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Some extensions for generalized (ϕ, ψ)-almost contractions

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Abstract

In this paper, we derive a new fixed point results in partially ordered b-metric-like spaces. Our results generalize and extend several well-known comparable results in the literature. Further, two examples are also given to show that our results are influential.

Keywords: b-metric-like, common fixed point, fixed point, ordered set.
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1. Introduction

In 1922, the Polish mathematician Stefan Banach established an attention grabbing fixed point theorem known as the "Banach Contraction Principle" (BCP) [3] which is one of the pivotal results of analysis and considered as the pivotal source of metric fixed point theory. Generalization of this BCP have been studied excessively (see [5, 6, 8] and [10]). Nominately, Jaggi [5] proved a theorem satisfying a contractive condition of rational type on a complete metric space. In 2010, Harjani et al. [6] showed the ordered version of this theorem proved by Jaggi. Luong and Thuan [8], in 2011, generalized the results of Harjani et al. [6]. Recently, Mustafa et al. [10] proved the following theorem involving a generalized (ϕ, ψ) $-$almost contraction.

Theorem 1.1 ([10]). *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric d such that (X, d) is a complete metric space. Let $f : X \to X$ be a nondecreasing mapping which satisfies the inequality*

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$$\phi(d(fx,fy)) \leq \phi(M(x,y)) - \psi(M(x,y)) + L\phi(\min\{d(x,fx),\ d(y,fy),\ d(x,fy),\ d(y,fx)\})$$

for all distinct points $x,\ y \in X$ *with* $y \leq x$ *where* $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ *and*

$$M(x,y) = \max\left\{\frac{d(x,fx)\,d(y,fy)}{d(x,y)},\ d(x,y)\right\}.$$

Also, assume either

(i) f *is continuous or;*

(ii) *if* $\{x_n\}$ *is a nondecreasing sequence in* X *such that* $x_n \to \omega$, *then* $\omega = \sup\{x_n\}$.

If there exists $x_0 \in X$ *such that* $x_0 \leq fx_0$, *then* f *has a fixed point.*

Inspired and motivated by this facts, we are to generalize the above results for a mapping $f : X \to X$ satisfying a generalized $(\phi,\psi)-$almost contraction in partially ordered b-metric-like spaces. Two examples are also given to show that our results are influential.

2. Preliminaries

Definition 2.1 ([1])**.** Let X be a nonempty set and $\kappa \geq 1$ a given real number. A function $A : X \times X \to \mathbb{R}^+$ is b-metric-like if, for all $x,\ y,\ z \in X$, the following conditions are satisfied:

(A1) if $A(x,y) = 0 \Rightarrow x = y$;

(A2) $A(x,y) = A(y,x)$;

(A3) $A(x,y) \leq \kappa[A(x,z) + A(y,z)]$.

A b-metric-like space is a pair (X,A) such that X is nonempty set and A is b-metric-like on X. The number κ is called the coefficient of (X,A).

Proposition 2.2 ([1])**.** *Let* (X,A) *be a b-metric-like space. Define* $A^p : X \times X \to [0,\infty)$ *by* $A^p(x,y) = |2A(x,y) - A(x,x) - A(y,y)|$. *Frankly,* $A^p(x,x) = 0$ *for all* $x \in X$.

Each b-metric-like A on X generates a topology τ_A on X whose base is the family of all open $A-$balls $\{D_A(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$, where

$$D_A(x,\varepsilon) = \{a \in X : |A(x,a) - A(x,x)| < \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 2.3 ([1])**.** Let (X,A) be a b-metric-like space with coefficient κ, and let $\{x_n\}$ be any sequence in X and $x \in X$. Then,

(a) a sequence $\{x_n\}$ is convergent to x with respect to τ_A, if $\lim_{n\to\infty} A(x_n,x) = A(x,x)$;

(b) a sequence $\{x_n\}$ is a Cauchy sequence in (X,A) if $\lim_{n,m\to\infty} A(x_n,x_m)$ exists and is finite;

(c) (X,A) is a complete b-metric-like space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that $\lim_{n,m\to\infty} A(x_n,x_m) = \lim_{n\to\infty} A(x_n,x) = A(x,x)$.

It is obvious that the limit of a sequence in b-metric-like space is usually not unique (see [7]).

Lemma 2.4 ([1]). *Let (X, A) be a b-metric-like space with coefficient κ, and let $\{x_n\}$ be sequence in X such that*

$$A(x_n, x_{n+1}) \leq \lambda A(x_{n-1}, x_n)$$

for some λ, $0 < \lambda < \frac{1}{\kappa}$, and each $n \in \mathbb{N}$. Then $\lim_{m,n\to\infty} A(x_m, x_n) = 0$.

Let Φ be a family of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ϕ is continuous and nondecreasing;

(ii) $\phi(t) = 0$ if and only if $t = 0$;

(iii) $\phi(0) = 0 < \phi(t)$ for all $t > 0$.

We denote by Ψ the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying

(a) ψ is lower semi continuous;

(b) $\psi(t) > 0$ for all $t > 0$ and $\psi(0) = 0$.

3. Main Results

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a function A such that (X, A) is a complete b-metric-like space with the constant $\kappa \geq 1$. Let $f : X \to X$ be a nondecreasing mapping which satisfies the inequality*

$$\phi\left(\kappa^2 A(fx, fy)\right) \leq \phi(M(x, y)) - \psi(M(x, y)) + L\phi(N^p(x, y)) \tag{3.1}$$

for all distinct points x, $y \in X$ with $y \leq x$, where $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ and

$$M(x, y) = \max\left\{\frac{A(x, fx)\, A(y, fy)}{A(x, y)},\ A(x, y)\right\},$$
$$N^p(x, y) = \min\left\{A^p(x, fx),\ A^p(y, fy),\ A^p(x, fy),\ A^p(y, fx)\right\}.$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to \omega$, then $\omega = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Proof. Let $x_0 \in X$ such that $x_0 \leq fx_0$. Define $x_n = fx_{n-1}$ for all $n \geq 1$. Using that f is a nondecreasing, we can construct inductively, starting with arbitrary $x_0 \in X$, a sequence $\{x_n\}$ such that $x_n \leq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1}$ for some n_0, then it is clear that x_{n_0} is a fixed point of f. Suppose that $x_n \neq x_{n+1}$ for all n. Therefore, by $x_n \leq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, we have $x_n > x_{n-1}$ for all $n \geq 1$.

Owing to $x_n > x_{n-1}$ for all $n \geq 1$, from (3.1), we have

$$\begin{aligned}
\phi\left(\kappa^2 A\left(x_n, x_{n+1}\right)\right) =&\phi\left(\kappa^2 A\left(f x_{n-1}, f x_n\right)\right) \\
\leq&\phi\left(\max\left\{\frac{A\left(x_{n-1}, f x_{n-1}\right) A\left(x_n, f x_n\right)}{A\left(x_{n-1}, x_n\right)},\ A\left(x_{n-1}, x_n\right)\right\}\right) \\
&-\psi\left(\max\left\{\frac{A\left(x_{n-1}, f x_{n-1}\right) A\left(x_n, f x_n\right)}{A\left(x_{n-1}, x_n\right)},\ A\left(x_{n-1}, x_n\right)\right\}\right) \\
&+L\phi\left(\min\left\{A^p\left(x_{n-1}, f x_{n-1}\right),\ A^p\left(x_n, f x_n\right),\right.\right. \\
&\left.\left.A^p\left(x_{n-1}, f x_n\right),\ A^p\left(x_n, f x_{n-1}\right)\right\}\right) \\
=&\phi\left(\max\left\{A\left(x_n, x_{n+1}\right),\ A\left(x_{n-1}, x_n\right)\right\}\right) \\
&-\psi\left(\max\left\{A\left(x_n, x_{n+1}\right),\ A\left(x_{n-1}, x_n\right)\right\}\right) \\
&+L\phi\left(\min\left\{A^p\left(x_{n-1}, f x_{n-1}\right),\ A^p\left(x_n, f x_n\right),\right.\right. \\
&\left.\left.A^p\left(x_{n-1}, f x_n\right),\ A^p\left(x_n, f x_{n-1}\right)\right\}\right) \\
=&\phi\left(\max\left\{A\left(x_n, x_{n+1}\right),\ A\left(x_{n-1}, x_n\right)\right\}\right) \\
&-\psi\left(\max\left\{A\left(x_n, x_{n+1}\right),\ A\left(x_{n-1}, x_n\right)\right\}\right).
\end{aligned} \tag{3.2}$$

If $A\left(x_{n-1}, x_n\right) < A\left(x_n, x_{n+1}\right)$ for some $n \geq 1$, then from (3.2) we get that

$$\phi\left(\kappa^2 A\left(x_n, x_{n+1}\right)\right) \leq \phi\left(A\left(x_n, x_{n+1}\right)\right) - \psi\left(A\left(x_n, x_{n+1}\right)\right), \tag{3.3}$$

or equivalently,

$$\kappa^2 A\left(x_n, x_{n+1}\right) < A\left(x_n, x_{n+1}\right).$$

This is a contradiction. Thus from (3.2) it follows that

$$\phi\left(\kappa^2 A\left(x_n, x_{n+1}\right)\right) \leq \phi\left(A\left(x_{n-1}, x_n\right)\right) - \psi\left(A\left(x_{n-1}, x_n\right)\right) < \phi\left(A\left(x_{n-1}, x_n\right)\right) \tag{3.4}$$

or

$$\begin{aligned}
\kappa^2 A\left(x_n, x_{n+1}\right) &< A\left(x_{n-1}, x_n\right), \\
A\left(x_n, x_{n+1}\right) &< \lambda A\left(x_{n-1}, x_n\right), \quad \text{where } \lambda = \frac{1}{\kappa^2} < \frac{1}{\kappa}.
\end{aligned}$$

Then by Lemma 2.4 we have $\lim_{m,n\to\infty} A\left(x_m, x_n\right) = 0$. Due to Definition 2.3 part (b), $\{x_n\}$ is a Cauchy sequence. Since (X, A) is a complete b-metric-like space, $\{x_n\}$ in X converges to $\omega \in X$ so that

$$\lim_{n,m\to\infty} A\left(x_n, x_m\right) = \lim_{n\to\infty} A\left(x_n, \omega\right) = A(\omega, \omega) = 0. \tag{3.5}$$

Now, suppose that the assumption (i) holds. The continuity of f implies

$$\omega = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f x_{n-1} = f\left(\lim_{n\to\infty} x_{n-1}\right) = f\omega$$

and this proved that ω is a fixed point of f. Eventually, suppose that the assumption (ii) holds. Since $\{x_n\}$ is a nondecreasing sequence and $x_n \to \omega$, then $\omega = \sup\{x_n\}$. Thus, $x_n \leq \omega$ for all n. Since f is nondecreasing, $f x_n \leq f\omega$ for all n, that is to say, $x_{n+1} \leq f\omega$ for all n. Further, as $x_n \leq x_{n+1} \leq f\omega$ for all n and $\omega = \sup\{x_n\}$, we get $\omega \leq f\omega$. For this purpose, we establish the sequence $\{y_n\}$ as follows:

$$y_0 = \omega, \quad y_n = f y_{n-1}, \quad n \geq 1.$$

Due to $\omega \leq f\ \omega$, we obtain that $y_0 \leq fy_0 = y_1$. Therefore, $\{y_n\}$ is a nondecreasing sequence and $\lim_{n\to\infty} y_n = u$ for certain $u \in X$. By the assumption (ii), we get $u = \sup\{y_n\}$.

Due to $x_n < \omega = y_0 \leq f\omega = fy_0 \leq y_n \leq u$ for all n, suppose that $\omega \neq u$, from (3.1), we have

$$\begin{aligned}\phi\left(\kappa^2 A(x_{n+1}, y_{n+1})\right) =& \phi\left(\kappa^2 A(fx_n, fy_n)\right)\\ \leq& \phi\left(\max\left\{\frac{A(x_n, fx_n)\, A(y_n, fy_n)}{A(x_n, y_n)},\ A(x_n, y_n)\right\}\right)\\ &- \psi\left(\max\left\{\frac{A(x_n, fx_n)\, A(y_n, fy_n)}{A(x_n, y_n)},\ A(x_n, y_n)\right\}\right)\\ &+ L\phi\left(\min\left\{A^p(x_n, fy_n),\ A^p(y_n, fx_n),\right.\right.\\ &\left.\left.A^p(x_n, fx_n),\ A^p(y_n, fy_n)\right\}\right)\\ =& \phi\left(\max\left\{\frac{A(x_n, x_{n+1})\, A(y_n, y_{n+1})}{A(x_n, y_n)},\ A(x_n, y_n)\right\}\right)\\ &- \psi\left(\max\left\{\frac{A(x_n, x_{n+1})\, A(y_n, y_{n+1})}{A(x_n, y_n)},\ A(x_n, y_n)\right\}\right)\\ &+ L\phi\left(\min\left\{A^p(x_n, y_{n+1}),\ A^p(y_n, x_{n+1}),\right.\right.\\ &\left.\left.A^p(x_n, x_{n+1}),\ A^p(y_n, y_{n+1})\right\}\right).\end{aligned}$$

Taking the upper limit as $n \to \infty$ in (3.5), we have

$$\phi\left(\kappa^2 A(\omega, u)\right) \leq \phi(\max\{0, A(\omega, u)\}) - \psi(\max\{0, A(\omega, u)\}) + L\phi(0) < \phi(A(\omega, u)),$$

which is a contradiction. Hence, $\omega = u$. We have $u \leq fu \leq u$, consequently, $fu = u$. For this reason, x is a fixed point of f. □

If we take $L = 0$ in Theorem 3.1, we have the following result.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a function A such that (X, A) is a complete b-metric-like space with the constant $\kappa \geq 1$. Let $f : X \to X$ be a nondecreasing mapping which satisfies the inequality*

$$\phi\left(\kappa^2 A(fx, fy)\right) \leq \phi(M(x, y)) - \psi(M(x, y)) \tag{3.6}$$

for all distinct points x, $y \in X$ with $y \leq x$ where $\phi \in \Phi$, $\psi \in \Psi$ and

$$M(x, y) = \max\left\{\frac{A(x, fx)\, A(y, fy)}{A(x, y)},\ A(x, y)\right\}.$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to \omega$, then $\omega = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Proof. We omit the proof due to the analogy to Theorem 3.1. □

If we take $\phi(t) = t$ and $\psi(t) = (1 - k)t$ in Theorem 3.1, we have the following result.

Theorem 3.3. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a function A such that (X, A) is a complete b-metric-like space with the constant $\kappa \geq 1$. Let $f : X \to X$ be a nondecreasing mapping which satisfies the inequality*

$$\kappa^2 A(fx, fy) \leq kM(x, y) + LN^p(x, y) \tag{3.7}$$

for all distinct points x, $y \in X$ with $y \leq x$ where $k \in (0, 1)$, $L \geq 0$ and

$$M(x, y) = \max\left\{\frac{A(x, fx) A(y, fy)}{A(x, y)},\ A(x, y)\right\},$$
$$N^p(x, y) = \min\left\{A^p(x, fx),\ A^p(y, fy),\ A^p(x, fy),\ A^p(y, fx)\right\}.$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to \omega$, then $\omega = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Proof. Along the lines of the proof of Theorem 3.1, we obtain the desired results. Due to the analogy, we skip the details of the proof. □

Theorem 3.4. *In addition to the hypotheses of* Theorem 3.1*, assume that*

for every $\omega, u \in X$ there exists $v \in X$ that is comparable to ω and u,

then f has a unique fixed point.

Proof. Suppose to the contrary that ω and u are fixed points of f where $\omega \neq u$. From (3.4), there exists $v \in X$ which is comparable with ω and u. Define the sequence $v_{n+1} = fv_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since v is comparable with ω, we obtain $v \leq \omega$. By induction, we have $v_n \leq \omega$.

If $v_{n_0} = \omega$ for some $n_0 \geq 1$, then $v_n = fv_{n-1} = f\omega = \omega$ for all $n \geq n_0 - 1$, that is, $v_n \to \omega$ as $n \to \infty$.

On the other hand, if $v_n \neq \omega$ for all n, from (3.1), we observe that

$$\begin{aligned}\phi\left(\kappa^2 A(\omega, v_n)\right) =& \phi\left(\kappa^2 A(f\omega, fv_{n-1})\right)\\ \leq& \phi(M(\omega, v_{n-1})) - \psi(M(\omega, v_{n-1}))\\ &+ L\phi(\min\{A^p(\omega, f\omega),\ A^p(v_{n-1}, fv_{n-1}),\\ &A^p(\omega, fv_{n-1}),\ A^p(v_{n-1}, f\omega)\})\\ =& \phi(M(\omega, v_{n-1})) - \psi(M(\omega, v_{n-1}))\end{aligned}$$

for all distinct points $\omega, u \in X$ with $u \leq \omega$ where $\phi \in \Phi$, $\psi \in \Psi$ and

$$\begin{aligned}M(\omega, v_{n-1}) =& \max\left\{\frac{A(\omega, f\omega) A(v_{n-1}, fv_{n-1})}{A(\omega, v_{n-1})},\ A(\omega, v_{n-1})\right\}\\ =& \max\left\{\frac{A(\omega, \omega) A(v_{n-1}, v_n)}{A(\omega, v_{n-1})},\ A(\omega, v_{n-1})\right\}.\end{aligned}$$

We assume that $A(\omega, \omega) = 0$ in the above inequality. Then, we have

$$M(\omega, v_{n-1}) = A(\omega, v_{n-1}). \tag{3.8}$$

Thus,

$$\phi\left(\kappa^2 A(\omega, v_n)\right) \leq \phi(A(\omega, v_{n-1})) - \psi(A(\omega, v_{n-1})),$$

which is a contradiction. This completes the proof. □

4. Consequences of the Main Results

We know that b-metric-like spaces are a proper extension of partial metric space, metric-like and b-metric spaces. Therefore, we can deduce the following corollaries in the settings of metric-like, partial metric and b-metric spaces, respectively.

4.1. Fixed Point Results in Metric-Like Spaces.

The notion of metric-like spaces which is an interesting generalization of partial metric space was introduced by Amini-Harandi (see [[2]-Definition 2.1]).

Corollary 4.1. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a function ξ such that (X, ξ) is a complete metric-like space. Let $f : X \to X$ be a nondecreasing mapping which satisfies the inequality*

$$\phi(\xi(fx, fy)) \leq \phi(M^*(x, y)) - \psi(M^*(x, y)) + L\phi(N^*(x, y))$$

for all distinct points x, $y \in X$ with $y \leq x$ where $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ and

$$M^*(x, y) = \max\left\{\frac{\xi(x, fx)\,\xi(y, fy)}{\xi(x, y)},\ \xi(x, y)\right\},$$
$$N^*(x, y) = \min\{\xi(x, fx),\ \xi(y, fy),\ \xi(x, fy),\ \xi(y, fx)\}.$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to \omega$, then $\omega = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Corollary 4.2. *In addition to the hypotheses of* Corollary 4.1*, assume that*

for every $\omega, u \in X$ there exists $v \in X$ that is comparable to ω and u,

then f has a unique fixed point.

4.2. Fixed Point Results in Partial Metric Spaces.

Matthews [9] established the notation of a partial metric space (see [6, Definition 3.1]).

Corollary 4.3. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a metric D such that (X, D) is a complete partial metric space and let $f : X \to X$ be a nondecreasing mapping which satisfies the inequality*

$$\phi(D(fx, fy)) \leq \phi(M^{**}(x, y)) - \psi(M^{**}(x, y)) + L\phi(N^{**}(x, y))$$

for all distinct points x, $y \in X$ with $y \leq x$ where $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ and

$$M^{**}(x, y) = \max\left\{\frac{D(x, fx)\,D(y, fy)}{D(x, y)},\ D(x, y)\right\},$$
$$N^{**}(x, y) = \min\{D(x, fx),\ D(y, fy),\ D(x, fy),\ D(y, fx)\}.$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to \omega$, then $\omega = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Corollary 4.4. *In addition to the hypotheses of* Corollary 4.3*, assume that*

$$\textit{for every } \omega, u \in X \textit{ there exists } v \in X \textit{ that is comparable to } \omega \textit{ and } u,$$

then f has a unique fixed point.

4.3. Fixed Point Results in $b-$Metric Spaces.

The concept of b-metric space was introduced by Czerwik (for more details and definition, see [4]).

Corollary 4.5. *Let $(X, \leq)$ be a partially ordered set. Suppose there exists a function b such that (X, b) is a complete b-metric space with the constant $s \geq 1$ and let $f : X \to X$ be a nondecreasing mapping which satisfies the inequality*

$$\phi(\kappa b(fx, fy)) \leq \phi(m(x, y)) - \psi(m(x, y)) + L\phi(n(x, y))$$

for all distinct points x, $y \in X$ with $y \leq x$ where $\phi \in \Phi$, $\psi \in \Psi$, $L \geq 0$ and

$$m(x, y) = \max\left\{\frac{b(x, fx)\, b(y, fy)}{b(x, y)},\ b(x, y)\right\},$$
$$n(x, y) = \min\{b(x, fx),\ b(y, fy),\ b(x, fy),\ b(y, fx)\}.$$

Also, assume either

(i) *f is continuous or;*

(ii) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to \omega$, then $\omega = \sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \leq fx_0$, then f has a fixed point.

Corollary 4.6. *In addition to the hypotheses of* Corollary 4.5*, assume that*

$$\textit{for every } \omega, u \in X \textit{ there exists } v \in X \textit{ that is comparable to } \omega \textit{ and } u,$$

then f has a unique fixed point.

Remark 4.7.

1. Theorem 3.2 is a generalization of Theorem 4 in [10].
2. If we take $L = 0$, $\phi(t) = t$, $\psi(t) = (1-k)t$ and $k = \alpha + \beta$ where $\alpha, \beta \in [0, 1)$ and $t \in [o, \infty)$ in Theorem 3.1, this a generalization of Corollary 7 in [10].
3. If we take $L = 0$ and $\phi(t) = t$ in Theorem 3.1, this a generalization of Theorem 2.1. in [8].
4. If we take $L = 0$, $\phi(t) = t$, $\psi(t) = (1-k)t$ and $k = \alpha + \beta$ where $\alpha, \beta \in [0, 1)$ and $t \in [o, \infty)$ in Corollary 4.3, this corresponds to Theorem 2.2 and Theorem 2.3 in [6].

5. Examples

We present the following two examples to support our results.

Example 5.1. Let $X = [0, \infty)$. Define $A : X \times X \to \mathbb{R}^+$ by $A(x, y) = x^2 + y^2 + |x - y|^2$ for all $x, y \in X$. Define an ordering "$\preceq$" on X as follows:

$$x \preceq y \Leftrightarrow x \leq y, \quad \forall x, y \in X.$$

$(X, \preccurlyeq)$ is a partially ordered set and (X, A) is a complete b-metric-like space with coefficient $\kappa = 2$ (see [7], Example 14).

Define self-map f on X by $fx = \ln\left(\sqrt{\left(\frac{x}{3}\right)^2 + 1} + \frac{x}{3}\right) = \sinh^{-1}\frac{x}{3}$. By (3.8), we have

$$\begin{aligned}
2^2 A(fx, fy) =& 4\left(f^2x + f^2y + |fx - fy|^2\right) \\
=& 4\left(\left(\sinh^{-1}\frac{x}{3}\right)^2 + \left(\sinh^{-1}\frac{y}{3}\right)^2\right. \\
&\left. + \left|\sinh^{-1}\frac{x}{3} - \sinh^{-1}\frac{y}{3}\right|^2\right) \\
\leq& 4\left(\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 + \left|\frac{x}{3} - \frac{y}{3}\right|^2\right) \\
\leq& \frac{4}{9}\left(x^2 + y^2 + |x - y|^2\right) \\
=& \frac{4}{9}A(x, y) \\
\leq& \frac{4}{9}\max\left\{\frac{A(x, fx)\, A(y, fy)}{A(x, y)},\ A(x, y)\right\} \\
&+ L\left\{\min\left\{A^p(x, fx),\ A^p(y, fy),\ A^p(x, fy),\ A^p(y, fx)\right\}\right\} \\
=& kM(x, y) + LN^p(x, y),
\end{aligned}$$

which implies that $\kappa^2 A(fx, fy) \leq kM(x, y) + LN^p(x, y)$ where $k = \frac{4}{9} \in (0, 1)$.

Now, all the conditions of Theorem 3.3 hold and f has a unique fixed point $0 \in X = [0, \infty)$.

Example 5.2. Let $X = \{0, 1, 2\}$ and $A : X \times X \to \mathbb{R}^+$ be defined by

$$\begin{aligned}
A(x, x) =& 0 \text{ for } x \in X, \\
A(0, 1) =& A(1, 2) = 1, \\
A(0, 2) =& \frac{9}{4}, \\
A(x, y) =& A(y, x) \text{ for } x, y \in X.
\end{aligned}$$

Then, (X, A) is a b-metric-like space (with $\kappa = \frac{9}{8}$). Define an order on X by

$$\preccurlyeq := \{(0, 0),\ (1, 1),\ (2, 2),\ (2, 0)\}$$

and obtain a complete ordered b-metric-like space. Consider the mapping

$$f = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Define the mappings ϕ, $\psi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t$ and $\psi(t) = \frac{t}{2}$.

We know that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to \omega$, then $\omega = \sup\{x_n\}$. Virtually, let $\{v_n\}$ be a nondecreasing sequence in X in terms of $\preccurlyeq$ such that $v_n \to u \in X$ as $n \to \infty$. We get $v_n \preccurlyeq v_{n+1}$ for all $n \in \mathbb{N}$.

(i) If $v_0 = 0$, then $v_0 = 0 \preccurlyeq v_1$. From the definition of $\preccurlyeq$, we have $v_1 = 0$. By induction, we have $v_n = 0$ for all $n \in \mathbb{N}$ and $u = 0$. Then $v_n \preccurlyeq u$ for all $n \in \mathbb{N}$ and $u = \sup\{v_n\}$.

(ii) If $v_0 = 1$, then $v_0 = 1 \preccurlyeq v_1$. From the definition of $\preccurlyeq$, we have $v_1 = 1$. By induction, we have $v_n = 1$ for all $n \in \mathbb{N}$ and $u = 1$. Then $v_n \preccurlyeq u$ for all $n \in \mathbb{N}$ and $u = \sup\{v_n\}$.

(iii) If $v_0 = 2$, then $v_0 = 2 \preccurlyeq v_1$. From the definition of $\preccurlyeq$, we have $v_1 \in \{2,0\}$. By induction, we have $v_n \in \{2,0\}$ for all $n \in \mathbb{N}$. Suppose that there exists $q \geq 1$ such that $v_q = 0$. From the definition of $\preccurlyeq$, we have $v_n = v_q = 0$ for all $n \geq q$. Therefore, we get $u = 0$. and $v_n \preccurlyeq u$ for all $n \in \mathbb{N}$. Now, suppose that $v_n = 2$ for all $n \in \mathbb{N}$. In the circumstances, we have $u = 2$ and $v_n \preccurlyeq u$ for all $n \in \mathbb{N}$ and $u = \sup\{v_n\}$.

Now, we proved that in all situations, we have $u = \sup\{v_n\}$.

Let $x, y \in X$ such that $x \preccurlyeq y$ and $x \neq y$, then, we get only $x = 2$ and $y = 0$. Especially,

$$A(f2, f0) = A(0,0) = 0$$

and

$$\begin{aligned} M(2,0) =& \max\left\{\frac{A(2,f2)\,A(0,f0)}{A(2,0)},\ A(2,0)\right\} \\ =& \max\left\{\frac{A(2,0)\,A(0,0)}{A(2,0)},\ A(2,0)\right\} \\ =& A(2,0) = \frac{9}{4}. \end{aligned}$$

Thus (3.7) holds. After all, it is clear that f is a nondecreasing mapping in terms of $\preccurlyeq$ and there exists $x_0 = 2$ such that $x_0 \preccurlyeq fx_0$. All the conditions of Theorem 3.2 are confirmed in terms of $\preccurlyeq$ and $u = 0$ is a fixed point of f.

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