1E_1(c_1-d_1)+B_0E_1(c_2-d_2)}{4(2\vartheta+\rho)\Lambda_3}.$$

By (2.21) we easily obtain that

$$\begin{aligned} a_3-\delta a_2^2 &= \frac{B_1E_1(c_1-d_1)}{4(2\vartheta+\rho)\Lambda_3} + \frac{B_0E_1[2B_0E_1^2(2\vartheta+\rho)(\Lambda_2^2-\delta\Lambda_3)+\Pi\Lambda_2^2]c_2}{4(2\vartheta+\rho)\Pi\Lambda_2^2\Lambda_3} \\ &\quad + \frac{B_0E_1[2B_0E_1^2(2\vartheta+\rho)(\Lambda_2^2-\delta\Lambda_3)-\Pi\Lambda_2^2]d_2}{4(2\vartheta+\rho)\Pi\Lambda_2^2\Lambda_3}. \end{aligned}$$

Hence, from Lemma 2.1 we imply that

$$|a_3-\delta a_2^2| \leqslant \frac{(|B_0|+|B_1|)E_1}{(2\vartheta+\rho)|\Lambda_3|}$$

if $2(2\vartheta+\rho)|B_0|E_1^2|\Lambda_2^2-\delta\Lambda_3| \leqslant |\Pi\Lambda_2^2|$, and

$$|a_3-\delta a_2^2| \leqslant \frac{|B_1|E_1}{(2\vartheta+\rho)|\Lambda_3|} + \frac{2|B_0|^2E_1^3|\Lambda_2^2-\delta\Lambda_3|}{|\Pi\Lambda_2^2\Lambda_3|}$$

if $2(2\vartheta+\rho)|B_0|E_1^2|\Lambda_2^2-\delta\Lambda_3| \geqslant |\Pi\Lambda_2^2|$. □

By fixing $\delta = 0$, we may formulate the following corollary.

Corollary 2.7. *Let* $f(z)$ *be given by* (1.1)*. If* $f \in \mathcal{Q}\Sigma^{\gamma,k}_{\alpha,\beta}(\vartheta,\rho;\phi)$*, then*

$$|a_3| \leqslant \begin{cases} \frac{(|B_0|+|B_1|)E_1}{(2\vartheta+\rho)|\Lambda_3|}, & \text{if } 2(2\vartheta+\rho)|B_0|E_1^2 \leqslant |\Pi|, \\ \frac{|B_1|E_1}{(2\vartheta+\rho)|\Lambda_3|} + \frac{2|B_0|^2E_1^3}{|\Pi\Lambda_3|}, & \text{if } 2(2\vartheta+\rho)|B_0|E_1^2 \geqslant |\Pi|, \end{cases}$$

where $\Pi = \Pi(\vartheta,\rho,B_0,E_1,E_2)$ *is given in* (2.3)*.*

3. Functional estimates for $f \in \mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(\tau,\vartheta,\rho;\phi)$

Now we consider the coefficients for the class $\mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(\tau,\vartheta,\rho;\phi)$ and give the next theorem for coefficient bounds.

Theorem 3.1. *If* $f(z)$ *is assumed as in* (1.1) *and* $f \in \mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(\tau,\vartheta,\rho;\phi)$*, then*

$$|a_2| \leqslant \min\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$$

for

$$\mathcal{F}_1 = \frac{|B_0\tau|E_1}{|(2\vartheta\rho+\vartheta-\rho+1)\Lambda_2|}, \quad \mathcal{F}_2 = \sqrt{\frac{|B_0\tau|(|E_2-E_1|+E_1)}{|\Omega(\vartheta,\rho)\Lambda_2^2|}}$$

and

$$\mathcal{F}_3 = \frac{|B_0\tau|E_1\sqrt{E_1}}{\sqrt{|B_0E_1^2\tau\Omega(\vartheta,\rho)+(E_1-E_2)(2\vartheta\rho+\vartheta-\rho+1)^2||\Lambda_2|}},$$

and

$$|a_3| \leqslant \min\{\mathcal{G}_1, \mathcal{G}_2\}$$

for

$$\mathcal{G}_1 = \frac{E_1|\tau|(|B_0|+|B_1|)}{2|6\vartheta\rho+2\vartheta-2\rho+1||\Lambda_3|} + \frac{|B_0\tau|^2E_1^2}{(2\vartheta\rho+\vartheta-\rho+1)^2|\Lambda_3|}$$

and

$$\mathcal{G}_2 = \frac{E_1|\tau|(|B_0|+|B_1|)}{2|6\vartheta\rho+2\vartheta-2\rho+1||\Lambda_3|} + \frac{|\tau B_0|(|E_2-E_1|+E_1)}{|\Omega(\vartheta,\rho)\Lambda_3|},$$

where

$$\Omega(\vartheta,\rho) = 2(6\vartheta\rho+2\vartheta-2\rho+1)-(2\vartheta\rho+\vartheta-\rho+1)^2. \tag{3.1}$$

Proof. Since $f(z) \in \mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\vartheta,\rho;\phi)$, there exist two analytic functions $u(z), v(z) : \mathbb{U} \to \mathbb{U}$ with $u(0) = 0$ and $v(0) = 0$ such that

$$\frac{1}{\tau}\left[\frac{z(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'(z)+(2\vartheta\rho+\vartheta-\rho)z^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)''(z)+\vartheta\rho z^3(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'''(z)}{(1-\vartheta+\rho)(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)(z)+(\vartheta-\rho)z(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'(z)+\vartheta\rho z^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)''(z)}-1\right] = \varphi(z)[\phi(u(z))-1] \tag{3.2}$$

and

$$\frac{1}{\tau}\left[\frac{w(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'(z)+(2\vartheta\rho+\vartheta-\rho)w^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)''(w)+\vartheta\rho w^3(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'''(w)}{(1-\vartheta+\rho)(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)(w)+(\vartheta-\rho)w(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'(w)+\vartheta\rho w^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)''(w)}-1\right] = \varphi(w)[\phi(v(w))-1]. \tag{3.3}$$

Expanding the left hand sides of (3.2) and (3.3), we obtain that

$$\begin{aligned}&\frac{1}{\tau}\left[\frac{z(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'(z)+(2\vartheta\rho+\vartheta-\rho)z^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)''(z)+\vartheta\rho z^3(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'''(z)}{(1-\vartheta+\rho)(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)(z)+(\vartheta-\rho)z(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)'(z)+\vartheta\rho z^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}f)''(z)}-1\right]\\ &= \frac{2\vartheta\rho+\vartheta-\rho+1}{\tau}\Lambda_2a_2z + \left[\frac{2(6\vartheta\rho+2\vartheta-2\rho+1)}{\tau}\Lambda_3a_3 - \frac{(2\vartheta\rho+\vartheta-\rho+1)^2}{\tau}\Lambda_2^2a_2^2\right]z^2+\cdots\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\frac{1}{\tau}\left[\frac{w(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'(z)+(2\vartheta\rho+\vartheta-\rho)w^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)''(w)+\vartheta\rho w^3(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'''(w)}{(1-\vartheta+\rho)(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)(w)+(\vartheta-\rho)w(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)'(w)+\vartheta\rho w^2(\mathcal{T}_{\alpha,\beta}^{\gamma,k}g)''(w)}-1\right]\\ &= -\frac{2\vartheta\rho+\vartheta-\rho+1}{\tau}\Lambda_2a_2w\\ &\quad+\left[\frac{-2(6\vartheta\rho+2\vartheta-2\rho+1)}{\tau}\Lambda_3a_3 + \frac{4(6\vartheta\rho+2\vartheta-2\rho+1)-(2\vartheta\rho+\vartheta-\rho+1)^2}{\tau}\Lambda_2^2a_2^2\right]w^2+\cdots.\end{aligned} \tag{3.5}$$

Furthermore, from (2.11)-(2.12) and (3.2)-(3.5), by the coefficient comparison method we know that

$$\frac{2\vartheta\rho+\vartheta-\rho+1}{\tau}\Lambda_2a_2 = \frac{1}{2}B_0E_1c_1, \tag{3.6}$$

$$\frac{2(6\vartheta\rho+2\vartheta-2\rho+1)}{\tau}\Lambda_3 a_3-\frac{(2\vartheta\rho+\vartheta-\rho+1)^2}{\tau}\Lambda_2^2 a_2^2=\frac{1}{2}B_1E_1c_1+\frac{1}{2}B_0E_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_0E_2c_1^2, \tag{3.7}$$

$$-\frac{2\vartheta\rho+\vartheta-\rho+1}{\tau}\Lambda_2 a_2=\frac{1}{2}B_0E_1d_1, \tag{3.8}$$

and

$$\begin{aligned}\frac{-2(6\vartheta\rho+2\vartheta-2\rho+1)}{\tau}\Lambda_3 a_3+\frac{4(6\vartheta\rho+2\vartheta-2\rho+1)-(2\vartheta\rho+\vartheta-\rho+1)^2}{\tau}\Lambda_2^2 a_2^2\\ =\frac{1}{2}B_1E_1d_1+\frac{1}{2}B_0E_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}B_0E_2d_1^2.\end{aligned} \tag{3.9}$$

Now proceeding on similar lines of Theorem 2.2 and using (3.6) and (3.8), we get

$$a_2=\frac{B_0E_1c_1\tau}{2(2\vartheta\rho+\vartheta-\rho+1)\Lambda_2}=-\frac{B_0E_1d_1\tau}{2(2\vartheta\rho+\vartheta-\rho+1)\Lambda_2}.$$

Then, it infers that

$$c_1=-d_1$$

and

$$8(2\vartheta\rho+\vartheta-\rho+1)^2\Lambda_2^2a_2^2=B_0^2E_1^2\tau^2(c_1^2+d_1^2). \tag{3.10}$$

By (3.7) and (3.9), we have that

$$\frac{2\Lambda_2^2a_2^2}{\tau}\Omega(\vartheta,\rho)=\frac{1}{4}B_0(c_1^2+d_1^2)(E_2-E_1)+\frac{1}{2}B_0E_1(c_2+d_2), \tag{3.11}$$

where

$$\Omega(\vartheta,\rho)=2(6\vartheta\rho+2\vartheta-2\rho+1)-(2\vartheta\rho+\vartheta-\rho+1)^2.$$

Therefore, by (3.10)-(3.11) we obtain that

$$a_2^2=\frac{\frac{1}{4}B_0^2E_1^3\tau^2(c_2+d_2)}{[B_0E_1^2\tau\Omega(\vartheta,\rho)+(E_1-E_2)(2\vartheta\rho+\vartheta-\rho+1)^2]\Lambda_2^2}. \tag{3.12}$$

Hence, from (3.10)-(3.12) and Lemma 2.1 we know that

$$|a_2|\leqslant\frac{|B_0\tau|E_1}{|(2\vartheta\rho+\vartheta-\rho+1)\Lambda_2|},$$

$$|a_2|\leqslant\sqrt{\frac{|B_0\tau|(|E_2-E_1|+E_1)}{|\Omega(\vartheta,\rho)\Lambda_2^2|}},$$

and

$$|a_2|\leqslant\frac{|B_0\tau|E_1\sqrt{E_1}}{\sqrt{|B_0E_1^2\tau\Omega(\vartheta,\rho)+(E_1-E_2)(2\vartheta\rho+\vartheta-\rho+1)^2||\Lambda_2|}}.$$

Correspondingly, by detracting (3.9) from (3.7), it infers that

$$\frac{1}{2}B_1E_1(c_1-d_1)+\frac{1}{2}B_0E_1(c_2-d_2)=\frac{4(6\vartheta\rho+2\vartheta-2\rho+1)(\Lambda_3a_3-\Lambda_2^2a_2^2)}{\tau}. \tag{3.13}$$

Hence, by (3.10) and (3.13), it trails that

$$a_3=\frac{E_1\tau[B_1(c_1-d_1)+B_0(c_2-d_2)]}{8(6\vartheta\rho+2\vartheta-2\rho+1)\Lambda_3}+\frac{B_0^2E_1^2\tau^2(c_1^2+d_1^2)}{8(2\vartheta\rho+\vartheta-\rho+1)^2\Lambda_3}.$$

So, we attain from Lemma 2.1 that

$$|a_3| \leqslant \frac{E_1|\tau|(|B_0|+|B_1|)}{2|6\vartheta\rho+2\vartheta-2\rho+1||\Lambda_3|} + \frac{|B_0\tau|^2E_1^2}{(2\vartheta\rho+\vartheta-\rho+1)^2|\Lambda_3|}.$$

On the other hand, by (3.11) and (3.13) we infer that

$$a_3 = \frac{E_1\tau[B_1(c_1-d_1)+B_0(c_2-d_2)]}{8(6\vartheta\rho+2\vartheta-2\rho+1)\Lambda_3} + \frac{\tau B_0[(E_2-E_1)(c_1^2+d_1^2)+2E_1(c_2+d_2)]}{8\Omega(\vartheta,\rho)\Lambda_3}.$$

Thus, from Lemma 2.1 we obtain that

$$|a_3| \leqslant \frac{E_1|\tau|(|B_0|+|B_1|)}{2|6\vartheta\rho+2\vartheta-2\rho+1||\Lambda_3|} + \frac{|\tau B_0|(|E_2-E_1|+E_1)}{|\Omega(\vartheta,\rho)\Lambda_3|}.$$

Then, we complete the proof of Theorem 3.1. □

If we choose $\vartheta=1$ and $\rho=0$, respectively, we let $\mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,1,\rho;\phi)=\mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\rho;\phi)$, $\mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\vartheta,0;\phi) = \mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\vartheta;\phi)$ and have two corollaries below.

Corollary 3.2. *If* $f(z)$ *is as assumed in* (1.1) *and* $f \in \mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\rho;\phi)$, *then*

$$|a_2| \leqslant \min\left\{\frac{|B_0\tau|E_1}{(\rho+2)|\Lambda_2|}, \sqrt{\frac{|B_0\tau|(|E_2-E_1|+E_1)}{|\Omega(\rho)\Lambda_2^2|}}, \frac{|B_0\tau|E_1\sqrt{E_1}}{\sqrt{|B_0E_1^2\tau\Omega(\rho)+(E_1-E_2)(\rho+2)^2||\Lambda_2|}}\right\}$$

and

$$|a_3| \leqslant \frac{E_1|\tau|(|B_0|+|B_1|)}{2(4\rho+3)|\Lambda_3|} + \min\left\{\frac{|B_0\tau|^2E_1^2}{(\rho+2)^2|\Lambda_3|}, \frac{|\tau B_0|(|E_2-E_1|+E_1)}{|\Omega(\rho)\Lambda_3|}\right\},$$

where

$$\Omega(\rho) = 6-(\rho-2)^2.$$

Corollary 3.3. *If* $f(z)$ *is as assumed in* (1.1) *and* $f \in \mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\vartheta;\phi)$, *then*

$$|a_2| \leqslant \min\left\{\frac{|B_0\tau|E_1}{(\vartheta+1)|\Lambda_2|}, \sqrt{\frac{|B_0\tau|(|E_2-E_1|+E_1)}{|\Omega(\vartheta)\Lambda_2^2|}}, \mathcal{F}\right\}$$

for

$$\mathcal{F} = \frac{|B_0\tau|E_1\sqrt{E_1}}{\sqrt{|B_0E_1^2\tau\Omega(\vartheta)+(E_1-E_2)(\vartheta+1)^2||\Lambda_2|}};$$

and

$$|a_3| \leqslant \frac{E_1|\tau|(|B_0|+|B_1|)}{2(2\vartheta+1)|\Lambda_3|} + \min\left\{\frac{|B_0\tau|^2E_1^2}{(\vartheta+1)^2|\Lambda_3|}, \frac{|\tau B_0|(|E_2-E_1|+E_1)}{|\Omega(\vartheta)\Lambda_3|}\right\},$$

where

$$\Omega(\vartheta) = 2-(\vartheta-1)^2.$$

Remark 3.4. If the quasi-subordination reduces to the subordination (i.e., fixing $\varphi(z)\equiv 1$), then our result in Theorem 3.1, partly belongs to Tang et al. [64]. Furthermore, suitably choosing the parameter ϑ, ρ, and ϕ, we afford the subsequent condensed forms for $\mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\vartheta,\rho;\phi)$ and analogues results as in Theorem 3.1:

1. $\mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau,\vartheta,0;\phi) = \mathcal{N}_{\Sigma,\tau}^{\vartheta}(\phi)$ $(\vartheta \geqslant 0, \tau \in \mathbb{C}^*)$, refer to Kumar et al. [32];

2. $\mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(1,0,0;\psi_\xi) = \mathcal{S}^*_\Sigma(\xi)$ (see Remark 2.5 for ψ_ξ), refer to Taha [62];

3. $\mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(1,1,0;\phi) = H_\Sigma(\phi)$, refer to Lashin [33], furthermore, $\mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(1,1,0;\chi_\zeta) = \mathcal{K}_\Sigma(\zeta)$ and $\mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(1,0,0;\chi_\zeta) = \mathcal{S}^*_\Sigma(\zeta)$ $(0 \leqslant \zeta < 1)$ (see Remark 2.5 for χ_ζ), refer to Brannan and Taha [12].

Next, we primarily cogitate Fekete-Szegö problems for $\mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(\tau,\vartheta,\rho;\phi)$.

Theorem 3.5. *Let* $f(z)$ *be fixed as in* (1.1) *and* $f \in \mathcal{M}\Sigma^{\gamma,k}_{\alpha,\beta}(\tau,\vartheta,\rho;\phi)$ *and* $\delta \in \mathbb{R}$. *Then*

$$|a_3 - \delta a_2^2| \leqslant \frac{|B_1\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|} + \frac{|B_0^2\tau^2|E_1^3|\Lambda_2^2 - \delta\Lambda_3|}{|B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_2|^2|\Lambda_3|}$$

if

$$2|B_0\tau|E_1^2|(6\vartheta\rho + 2\vartheta - 2\rho + 1)(\Lambda_2^2 - \delta\Lambda_3)| \geqslant |B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_2|^2,$$

or

$$|a_3 - \delta a_2^2| \leqslant \frac{(|B_0| + |B_1|)|\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|}$$

if

$$2|B_0\tau|E_1^2|(6\vartheta\rho + 2\vartheta - 2\rho + 1)(\Lambda_2^2 - \delta\Lambda_3)| \leqslant |B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_2|^2,$$

where $\Omega(\vartheta,\rho)$ *is given by* (3.1).

Proof. From (3.13), we have

$$a_3 - \frac{\Lambda_2^2}{\Lambda_3}a_2^2 = \frac{E_1\tau[B_1(c_1 - d_1) + B_0(c_2 - d_2)]}{8(6\vartheta\rho + 2\vartheta - 2\rho + 1)\Lambda_3}.$$

By (3.12) we obtain that

$$\begin{aligned} a_3 - \delta a_2^2 &= \frac{B_1E_1\tau(c_1 - d_1)}{8(6\vartheta\rho + 2\vartheta - 2\rho + 1)\Lambda_3} \\ &+ \frac{B_0E_1\tau\{2B_0E_1^2\tau(6\vartheta\rho + 2\vartheta - 2\rho + 1)(\Lambda_2^2 - \delta\Lambda_3) + [B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2]\Lambda_2^2\}c_2}{8(6\vartheta\rho + 2\vartheta - 2\rho + 1)[B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2]\Lambda_2^2\Lambda_3} \\ &+ \frac{B_0E_1\tau\{2B_0E_1^2\tau(6\vartheta\rho + 2\vartheta - 2\rho + 1)(\Lambda_2^2 - \delta\Lambda_3) - [B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2]\Lambda_2^2\}d_2}{8(6\vartheta\rho + 2\vartheta - 2\rho + 1)[B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2]\Lambda_2^2\Lambda_3}. \end{aligned}$$

Hence, from Lemma 2.1 we imply that

$$|a_3 - \delta a_2^2| \leqslant \frac{|B_1\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|} + \frac{|B_0^2\tau^2|E_1^3|\Lambda_2^2 - \delta\Lambda_3|}{|B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_2|^2|\Lambda_3|}$$

when

$$2|B_0\tau|E_1^2|(6\vartheta\rho + 2\vartheta - 2\rho + 1)(\Lambda_2^2 - \delta\Lambda_3)| \geqslant |B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_2|^2,$$

or

$$|a_3 - \delta a_2^2| \leqslant \frac{(|B_0| + |B_1|)|\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|}$$

when

$$2|B_0\tau|E_1^2|(6\vartheta\rho + 2\vartheta - 2\rho + 1)(\Lambda_2^2 - \delta\Lambda_3)| \leqslant |B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_2|^2.$$

□

By taking $\delta = 0$, we provide the following corollary.

Corollary 3.6. *Let* $f(z)$ *given by* (1.1) *belongs to the class* $\mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau, \vartheta, \rho; \phi)$. *Then*

$$|a_3| \leqslant \frac{|B_1\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|} + \frac{|B_0^2\tau^2|E_1^3}{|B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2||\Lambda_3|}$$

if

$$2|B_0\tau|E_1^2|6\vartheta\rho + 2\vartheta - 2\rho + 1| \geqslant |B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2|,$$

or

$$|a_3| \leqslant \frac{(|B_0| + |B_1|)|\tau|E_1}{2|6\vartheta\rho + 2\vartheta - 2\rho + 1||\Lambda_3|}$$

if

$$2|B_0\tau|E_1^2|6\vartheta\rho + 2\vartheta - 2\rho + 1| \leqslant |B_0E_1^2\tau\Omega(\vartheta,\rho) + (E_1 - E_2)(2\vartheta\rho + \vartheta - \rho + 1)^2|,$$

where $\Omega(\vartheta, \rho)$ *is assumed by* (3.1).

4. Conclusions

By fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \cdot \operatorname{erfc}(-z) = e^{z^2}\left(1 + \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n+1)}z^{2n+1}\right).$$

We can define and explore new subclasses of the function class Σ of analytic and bi-univalent functions concomitant with the quasi-subordination and convolution operator concerning error function, and study the analogous estimates of the coefficients a_2 and a_3 as given for $f \in \mathcal{M}\Sigma_{\alpha,\beta}^{\gamma,k}(\tau, \vartheta, \rho; \phi)$ or $f \in \mathcal{Q}\Sigma_{\alpha,\beta}^{\gamma,k}(\vartheta, \rho; \phi)$. Instantaneously, one can obtain the corresponding Fekete-Szegö functional inequalities. Besides, the consequences and connections to some earlier known results would be expounded which are new and has not been studied so far, we left this as an exercise to interested readers.

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