

Development and analysis of a proposed scheme to solve initial value problems



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Abstract

This paper presents the development and analysis of a proposed scheme to solve Initial Value Problems (IVPs). The proposed scheme is devised by means of the interpolating function. The properties of the proposed scheme such as the local truncation error, order of accuracy, stability, consistency, and convergence are analyzed. Furthermore, the performance of the proposed scheme is tested on five numerical examples. Moreover, the comparative study of the results generated via the proposed scheme and the exact solution is presented. Hence, the proposed scheme has fifth order convergence and is a good tool for approximating the solution of IVPs.

Keywords: Consistency, convergence, error, initial value problem, stability.

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1. Introduction

It is a known fact that a huge number of differential equations that model real life problem can not be solved analytically, one way is to use numerical approaches to obtain an approximate solution of the differential equations. In the recent years, several numerical methods have been developed to solve IVPs in Ordinary Differential Equations (ODEs) of the form

$$\frac{dy}{dx} = f(x, y), \quad y(a) = y_0, \quad x \in [a, b], y \in (-\infty, \infty) \quad (1.1)$$

such as the explicit methods, implicit methods and so on. In [4], the authors developed a new numerical method for the solution of IVPs in ODEs. Fadugba and Qureshi [9] discussed extensively on the properties

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of a one-step numerical method of order two for the solution of continuous dynamical systems. Fadugba and Idowu [5] applied a new numerical method of third order accuracy for the solution of IVPs in ODEs. They also analysed the properties of the method. Islam [10] applied fourth-order Runge-Kutta method to solve some IVPs in ODEs. For more details on the numerical solutions of IVPs in ODEs, see [1–3, 6–8, 11, 13, 15], just to mention a few. In this paper, a proposed scheme is developed and analyzed for the solution of IVPs in ODEs. The error analysis of the scheme in terms of maximum absolute relative error, absolute relative error and ℓ^2 -error norm is investigated. The rest of the paper is outlined as follows. In Section 2, the derivation of the proposed scheme is presented. The properties of the proposed scheme were analyzed in Section 3. Section 4 presents numerical examples and discussion of results. Section 5 concludes the paper.

2. Derivation of the Proposed Scheme

Consider an interpolating function of the form

$$F(x) = \sum_{j=0}^5 \beta_j x^j + \beta_6 e^c, \quad (2.1)$$

where $\beta_0, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ are undetermined constants and c is a constant. The integration interval of $[a, b]$ is defined as $a = x_0 \leq x \leq x_n = b$. The step length is defined as

$$h = \frac{b - a}{N}. \quad (2.2)$$

The mesh point is defined as

$$x_n = nh, \quad n = 1(1)N \quad (2.3)$$

or

$$x_{n+1} = (n + 1)h, \quad n = 0(1)N - 1 \quad (2.4)$$

with $x_0 = 0$. Expanding (2.1) at the points x_n and x_{n+1} yields

$$F(x_n) = \sum_{j=0}^5 \beta_j x_n^j + \beta_6 e^c \quad (2.5)$$

and

$$F(x_{n+1}) = \sum_{j=0}^5 \beta_j x_{n+1}^j + \beta_6 e^c, \quad (2.6)$$

respectively. Differentiating (2.5) five times yields

$$\begin{aligned} f_n &= \beta_1 + 2x_n \beta_2 + 3x_n^2 \beta_3 + 4x_n^3 \beta_4 + 5x_n^4 \beta_5, \\ f_n^{(1)} &= 2\beta_2 + 6x_n \beta_3 + 12x_n^2 \beta_4 + 20x_n^3 \beta_5, \\ f_n^{(2)} &= 6\beta_3 + 24x_n \beta_4 + 60x_n^2 \beta_5, \\ f_n^{(3)} &= 24\beta_4 + 120x_n \beta_5, \\ f_n^{(4)} &= 120\beta_5. \end{aligned} \quad (2.7)$$

Equation (2.7) can be written in the form $AX = b$, where

$$A = \begin{bmatrix} 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 6 & 24x_n & 60x_n^2 \\ 0 & 0 & 0 & 24 & 120x_n \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix}, \quad X = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix}, \quad b = \begin{bmatrix} f_n \\ f_n^{(1)} \\ f_n^{(2)} \\ f_n^{(3)} \\ f_n^{(4)} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 6 & 24x_n & 60x_n^2 \\ 0 & 0 & 0 & 24 & 120x_n \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} = \begin{bmatrix} f_n \\ f_n^{(1)} \\ f_n^{(2)} \\ f_n^{(3)} \\ f_n^{(4)} \end{bmatrix}. \quad (2.8)$$

Solving (2.8) by means of the Gauss Jordan method yields

$$\begin{aligned} \beta_1 &= \frac{1}{24} \left(24f_n - 24x_n f_n^{(1)} + 12x_n^2 f_n^{(2)} - 4x_n^3 f_n^{(3)} + x_n^4 f_n^{(4)} \right), \\ \beta_2 &= \frac{1}{12} \left(6f_n^{(1)} - 6x_n f_n^{(2)} + 3x_n^2 f_n^{(3)} - x_n^3 f_n^{(4)} \right), \\ \beta_3 &= \frac{1}{12} \left(2f_n^{(2)} - 2x_n f_n^{(3)} + x_n^2 f_n^{(4)} \right), \\ \beta_4 &= \frac{1}{24} \left(f_n^{(3)} - x_n f_n^{(4)} \right), \\ \beta_5 &= \frac{1}{120} f_n^{(4)}. \end{aligned} \quad (2.9)$$

Subtracting (2.5) from (2.6), one obtains

$$F(x_{n+1}) - F(x_n) = \sum_{j=1}^5 \beta_j (x_{n+1}^j - x_n^j). \quad (2.10)$$

Therefore, (2.10) yields

$$\begin{aligned} F(x_{n+1}) - F(x_n) &= \beta_1(x_{n+1} - x_n) + \beta_2(x_{n+1}^2 - x_n^2) + \beta_3(x_{n+1}^3 - x_n^3) \\ &\quad + \beta_4(x_{n+1}^4 - x_n^4) + \beta_5(x_{n+1}^5 - x_n^5). \end{aligned} \quad (2.11)$$

Using (2.3) and (2.4), one obtains the following

$$\begin{aligned} x_{n+1} - x_n &= h, \\ x_{n+1}^2 - x_n^2 &= (2n+1)h^2, \\ x_{n+1}^3 - x_n^3 &= (3n^2 + 3n + 1)h^3, \\ x_{n+1}^4 - x_n^4 &= (4n^3 + 6n^2 + 4n + 1)h^4, \\ x_{n+1}^5 - x_n^5 &= (5n^4 + 10n^3 + 10n^2 + 5n + 1)h^5. \end{aligned} \quad (2.12)$$

Substituting (2.3) into (2.9), one gets

$$\begin{aligned} \beta_1 &= \frac{1}{24} \left(24f_n - 24nhf_n^{(1)} + 12n^2h^2f_n^{(2)} - 4n^3h^3f_n^{(3)} + n^4h^4f_n^{(4)} \right), \\ \beta_2 &= \frac{1}{12} \left(6f_n^{(1)} - 6nhf_n^{(2)} + 3n^2h^2f_n^{(3)} - n^3h^3f_n^{(4)} \right), \\ \beta_3 &= \frac{1}{12} \left(2f_n^{(2)} - 2nhf_n^{(3)} + n^2h^2f_n^{(4)} \right), \\ \beta_4 &= \frac{1}{24} \left(f_n^{(3)} - nhf_n^{(4)} \right), \\ \beta_5 &= \frac{1}{120} f_n^{(4)}. \end{aligned} \quad (2.13)$$

Since one-step numerical method shall be derived, then let

$$y_{n+1} - y_n \equiv F(x_{n+1}) - F(x_n). \quad (2.14)$$

Substituting (2.11), (2.12), and (2.13) into (2.14), yields

$$\begin{aligned} y_{n+1} - y_n = & \frac{h}{24} \left(24f_n - 24nhf_n^{(1)} + 12n^2h^2f_n^{(2)} - 4n^3h^3f_n^{(3)} + n^4h^4f_n^{(4)} \right), \\ & + \frac{h^2}{12} \left(6f_n^{(1)} - 6nhf_n^{(2)} + 3n^2h^2f_n^{(3)} - n^3h^3f_n^{(4)} \right) (2n + 1), \\ & + \frac{h^3}{12} \left(2f_n^{(2)} - 2nhf_n^{(3)} + n^2h^2f_n^{(4)} \right) (3n^2 + 3n + 1), \\ & + \frac{h^4}{24} \left(f_n^{(3)} - nhf_n^{(4)} \right) (4n^3 + 6n^2 + 4n + 1), \\ & + \frac{h^5}{120} f_n^{(4)} (5n^4 + 10n^3 + 10n^2 + 5n + 1). \end{aligned}$$

Setting

$$\begin{aligned} S_1 &= \frac{1}{2} \left(24f_n - 24nhf_n^{(1)} + 12n^2h^2f_n^{(2)} - 4n^3h^3f_n^{(3)} + n^4h^4f_n^{(4)} \right), \\ S_2 &= h \left(6f_n^{(1)} - 6nhf_n^{(2)} + 3n^2h^2f_n^{(3)} - n^3h^3f_n^{(4)} \right) (2n + 1), \\ S_3 &= h^2 \left(2f_n^{(2)} - 2nhf_n^{(3)} + n^2h^2f_n^{(4)} \right) (3n^2 + 3n + 1), \\ S_4 &= \frac{h^3}{2} \left(f_n^{(3)} - nhf_n^{(4)} \right) (4n^3 + 6n^2 + 4n + 1), \\ S_5 &= \frac{h^4}{10} f_n^{(4)} (5n^4 + 10n^3 + 10n^2 + 5n + 1), \end{aligned} \quad (2.15)$$

therefore,

$$y_{n+1} = y_n + \frac{h}{12} (S_1 + S_2 + S_3 + S_4 + S_5). \quad (2.16)$$

Equation (2.16) is the newly proposed scheme.

3. Analysis of the properties of the proposed scheme

3.1. Order of accuracy of the proposed scheme

Consider the Taylor's series expansion of the form

$$\begin{aligned} y(x_n + h) = & y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) \\ & + \frac{h^3}{3!} f^{(2)}(x_n, y(x_n)) + \frac{h^4}{4!} f^{(3)}(x_n, y(x_n)) + \frac{h^5}{5!} f^{(4)}(x_n, y(x_n)) + O(h^6). \end{aligned} \quad (3.1)$$

The local truncation error for the proposed scheme is given by

$$\tau_{n+1} = y(x_n + h) - y_{n+1}. \quad (3.2)$$

Substituting (2.16) and (3.1) into (3.2) yields

$$\begin{aligned} \tau_{n+1} = & y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) + \frac{h^3}{3!} f^{(2)}(x_n, y(x_n)) + \frac{h^4}{4!} f^{(3)}(x_n, y(x_n)) \\ & + \frac{h^5}{5!} f^{(4)}(x_n, y(x_n)) + O(h^6) - y_n - \frac{h}{12} (S_1 + S_2 + S_3 + S_4 + S_5) \end{aligned} \quad (3.3)$$

with S_1, S_2, S_3, S_4, S_5 given by (2.15). Solving further, (3.3) becomes

$$\begin{aligned} \tau_{n+1} = & y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) \\ & + \frac{h^3}{3!} f^{(2)}(x_n, y(x_n)) + \frac{h^4}{4!} f^{(3)}(x_n, y(x_n)) + \frac{h^5}{5!} f^{(4)}(x_n, y(x_n)) + O(h^6) \\ & - \left[y_n + \frac{h}{5!} \left(120f_n + 60hf_n^{(1)} + 20h^2f_n^{(2)} + 5h^3f_n^{(3)} + h^4f_n^{(4)} \right) \right]. \end{aligned}$$

By means of the localizing assumptions, the terms up to h^5 have been cancelled, thus

$$\tau_{n+1} = O(h^6). \quad (3.4)$$

Equation (3.4) shows that the order of accuracy of the proposed scheme is 5.

3.2. Consistency property of the proposed scheme

According to [12], a numerical method is said to be consistent if it has at least order $p = 1$. It is clearly seen from (3.4) that the proposed scheme is consistent since

- it has an order of accuracy of $p = 5$;
- $\lim_{h \rightarrow 0} \frac{\tau_{n+1}}{h} = \lim_{h \rightarrow 0} \left[\frac{(h^7)}{h} \right] = 0$;
- the increment function $\phi(x_n, y_n; 0) = f_n = f(x_n, y_n)$.

3.3. Stability property of the proposed scheme

A method is said to be numerically stable if it is capable of damping out small fluctuation carried out in input data [14]. To discuss the stability of the proposed scheme, consider the IVP of the form

$$y' = ry, \quad y(0) = 1,$$

whose exact solution is given by

$$y(x) = \exp(rx), \quad r < 0, \quad (3.5)$$

where r is a complex constant. Expanding (3.5) at the point $x = x_{n+1}$ and using the fact that $h = x_{n+1} - x_n$, one obtains

$$y(x_{n+1}) = \exp(rx_{n+1}) = \exp(rx_n) \cdot \exp(rh) = y(x_n) \exp(rh). \quad (3.6)$$

Using the proposed scheme (2.16), the numerical approximation is obtained as

$$y_{n+1} = y_n \left[1 + \frac{h}{120} (120r + 60r^2h + 20r^3h^2 + 5r^4h^3 + r^5h^4) \right]. \quad (3.7)$$

Setting

$$\zeta = \left[1 + \frac{h}{120} (120r + 60r^2h + 20r^3h^2 + 5r^4h^3 + r^5h^4) \right], \quad (3.8)$$

equation (3.7) becomes

$$y_{n+1} = \zeta y_n. \quad (3.9)$$

Comparing (3.6) and (3.9), it is clearly seen that (3.8) is the sixth-term of the series expansion of $\exp(rh)$. Hence, the stability of the proposed scheme requires that

$$\|\zeta\| < 1. \quad (3.10)$$

Also, setting $z = rh$ in (3.8) and simplifying further, the region of stability of the proposed scheme satisfies

$$\zeta = \left[1 + \frac{1}{120} (120z + 60z^2 + 20z^3 + 5z^4 + z^5) \right]. \quad (3.11)$$

Equations (3.10) and (3.11) show that the proposed fifth order scheme is stable. The stability region of the proposed scheme is plotted in the Figure 1.

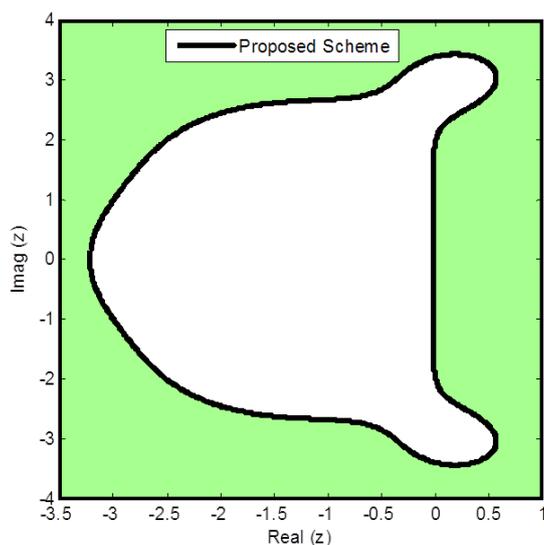


Figure 1: Stability region for the proposed scheme (unshaded).

3.4. Convergence property of the proposed scheme

The convergence of the proposed scheme is given by the following result.

Theorem 3.1. *Given any well-posed initial value problem, then the proposed scheme is convergent, since it satisfies the following conditions:*

- a. consistency;
- b. stability.

4. Numerical examples and discussion of results

This section presents some numerical experiments and discussion of results.

4.1. Numerical examples

Consider the following IVPs with their exact solutions.

Problem 4.1. $\frac{dy}{dx} = y$, $y(0) = 1$, $0 \leq x \leq 1$, $y(x) = \exp(x)$.

Problem 4.2. $\frac{dy}{dx} = 2y$, $y(0) = 2$, $0 \leq x \leq 1$, $y(x) = 2 \exp(2x)$.

Problem 4.3. $\frac{dy}{dx} = 1 + x - y$, $y(0) = 0$, $0 \leq x \leq 1$, $y(x) = x$.

Problem 4.4. $\frac{dy}{dx} = y^2$, $y(0) = 1$, $0 \leq x \leq 0.9$, $y(x) = \frac{1}{1-x}$.

Problem 4.5. $\frac{dy}{dx} = -y$, $y(0) = 1$, $0 \leq x \leq 1$, $y(x) = \exp(-x)$.

The results generated via the proposed scheme against exact solution for Problems 1-5 were displayed in Figures 2, 4, 6, 8, and 10, respectively. The Absolute relative errors generated via the proposed scheme are plotted in Figures 3, 5, 7, 9, and 11, respectively. By varying the step length h , the Maximum Absolute Relative Errors (MABRE) on $[a, b]$ defined by $MABRE = \max_{a \leq n \leq b} |y(x_{n+1}) - y_{n+1}|$, Absolute Relative Errors (ABRE) at $x = b$ defined by $ABRE = |y(b) - y_N|$, and ℓ^2 -error norm defined by $\ell^2 = \sqrt{\sum_{n=0}^{N-1} |y(x_{n+1}) - y_{n+1}|^2}$ for the Problems 4.1-4.5, were presented in Tables 1, 2, 3, 4, and 5, respectively.

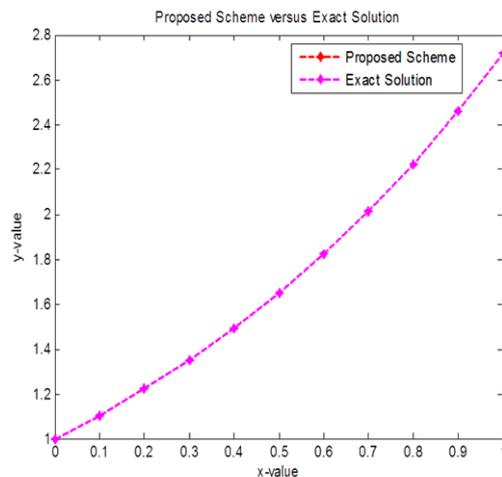


Figure 2: The results generated via the proposed scheme against exact solution for problem 4.1.

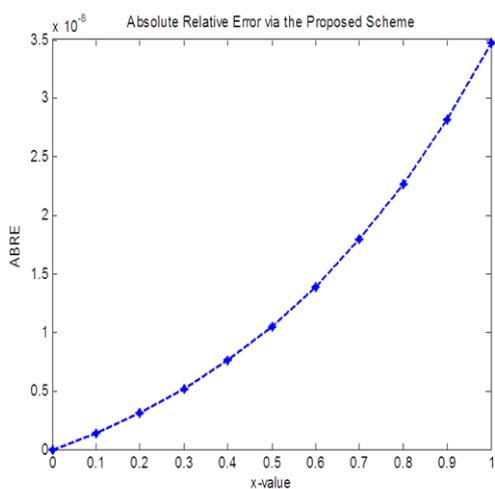


Figure 3: Absolute relative error generated via the proposed scheme for Problem 4.1.

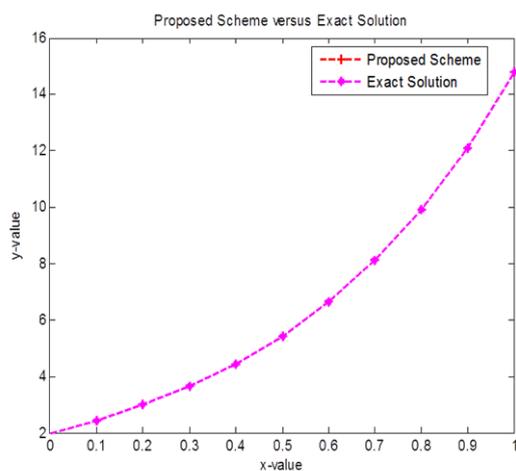


Figure 4: The results generated via the proposed scheme against exact solution for Problem 4.2.

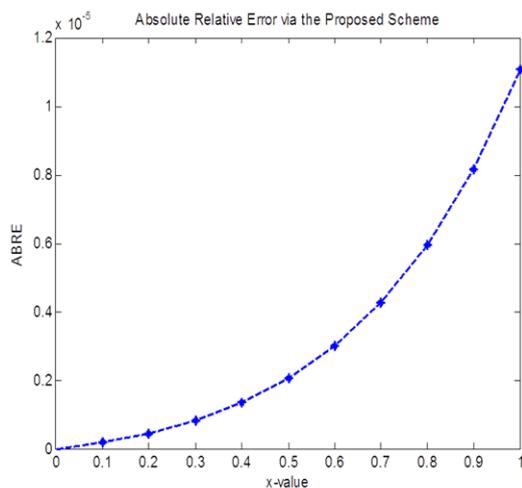


Figure 5: Absolute relative error generated via the proposed scheme for Problem 4.2.

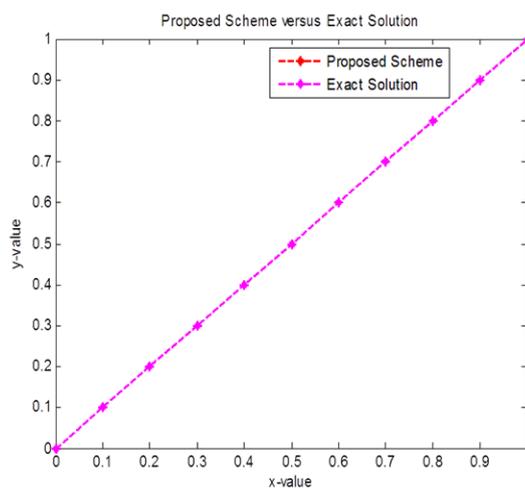


Figure 6: The results generated via the proposed scheme against exact solution for Problem 4.3.

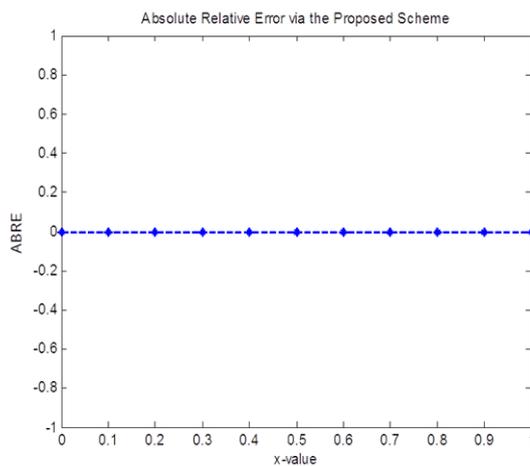


Figure 7: Absolute relative error generated via the proposed scheme for Problem 4.3.

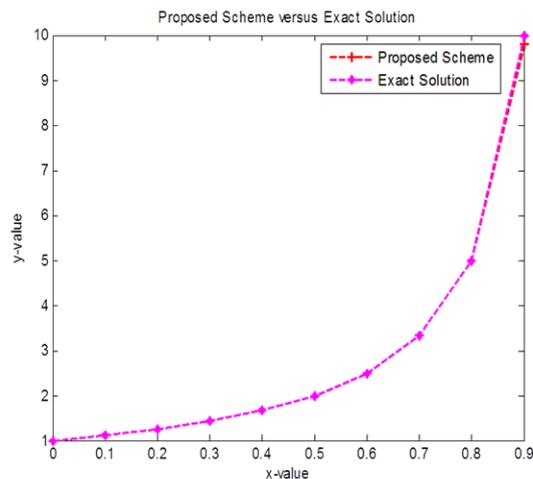


Figure 8: The results generated via the proposed scheme against exact solution for Problem 4.4.

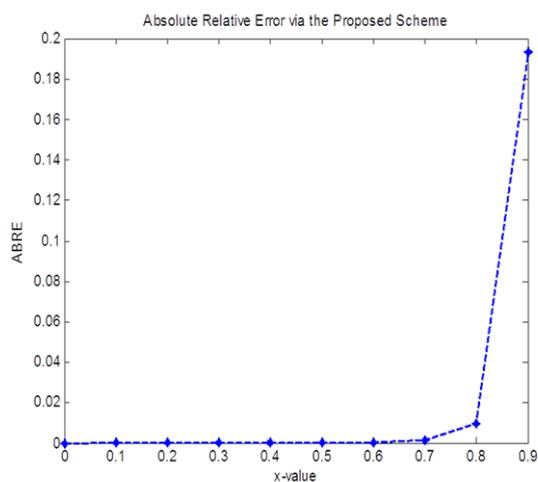


Figure 9: Absolute relative error generated via the proposed scheme for Problem 4.4.

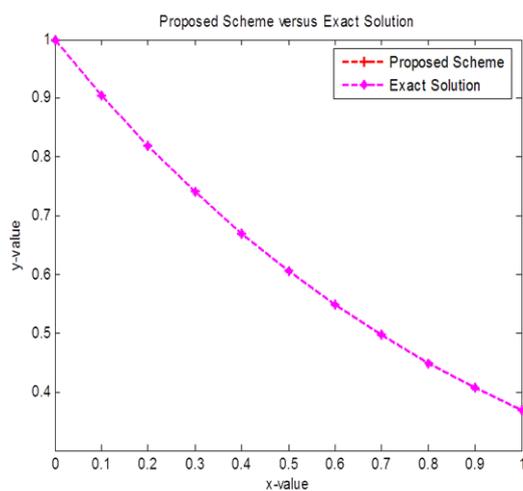


Figure 10: The results generated via the proposed scheme against exact solution for Problem 4.5.

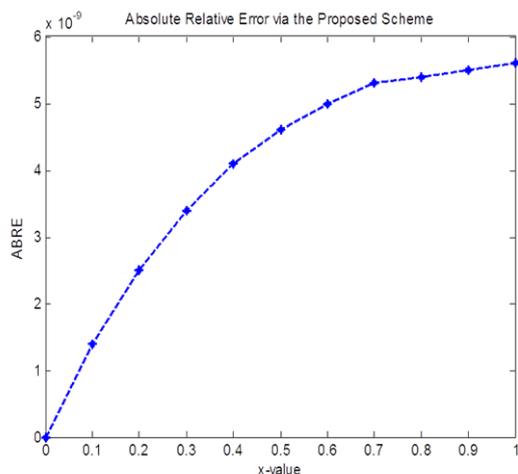


Figure 11: Absolute relative error generated via the proposed scheme for Problem 4.5.

Table 1: Errors with varying step length values for problem 4.1.

h	MABRE	ABRE at $x = 1$	ℓ^2 -error norm
0.1	3.47E-08	3.47E-08	5.69E-08
0.01	0	0	0
0.001	0	0	0
0.0001	0	0	0

Table 2: Errors with varying step length values for problem 4.2.

h	MABRE	ABRE at $x = 1$	ℓ^2 -error norm
0.1	1.11E-05	1.11E-05	1.61E-05
0.01	1.00E-10	1.00E-10	5.48E-10
0.001	0	0	0
0.0001	0	0	0

Table 3: Errors with varying step length values for problem 4.3.

h	MABRE	ABRE at $x = 1$	ℓ^2 -error norm
0.1	0	0	0
0.01	0	0	0
0.001	0	0	0
0.0001	0	0	0

Table 4: Errors with varying step length values for problem 4.4.

h	MABRE	ABRE at $x = 1$	ℓ^2 -error norm
0.1	0.193049567	0.193049567	0.193304442
0.01	1.89E-05	1.89E-05	2.35E-05
0.001	2.00E-09	2.00E-09	7.62E-10
0.0001	0	0	0

Table 5: Errors with varying step length values for problem 4.5.

h	MABRE	ABRE at $x = 1$	ℓ^2 -error norm
0.1	5.60E-09	5.60E-09	1.42E-08
0.01	0	0	0
0.001	0	0	0
0.0001	0	0	0

4.2. Discussion of Results

It is observed from Figures 2, 4, 6, 8, and 10 that the results generated via the proposed scheme agrees with that of the exact solution. It is also observed from Figures 3, 5, 7, 9, and 11, that the proposed scheme follows the exact solution curve more elegantly as shown by the absolute relative error curves. Tables 1, 2, 3, 4, and 5 show values of the MABRE on $[a, b]$, ABRE at $x = b$, and ℓ^2 -error norm generated via the proposed scheme for Problems 4.1-4.5, respectively by varying the step length h . It is observed from Tables 1-5 that the proposed scheme performs excellently and yields smaller error for every decreasing step length, h . It is also observed from Tables 1, 2, 3, 4, and 5 that the fifth order accuracy/convergence of the proposed scheme has been confirmed when applied on Problems 4.1-4.5 with the step length h having a first order decrease in its magnitude, that is $h = 0.1, 0.01, 0.001, 0.0001$. It is clearly seen in the Tables 1, 2, 3, 4, and 5 for every one-order decrease in h , there are five-order decrease in the magnitude of the computed errors (MABRE, ABRE, and ℓ^2 -error norm).

5. Conclusion

In this paper, a proposed scheme has been developed and analyzed to solve IVPs. The properties of the proposed scheme have been analyzed. Also the error analysis of the scheme in terms of MABRE, ABRE, and ℓ^2 -error norm have been examined. Five numerical examples were solved successfully by using the proposed scheme. The results generated via the scheme compare favorably with the exact solution. Furthermore, the error curves show that the proposed scheme follows the exact solution curves more elegantly. Moreover, it is observed that the proposed scheme converges faster to the exact solution for every one-order decrease in h . Also, it is observed that the proposed scheme is consistent, stable and has accuracy of order five. Hence, it can be concluded that the proposed scheme is a good approach to be included in the class of the explicit linear methods for the solution of IVPs in ODEs. Finally, the results were carried out via MATLAB R2014a, Version: 8.3.0.552, 32 bit (win 32) in double precision.

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