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p-Valent strongly starlike and strongly convex functions connected with linear differential Borel operator

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Abstract

In this paper, we define two subclasses $\mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta)$ and $\mathcal{K}(\alpha, m, \delta, p, q, \lambda, \gamma, \beta)$ of strongly starlike and strongly convex functions of order β and type γ by using the linear q-differential Borel operator. We also derive some interesting properties, such as inclusion relationships of these classes and the integral operator $\mathcal{J}_{\mu,p}$.

Keywords: p-Valent, strongly starlike, strongly convex, linear q-differential Borel operator.

2020 MSC: 05A30, 30C45, 11B65, 47B38.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}),$$

which are analytic and p-valent in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}^*(p, \beta)$ of p-valently starlike function of order β, $0 \leqslant \beta < p$, if $f(z) \neq 0$ and

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad (z \in \Delta).$$

The class $\mathcal{S}^*(p, \beta)$ was introduced and studied by Patel and Thakare [19] (see also [3, 11, 12]). Also, we note that $\mathcal{S}^*(p, 0) = \mathcal{S}^*_p$, where $\mathcal{S}^*_p$ is the class of p-valently starlike functions (see [8]).

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A function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{C}(p,\beta)$ of p-valently convex functions of order β, $0 \leqslant \beta < p$, if $f'(z) \neq 0$ and

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \beta, \quad (z \in \Delta).$$

The class $\mathcal{C}(p,\beta)$ was introduced and studied by Owa [18] (see also [11, 22]). Also, we note that $\mathcal{C}(p,0) = \mathcal{C}_p$, where $\mathcal{C}_p$ is the class of p-valently convex functions (see [8]).

A function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{STS}^*(p,\gamma,\beta)$ of strongly starlike functions of order γ and type β, if it satisfies the following inequality (see [13]):

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\gamma\right)\right| < \frac{\beta\pi}{2}, \quad (0<\beta\leqslant p,\ 0\leqslant\gamma<p,\ z\in\Delta).$$

Also, a function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{STC}(p,\gamma,\beta)$ of strongly convex function of order γ and type β, if it satisfies the following inequality (see [2, 13]):

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}-\gamma\right)\right| < \frac{\beta\pi}{2}, \quad (0<\beta\leqslant p,\ 0\leqslant\gamma<p,\ z\in\Delta).$$

It is obvious that:

$$f(z) \in \mathcal{STC}(p,\gamma,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{STS}^*(p,\gamma,\beta), \quad (0<\beta\leqslant p,\ 0\leqslant\gamma<p,\ z\in\Delta).$$

Remark 1.1. We note that:

(i) $\mathcal{STS}^*(p,\gamma,0) = \mathcal{STS}^*(p,\gamma)$ (see Nunokawa et al. [15]);
(ii) $\mathcal{STS}^*(1,\gamma,\beta) = \mathcal{S}_s^*(\gamma,\beta)$ and $\mathcal{STC}(1,\gamma,\beta) = \mathcal{C}_s(\gamma,\beta)$ (see Nunokawa et al. [16] and Prajapat and Goyal [20]);
(iii) $\mathcal{STS}^*(1,\gamma,0) = \mathcal{S}_s^*(\gamma)$ and $\mathcal{STC}(1,\gamma,0) = \mathcal{C}_s(\gamma)$ (see Nunokawa [14] and Obradovic and Joshi [17]);
(iv) $\mathcal{STS}^*(p,1,\beta) = \mathcal{S}^*(p,\beta)$ (see Patel and Thakare [19]);
(v) $\mathcal{STC}(p,1,\beta) = \mathcal{C}(p,\beta)$ (see Owa [18]).

A discrete random variable x is said to have a Borel distribution if it takes the values $1,2,3,\ldots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \ldots$, respectively, where λ is called the parameter.

Very recently, Wanas and Khuttar [24] introduced the Borel distribution (BD) whose probability mass function is

$$P(x=\rho) = \frac{(\rho\lambda)^{\rho-1} e^{-\lambda\rho}}{\rho!}, \quad \rho = 1,2,3,\ldots.$$

Wanas and Khuttar introduced a series $\mathcal{M}(\lambda;z)$ whose coefficients are probabilities of the Borel distribution (BD)

$$\mathcal{M}_p(\lambda;z) = z^p + \sum_{k=p+1}^{\infty} \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} z^k,\ (0<\lambda\leqslant 1) = z^p + \sum_{k=p+1}^{\infty} \phi_{k,p}(\lambda) z^k,\ (0<\lambda\leqslant 1),$$

where

$$\phi_{k,p}(\lambda) = \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!}.$$

We define a linear operator $\mathcal{D}(p,\lambda;z)f : \mathcal{A}_p \to \mathcal{A}_p$ as follows

$$\mathcal{D}(p,\lambda;z)f(z) = \mathcal{M}_p(\lambda;z) * f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{[\lambda(k-p)]^{k-p-1} e^{-\lambda(k-p)}}{(k-p)!} a_k z^k,\ (0<\lambda\leqslant 1).$$

Srivastava [21] made use of various operators of q-calculus and fractional q-calculus and recalling the definition and notations. The q-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\lambda; q)_k = \begin{cases} 1, & k = 0, \\ (1-\lambda)(1-\lambda q)\cdots\left(1-\lambda q^{k-1}\right), & k \in \mathbb{N}. \end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda}; q\right)_k = \frac{(1-q)^k \ \Gamma_q(\lambda + k)}{\Gamma_q(\lambda)}, \qquad (k \in \mathbb{N}_0),$$

where (see [7])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}, \qquad (|q| < 1).$$

Also, we note that

$$(\lambda; q)_\infty = \prod_{k=0}^{\infty} \left(1 - \lambda q^k\right), \qquad (|q| < 1),$$

and, the q-gamma function $\Gamma_q(z)$ is known as

$$\Gamma_q(z+1) = [z]_q \, \Gamma_q(z),$$

where $[k]_q$ denotes the basic q-number defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1 + \sum\limits_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{1.1}$$

Using the definition formula (1.1) we have the next two products.

(i) For any non negative integer k, the q-*shifted factorial* is given by

$$[k]_q! := \begin{cases} 1, & \text{if } k = 0, \\ \prod\limits_{n=1}^{k} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number r, the q-*generalized Pochhammer symbol* is defined by

$$[r]_{q,k} := \begin{cases} 1, & \text{if } k = 0, \\ \prod\limits_{n=r}^{r+k-1} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \to \Gamma(z) \quad \text{as} \quad q \to 1^-.$$

Also, we observe that

$$\lim_{q \to 1^-} \left\{ \frac{\left(q^\lambda; q\right)_k}{(1-q)^k} \right\} = (\lambda)_k.$$

For $0 < q < 1$, the q-*derivative operator* [10] (see also [1, 5, 9]) for $\mathcal{D}(p, \lambda; z)f$ is defined by

$$D_q\left(\mathcal{D}(p,\lambda;z)f(z)\right) := \frac{\mathcal{D}(p,\lambda;z)f(z) - \mathcal{D}(p,\lambda;z)f(qz)}{z(1-q)}$$

$$= [p]_q\, z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q \frac{[\lambda (k-p)]^{k-p-1}\, e^{-\lambda(k-p)}}{(k-p)!} a_k\, z^{k-1},$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \qquad [0,q] := 0.$$

For $\alpha > -1$ and $0 < q < 1$, we defined the linear operator $\mathcal{D}^{\alpha,q}_{\lambda,p} f : \mathcal{A}_p \to \mathcal{A}_p$ by

$$\mathcal{D}^{\alpha,q}_{p,\lambda} f(z) * \mathcal{N}^q_{p,\alpha+1}(z) = \frac{z}{[p]_q} D_q\left(\mathcal{D}(p,\lambda;z) f(z)\right),\ z \in \Delta,$$

where the function $\mathcal{N}^q_{p,\alpha+1}$ is given by

$$\mathcal{N}^q_{p,\alpha+1}(z) := z^p + \sum_{k=p+1}^{\infty} \frac{[\alpha+1]_{q,k-p}}{[k-1]_q!} z^k,\ z \in \Delta.$$

A simple computation shows that

$$\begin{aligned}\mathcal{D}^{\alpha,q}_{p,\lambda} f(z) :&= z^p + \sum_{k=p+1}^{\infty} \frac{[k]_q!\,[\lambda (k-p)]^{k-p-1}\, e^{-\lambda(k-p)}}{[p]_q\,[\alpha+1]_{q,k-p}\,(k-p)!} a_k\, z^k \\ &= z^p + \sum_{k=p+1}^{\infty} \phi_k a_k\, z^k \qquad (0 < \lambda \leqslant 1,\ \alpha > -1,\ 0 < q < 1,\ z \in \Delta).\end{aligned}$$

where

$$\phi_k = \frac{[k]_q!\,[\lambda (k-p)]^{k-p-1}\, e^{-\lambda(k-p)}}{[p]_q\,[\alpha+1]_{q,k-p}\,(k-p)!}.$$

For $\beta \geqslant 0$, with the aid of the operator $\mathcal{D}^{\alpha,q}_{p,\lambda}$ we will define the linear q-differential Borel operator $\mathcal{A}_p \to \mathcal{A}_p$ as follows:

$$\begin{aligned}
\mathcal{G}^{\alpha,0}_{p,q,\lambda,\delta} f(z) :&= \mathcal{D}^{\alpha,q}_{p,\lambda} f(z),\\
\mathcal{G}^{\alpha,1}_{p,q,\lambda,\delta} f(z) :&= (1-\delta)\,\mathcal{G}^{\alpha,0}_{p,q,\lambda,\delta} f(z) + \delta \frac{z}{p}\left(\mathcal{G}^{\alpha,0}_{p,q,\lambda,\delta} f(z)\right)'\\
&= z^p + \sum_{k=p+1}^{\infty} \frac{[k]_q!\,[\lambda (k-p)]^{k-p-1}\, e^{-\lambda(k-p)}}{[p]_q\,[\alpha+1]_{q,k-p}\,(k-p)!}\left[1+\delta\left(\frac{k}{p}-1\right)\right] a_k\, z^k,\\
\mathcal{G}^{\alpha,2}_{p,q,\lambda,\delta} f(z) :&= (1-\delta)\,\mathcal{G}^{\alpha,1}_{p,q,\lambda,\delta} f(z) + \delta \frac{z}{p}\left(\mathcal{G}^{\alpha,1}_{p,q,\lambda,\delta} f(z)\right)'\\
&= z^p + \sum_{k=p+1}^{\infty} \frac{[k]_q!\,[\lambda (k-p)]^{k-p-1}\, e^{-\lambda(k-p)}}{[p]_q\,[\alpha+1]_{q,k-p}\,(k-p)!}\left[1+\delta\left(\frac{k}{p}-1\right)\right]^2 a_k\, z^k,\\
&\ \ \vdots\\
\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z) :&= z^p + \sum_{k=p+1}^{\infty} \frac{[k]_q!\,[\lambda (k-p)]^{k-p-1}\, e^{-\lambda(k-p)}}{[\alpha+1]_{q,k-p}\,(k-p)!}\left[1+\delta\left(\frac{k}{p}-1\right)\right]^m a_k\, z^k,\\
&\left(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \delta \geqslant 0,\ 0 < \lambda \leqslant 1,\ \alpha > -1,\ 0 < q < 1\right).
\end{aligned} \tag{1.2}$$

From the definition relation (1.2), we can easily verify that the next relations hold for all $f \in \mathcal{A}_p$:

(i)

$$z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)\right)' = \alpha\, \mathcal{G}^{\alpha-1,m}_{p,q,\lambda,\delta}f(z) - (\alpha-p)\,\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z); \tag{1.3}$$

(ii)

$$\delta z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)\right)' = p\, \mathcal{G}^{\alpha,m+1}_{p,q,\lambda,\delta}f(z) - p\ (1-\delta)\,\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z). \tag{1.4}$$

Remark 1.2. By specializing the parameters p and m, we obtain the following operators based on Borel distribution.

1. Putting $p = 1$, we obtain that $\mathcal{G}^{\alpha,m}_{1,q,\lambda,\delta} =: \mathcal{I}^{\alpha,m}_{q,\lambda,\delta}$, where the operator $\mathcal{I}^{\alpha,m}_{q,\lambda,\delta}$ defined as follows

$$\mathcal{I}^{\alpha,m}_{q,\lambda,\delta}f(z) := z + \sum_{k=2}^{\infty} \frac{[k]_q!\,[\lambda\,(k-1)]^{k-2}\,e^{-\lambda(k-1)}}{[\alpha+1]_{q,k-1}\,(k-1)!}\,[1+\delta\,(k-1)]^m\,a_k\,z^k.$$

2. Putting $p = 1$ and $m = 0$, we obtain that $\mathcal{G}^{\alpha,0}_{1,q,\lambda,\delta} =: \mathcal{B}^{\alpha,q}_{\lambda}$, where the operator $\mathcal{B}^{\alpha,q}_{\lambda}$ studied by El-Deeb and Murugusundaramoorthy [6].
3. Putting $q \to 1^-$ and $p = 1$, we obtain that $\lim\limits_{q\to 1^-} \mathcal{G}^{\alpha,m}_{1,q,\lambda,\delta} := \mathcal{R}^{\alpha,m}_{\lambda,\delta}$, where the operator $\mathcal{R}^{\alpha,m}_{\lambda,\delta}$ defined as follows

$$\mathcal{R}^{\alpha,m}_{\lambda,\delta}f(z) := z + \sum_{k=2}^{\infty} \frac{k\,[\lambda\,(k-1)]^{k-2}\,e^{-\lambda(k-1)}}{(\alpha+1)_{k-1}}\,[1+\delta\,(k-1)]^m\,a_k z^k.$$

4. Putting $q \to 1^-$, $p = 1$ and $m = 0$, we obtain that $\lim\limits_{q\to 1^-} \mathcal{G}^{\alpha,0}_{1,q,\lambda,\delta} := \mathcal{R}^{\alpha}_{\lambda}$, where the operator $\mathcal{R}^{\alpha}_{\lambda}$ studied by El-Deeb and Murugusundaramoorthy [6].

By using the linear operator $\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}$, we now define new subclasses of $\mathcal{A}_p$ as follows:

$$\mathcal{S}^*\left(\alpha,m,\delta,p,q,\lambda,\gamma,\beta\right) = \left\{ f(z)\in\mathcal{A}_p:\ \mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)\in\mathcal{STS}^*\left(p,\gamma,\beta\right),\ \frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)} \neq \beta, \right. \tag{1.5}$$
$$\left.(m\in\mathbb{N},\ \delta\geqslant 0,\ 0<\lambda\leqslant 1,\ \alpha>-1,\ 0<q<1,\ 0<\beta\leqslant p,\ 0\leqslant\gamma<p)\right\},$$

and

$$\mathcal{K}\left(\alpha,m,\delta,p,q,\lambda,\gamma,\beta\right) = \left\{ f(z)\in\mathcal{A}_p:\ \mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)\in\ \mathcal{STC}\left(p,\gamma,\beta\right),\ \frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z)} \neq \beta, \right. \tag{1.6}$$
$$\left.(m\in\mathbb{N},\ \delta\geqslant 0,\ 0<\lambda\leqslant 1,\ \alpha>-1,\ 0<q<1,\ 0<\beta\leqslant p,\ 0\leqslant\gamma<p)\right\}.$$

It is obvious from the definitions (1.5) and (1.6) that:

$$f(z)\in\mathcal{K}\left(\alpha,m,\delta,p,q,\lambda,\gamma,\beta\right) \Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*\left(\alpha,m,\delta,p,q,\lambda,\gamma,\beta\right). \tag{1.7}$$

2. Main results

To establish our main results, we shall require the following lemma.

Lemma 2.1 ([14]). *Let a function* $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *be analytic in* Δ *and* $h(z) \neq 0$, $z \in \Delta$. *If there exists a point* $z_0 \in \Delta$ *such that*

$$|\arg h(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|) \quad \textit{and} \quad |\arg h(z_0)| = \frac{\pi}{2}\alpha, \quad (0 \leqslant \alpha < 1), \tag{2.1}$$

we have

$$\frac{z_0 h'(z_0)}{h(z_0)} = ik\alpha, \tag{2.2}$$

where

$$k \geqslant \frac{1}{2}\left(a + \frac{1}{a}\right) \ \textit{when} \ \arg h(z_0) = \frac{\pi}{2}\alpha, \qquad k \leqslant -\frac{1}{2}\left(a + \frac{1}{a}\right) \ \textit{when} \ \arg h(z_0) = -\frac{\pi}{2}\alpha, \tag{2.3}$$

and

$$(h(z_0))^{\frac{1}{\alpha}} = \pm ia, \quad (a > 0).$$

Theorem 2.2. *Let* $f(z) \in \mathcal{A}_p$, *then we have*

$$\mathcal{S}^*(\alpha, m+1, \delta, p, q, \lambda, \gamma, \beta) \subset \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) \subset \mathcal{S}^*(\alpha+1, m, \delta, p, q, \lambda, \gamma, \beta).$$

Proof. We first prove that

$$\mathcal{S}^*(\alpha, m+1, \delta, p, q, \lambda, \gamma, \beta) \subset \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta).$$

Let $f \in \mathcal{S}^*(\alpha, m+1, \delta, p, q, \lambda, \gamma, \beta)$ and suppose that

$$h(z) = \frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z)} - \gamma\right), \qquad (z \in \Delta), \tag{2.4}$$

where h is analytic in Δ with $h(0) = 1$. Combining (1.4) and (2.4), we find that

$$\frac{p}{\delta}\frac{\mathcal{G}^{\alpha,m+1}_{p,q,\lambda,\delta} f(z)}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z)} = (p-\gamma)\,h(z) + \frac{p(1-\delta)}{\delta} + \gamma. \tag{2.5}$$

Differentiating (2.5) logarithmically with respect to z, we have

$$\frac{z\left(\mathcal{G}^{\alpha,m+1}_{p,q,\lambda,\delta} f(z)\right)'}{\mathcal{G}^{\alpha,m+1}_{p,q,\lambda,\delta} f(z)} - \gamma = (p-\gamma)\,h(z) + \frac{(p-\gamma)\,zh'(z)}{(p-\gamma)\,h(z) + \frac{p(1-\delta)}{\delta} + \gamma}.$$

Suppose that there exists a point $z_0 \in \Delta$ such that the conditions (2.1)-(2.3) of Lemma 2.1 are satisfied. Thus, if $\arg(h(z_0)) = \frac{\pi}{2}\beta$ for $z_0 \in \Delta$, then

$$\begin{aligned}\frac{z_0\left(\mathcal{G}^{\alpha,m+1}_{p,q,\lambda,\delta} f(z_0)\right)'}{\mathcal{G}^{\alpha,m+1}_{p,q,\lambda,\delta} f(z_0)} - \gamma &= (p-\gamma)\,h(z_0)\left[1 + \frac{\frac{z_0 h'(z_0)}{h(z_0)}}{(p-\gamma)\,h(z_0) + \frac{p(1-\delta)}{\delta} + \gamma}\right] \\ &= (p-\gamma)\,a^{\beta} e^{-\frac{i\pi\beta}{2}}\left[1 + \frac{ik\beta}{(p-\gamma)\,a^{\beta} e^{-\frac{i\pi\beta}{2}} + \frac{p(1-\delta)}{\delta} + \gamma}\right].\end{aligned}$$

This implies that

$$\begin{aligned}
&\arg\left(\frac{z_0\left(\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m+1}f(z_0)\right)'}{\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m+1}f(z_0)}-\gamma\right)\\
&=-\frac{\pi\beta}{2}+\arg\left(1+\frac{ik\beta}{(p-\gamma)\,a^{\beta}e^{-\frac{i\pi\beta}{2}}+\frac{p(1-\delta)}{\delta}+\gamma}\right)\\
&=-\frac{\pi\beta}{2}+\tan^{-1}\left(\frac{k\alpha\left[\left(\frac{p(1-\lambda)}{\lambda}+\beta\right)+(p-\beta)\,a^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\right]}{\left(\frac{p(1-\delta)}{\delta}+\gamma\right)^2+2\left(\frac{p(1-\delta)}{\delta}+\gamma\right)(p-\gamma)\,a^{\beta}\cos\left(\frac{\pi\beta}{2}\right)+(p-\gamma)^2a^{2\beta}-k\beta\,(p-\gamma)\,a^{\beta}\sin\left(\frac{\pi\beta}{2}\right)}\right).
\end{aligned}$$

This gives that

$$\arg\left(\frac{z_0\left(\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m+1}f(z_0)\right)'}{\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m+1}f(z_0)}-\gamma\right)\leqslant-\frac{\pi\beta}{2},$$

since

$$k\leqslant-\frac{1}{2}\left(a+\frac{1}{a}\right)\leqslant-1\ \text{ and }\ z_0\in\Delta,$$

this contradicts the condition $f(z)\in\mathcal{S}^*(\alpha,m+1,\delta,p,q,\lambda,\gamma,\beta)$. On the other hand if we set $\arg(h(z_0))=\frac{\pi}{2}\beta$ for $z_0\in\Delta$, then it can similarly be shown that

$$\arg\left(\frac{z_0\left(\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m+1}f(z_0)\right)'}{\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m+1}f(z_0)}-\gamma\right)\geqslant\frac{\pi\beta}{2},$$

since

$$k\geqslant\frac{1}{2}\left(a+\frac{1}{a}\right)\geqslant1\ \text{ and }\ z_0\in\Delta,$$

which again contradicts the hypothesis that $f(z)\in\mathcal{S}^*(\alpha,m+1,\delta,p,q,\lambda,\gamma,\beta)$. Thus, the function defined by (2.4) has to satisfy the following inequality

$$\arg(h(z))\leqslant\frac{\pi}{2}\beta,\ (z\in\Delta),$$

which implies that

$$\left|\arg\left(\frac{z\left(\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m}f(z)\right)'}{\mathcal{G}_{p,q,\lambda,\delta}^{\alpha,m}f(z)}-\gamma\right)\right|\leqslant\frac{\pi}{2}\beta,\quad(z\in\Delta).$$

For the second inclusion relationship asserted by Theorem 2.2, using arguments similar to those detailed above with (1.3), we obtain

$$\mathcal{S}^*(\alpha,m,\delta,p,q,\lambda,\gamma,\beta)\subset\mathcal{S}^*(\alpha+1,m,\delta,p,q,\lambda,\gamma,\beta).$$

We thus complete the proof of Theorem 2.2. □

Theorem 2.3. *Let* $f(z)\in\mathcal{A}_p$, *then we have*

$$\mathcal{K}(\alpha,m+1,\delta,p,q,\lambda,\gamma,\beta)\subset\mathcal{K}(\alpha,m,\delta,p,q,\lambda,\gamma,\beta)\subset\mathcal{K}(\alpha+1,m,\delta,p,q,\lambda,\gamma,\beta).$$

Proof.

$$\begin{aligned} f(z) \in \mathcal{K}(\alpha, m+1, \delta, p, q, \lambda, \gamma, \beta) &\Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(\alpha, m+1, \delta, p, q, \lambda, \gamma, \beta) \\ &\Rightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) \\ &\Leftrightarrow f(z) \in \mathcal{K}(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} f(z) \in \mathcal{K}(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) &\Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) \\ &\Rightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(\alpha+1, m, \delta, p, q, \lambda, \gamma, \beta) \\ &\Leftrightarrow f(z) \in \mathcal{K}(\alpha+1, m, \delta, p, q, \lambda, \gamma, \beta). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we deduce that the assertion of Theorem 2.3 holds. □

For a function $f(z) \in \mathcal{A}_p$ the integral operator $\mathcal{J}_{\mu,p}$ is defined by (see Choi et al. [4]):

$$\mathcal{J}_{\mu,p}(f)(z) = \frac{\mu+p}{z^{\mu}} \int_0^z t^{\mu-1} f(t)\,dt = \left(z^p + \sum_{k=p+1}^{\infty} \frac{\mu+p}{\mu+k} z^k \right) * f(z), \ (\mu > -p, \ p \in \mathbb{N}),$$

which satisfies the following relationship:

$$z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)\right)' = (\mu+p)\,\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z) - \mu\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z). \tag{2.8}$$

Theorem 2.4. *Let $f(z) \in \mathcal{A}_p$. If $f(z) \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta)$ with*

$$\frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)} \neq \beta, \quad (z \in \Delta),$$

we have

$$\mathcal{J}_{\mu,p}(f)(z) \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta).$$

Proof. We begin by setting

$$\frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)} = \gamma + (p-\gamma)\,h(z), \quad (z \in \Delta), \tag{2.9}$$

where h is analytic in Δ with $h(0) = 1$. Combining (2.8) and (2.9), we find that

$$\frac{(\mu+p)\,\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z)}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)} = (\gamma+\mu) + (p-\gamma)\,h(z). \tag{2.10}$$

Differentiating (2.10) logarithmically with respect to z, we have

$$\frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta} f(z)} - \gamma = (p-\gamma)\,h(z) + \frac{(p-\gamma)\,zh'(z)}{(p-\gamma)\,h(z) + (\gamma+\mu)}.$$

Suppose now that there exists a point $z_0 \in \Delta$ such that the conditions (2.1)-(2.3) of Lemma 2.1 are satisfied. Thus, if $\arg(h(z_0)) = \frac{\pi}{2}\beta$ for $z_0 \in \Delta$, then

$$\frac{z_0\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)} - \gamma = (p-\gamma)\,h(z_0)\left[1 + \frac{\frac{z_0 h'(z_0)}{h(z_0)}}{(p-\gamma)\,h(z_0) + (\gamma+\mu)}\right]$$
$$= (p-\gamma)\,a^{\beta} e^{-\frac{i\pi\beta}{2}}\left[1 + \frac{ik\beta}{(p-\gamma)\,a^{\beta} e^{-\frac{i\pi\beta}{2}} + (\gamma+\mu)}\right].$$

This implies that

$$\arg\left(\frac{z_0\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)} - \gamma\right)$$
$$= -\frac{\pi\beta}{2} + \arg\left(1 + \frac{ik\beta}{(p-\gamma)\,a^{\beta} e^{-\frac{i\pi\beta}{2}} + (\gamma+\mu)}\right)$$
$$= -\frac{\pi\beta}{2} + \tan^{-1}\left(\frac{k\beta\left[(\gamma+\mu) + (p-\gamma)\,a^{\beta}\cos\left(\frac{\pi\beta}{2}\right)\right]}{(\gamma+\mu)^2 + 2(\gamma+\mu)(p-\gamma)\,a^{\beta}\cos\left(\frac{\pi\beta}{2}\right) + (p-\gamma)^2 a^{2\beta} - k\beta\,(p-\gamma)\,a^{\beta}\sin\left(\frac{\pi\beta}{2}\right)}\right).$$

This gives that

$$\arg\left(\frac{z_0\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)} - \gamma\right) \leqslant -\frac{\pi\beta}{2},$$

since

$$k \leqslant -\frac{1}{2}\left(a + \frac{1}{a}\right) \leqslant -1 \quad \text{and} \quad z_0 \in \Delta,$$

this contradicts the condition $f(z) \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta)$. On the other hand if we set $\arg(h(z_0)) = \frac{\pi}{2}\beta$ for $z_0 \in \Delta$, then it can similarly be shown that

$$\arg\left(\frac{z_0\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}f(z_0)} - \gamma\right) \geqslant \frac{\pi\beta}{2},$$

since

$$k \geqslant \frac{1}{2}\left(a + \frac{1}{a}\right) \geqslant 1 \quad \text{and} \quad z_0 \in \Delta,$$

which again contradicts the hypothesis that $f(z) \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta)$. Thus, the function defined by (2.9) has to satisfy the following inequality:

$$\arg(h(z)) \leqslant \frac{\pi}{2}\beta, \ (z \in \Delta).$$

This shows that

$$\left|\arg\left(\frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)\right)'}{\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)} - \gamma\right)\right| \leqslant \frac{\pi}{2}\beta, \quad (z \in \Delta).$$

This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let* $f(z) \in \mathcal{A}_p$. *If* $f(z) \in \mathcal{K}(\alpha, m, \delta, p, q, \lambda, \gamma, \beta)$ *with*

$$1 + \frac{z\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)\right)''}{\left(\mathcal{G}^{\alpha,m}_{p,q,\lambda,\delta}\mathcal{J}_{\mu,p}(f)(z)\right)'} \neq \beta, \quad (z \in \Delta),$$

we get

$$\mathcal{J}_{\mu,p}(f)(z) \in \mathcal{K}(\alpha, m, \delta, p, q, \lambda, \gamma, \beta).$$

Proof. In view of (1.7) and Theorem 2.4, we find that

$$\begin{aligned} f(z) \in \mathcal{K}(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) &\Leftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) \\ &\Rightarrow \frac{z(\mathcal{J}_{\mu,p}(f)(z))'}{p} \in \mathcal{S}^*(\alpha, m, \delta, p, q, \lambda, \gamma, \beta) \\ &\Leftrightarrow \mathcal{J}_{\mu,p}(f)(z) \in \mathcal{K}(\alpha, m, \delta, p, q, \lambda, \gamma, \beta), \end{aligned}$$

we deduce that the assertion of Theorem 2.5 holds. □

Remark 2.6. By specializing the parameters q, p, and m, we obtain various results for the subclasses stated in Remark 1.1 based on the different operators listed in Remark 1.2 in the introduction, we left this as an exercise to interested readers.

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