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Variation of parameters for local fractional nonhomogenous linear-differential equations

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Abstract

In this paper we study the method of variation of parameters to find a particular solution of a nonhomogenous linear fractional differential equations. A formula similar to that for usual ordinary differential equations is obtained.

Keywords: Conformable fractional derivative, fractional integral, fractional differential equation, variation of parameters.
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1. Introduction

There are many definitions available in the literature for fractional derivatives. The main ones are the Riemann Liouville definition and the Caputo definition, see [10, 11], and for some applications one can see [5], [8] and [12].

(i) Riemann - Liouville Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

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(ii) Caputo Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Such definitions have many setbacks such as:

(i) The Riemann-Liouville derivative does not satisfy $D_a^\alpha(1) = 0$ ($D_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^\alpha(fg) = f D_a^\alpha(g) + g D_a^\alpha(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^\alpha(f/g) = \frac{g D_a^\alpha(f) - f D_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^\alpha(f \circ g)(t) = f^{(\alpha)}\big(g(t)\big)\, g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$, in general.

(vi) All fractional derivatives, specially Caputo definition, assumes that the function f is differentiable.

In [9], the authors gave a new definition of fractional derivative which is a natural extension to the usual first derivative as follows:
Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then for all $t > 0, \alpha \in (0, 1)$, let

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

T_α is called the **conformable fractional derivative of f of order α**.
Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$.
If f is $\alpha-$differentiable in some $(0, b)$, $b > 0$ and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

According to this definition, we have the following properties, see [9],

1 . $T_\alpha(1) = 0$,

2 . $T_\alpha(t^p) = p t^{p-\alpha}$ for all $p \in \mathbb{R}$,

3 . $T_\alpha(\sin at) = a t^{1-\alpha} \cos at$, $a \in \mathbb{R}$,

4 .$T_\alpha(\cos at) = -a t^{1-\alpha} \sin at$, $a \in \mathbb{R}$,

5 .$T_\alpha(e^{at}) = at^{1-\alpha}e^{at},\ a \in \mathbb{R}$.

Further, many functions behave as in the usual derivative. Here are some formulas

$$T_\alpha(\frac{1}{\alpha}t^\alpha) = 1,$$
$$T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha},$$
$$T_\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos(\frac{1}{\alpha}t^\alpha),$$
$$T_\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin(\frac{1}{\alpha}t^\alpha).$$

For more applications on the conformable fractional derivative, one can see [1, 2, 3, 4, 6, 7].

In this paper we use the conformable fractional derivative to study methods for finding particular solutions of a certain class of nonhomogenous linear fractional differential equations.

2. Fractional linear differential equations

Definition 2.1. Let $0 < \alpha < 1$ and $n \in \{1, 2, 3, ...\}$. Then we write

$$T^{n\alpha}f = \underbrace{D^\alpha D^\alpha ... D^\alpha}_{ntimes} f.$$

For simplicity, we write $f^{(n\alpha)}$ for $T^{n\alpha}$. So $y^{(2\alpha)}(x)$ stands for $\frac{d^\alpha}{dx^\alpha}\left(\frac{d^\alpha y}{dx^\alpha}\right)$.

Definition 2.2. A differential equation of the form

$$T^{n\alpha}y + a_{n-1}T^{(n-1)\alpha}y + ... + a_1T^\alpha y + a_0y = f(x) \tag{2.1}$$

is called a linear fractional differential equation of order n. The coefficients $a_0,\ a_1,\ ...,\ a_{n-1}$ could be constants or variables.

Since $0 < \alpha < 1$, if y is $n-$times differentiable, then there are $n-$independent solutions $y_1,\ y_2,\ ...,\ y_n$ for the homogeneous differential equation

$$T^{n\alpha} + ... + a_0y = 0\,. \tag{2.2}$$

To find a particular solution for (2.1), one can use either undetermined coefficients (for special types of f and constant coefficients) or the variation of parameters method. In fact we find a complete formula for y_p when $n = 2$.

Definition 2.3. Let $y_1,\ y_2$ be two independent functions. The function

$$W^\alpha[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1^{(\alpha)} & y_2^{(\alpha)} \end{vmatrix}$$

will be called the $\alpha-$Wronskian of y_1 and y_2.

More generally, If $y_1,\ y_2,\ ...,\ y_n$ are n linearly independent functions, then

$$W^\alpha[y_1, y_2, ..., y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(\alpha)} & y_2^{(\alpha)} & \cdots & y_n^{(\alpha)} \\ . & & & \\ . & & & \\ . & & & \\ y_1^{((n-1)\alpha)} & y_2^{((n-1)\alpha)} & \cdots & y_n^{((n-1)\alpha)} \end{vmatrix}.$$

3. The Main Results

Let y_1 and y_2 be two independent solutions for

$$D^{\alpha}\left(D^{\alpha}y\right)+a_1D^{\alpha}y+a_0y=0\,. \tag{3.1}$$

Our goal is to find y_p for

$$D^{\alpha}\left(D^{\alpha}y\right)+a_1D^{\alpha}y+a_0y=f(x)\,. \tag{3.2}$$

Procedure:

Step(i). Let

$$y_p=c_1y_1+c_2y_2,$$

where c_1 and c_2 are functions of x. So

$$y_p(x)=c_1(x)y_1(x)+c_2(x)y_2(x),$$

$$y_p^{(\alpha)}=c_1^{(\alpha)}y_1+c_1y_1^{(\alpha)}+c_2^{(\alpha)}y_2+c_2y_2^{(\alpha)}.$$

Step(ii). Put

$$c_1^{(\alpha)}y_1+c_2^{(\alpha)}y_2=0\,. \tag{3.3}$$

Hence

$$\begin{aligned}D^{\alpha}\left(y_p^{(\alpha)}\right)&=D^{\alpha}\left(c_1y_1+c_2y_2\right)\\&=c_1^{(\alpha)}y_1^{(\alpha)}+c_1y_1^{(2\alpha)}+c_2^{(\alpha)}y_2^{(\alpha)}+c_2y_2^{(2\alpha)}\,.\end{aligned}$$

Step(iii). Substitute $D^{\alpha}\left(y_p^{(\alpha)}\right)$ in the equation (3.2) to get

$$c_1^{(\alpha)}y_1^{(\alpha)}+c_1y_1^{(2\alpha)}+c_2^{(\alpha)}y_2^{(\alpha)}+c_2y_2^{(2\alpha)}+a_1\left(c_1y_1^{(\alpha)}+c_2y_2^{(\alpha)}\right)+a_0\left(c_1y_1+c_2y_2\right)=f(x)\,.$$

Hence,

$$c_1\left(y_1^{(2\alpha)}+a_1y_1^{(\alpha)}+a_0y_1\right)+c_2\left(y_2^{(2\alpha)}+a_1y_2^{(\alpha)}+a_0y_2\right)+c_1^{(\alpha)}y_1^{(\alpha)}+c_2^{(\alpha)}y_2^{(\alpha)}=f(x).$$

Step(iv). Since y_1 and y_2 are solutions for (3.2), we get

$$c_1^{(\alpha)}y_1^{(\alpha)}+c_2^{(\alpha)}y_2^{(\alpha)}=f(x)\,. \tag{3.4}$$

Now we have to solve (3.3) and (3.4) to get

$$c_1^{(\alpha)}=-\frac{f(x)y_2(x)}{W^{\alpha}[y_1,y_2]}$$

and

$$c_2^{(\alpha)}=\frac{f(x)y_1(x)}{W^{\alpha}[y_1,y_2]}\,.$$

Thus

$$y_p(x)=\int_a^x\frac{\begin{vmatrix}y_1(t)&y_2(t)\\y_1(x)&y_2(x)\end{vmatrix}}{W^{\alpha}[y_1,y_2](t)}\,\frac{f(t)}{t^{1-\alpha}}\,dt\,,$$

that is to say,

$$
\begin{aligned}
y_p(x) &= I_\alpha^a \left[\frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W^\alpha[y_1, y_2](t)} f(t) \right] \\
&= I_\alpha^a \big(K(x,t) f(t) \big),
\end{aligned}
$$

where

$$
K(x,t) = \frac{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1(x) & y_2(x) \end{vmatrix}}{W^\alpha[y_1, y_2](t)}.
$$

Similarly we can consider the case of higher order linear fractional differential equations. Let $y_1,\ y_2,\ ...,\ y_n$ be linearly independent solutions of the homogeneous equation (2.2). As we have shown above, one can find a particular solution of the nonhomogeneous equation (2.1) of the form

$$
y_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x) + ... + c_n(x) y_n(x). \tag{3.5}
$$

One can apply the same steps in the previous process and obtain the following nonhomogeneous algebraic linear system of n equations for $c_1^{(\alpha)},\ c_2^{(\alpha)},\ ...,\ c_n^{(\alpha)}$

$$
\begin{aligned}
& y_1 c_1^{(\alpha)} + y_2 c_2^{(\alpha)} + ... + y_n c_n^{(\alpha)} = 0 \\
& y_1^{(\alpha)} c_1^{(\alpha)} + y_2^{(\alpha)} c_2^{(\alpha)} + ... + y_n^{(\alpha)} c_n^{(\alpha)} = 0 \\
& y_1^{(2\alpha)} c_1^{(\alpha)} + y_2^{(2\alpha)} c_2^{(\alpha)} + ... + y_n^{(2\alpha)} c_n^{(\alpha)} = 0 \\
& \vdots \\
& y_1^{((n-1)\alpha)} c_1^{(\alpha)} + y_2^{((n-1)\alpha)} c_2^{(\alpha)} + ... + y_n^{((n-1)\alpha)} c_n^{(\alpha)} = f.
\end{aligned}
$$

Using Cramer's rule we find

$$
c_m^{(\alpha)}(x) = \frac{f(x) W_m^\alpha(x)}{W^\alpha(x)}, \tag{3.6}
$$

where $W^\alpha(x) = W^\alpha(y_1,\ y_2,\ ...,\ y_n)(x)$ and W_m^α is the determinant obtained from W^α by replacing the mth column by the column (0, 0, ..., 0, 1). Thus

$$
y_p(x) = \sum_{m=1}^{n} y_m \int_a^x \frac{f(t) W_m^\alpha(t)}{W^\alpha(t) t^{1-\alpha}}\, dt, \tag{3.7}
$$

where a is an arbitrary positive constant.

4. Examples

Example 4.1. We first solve the following fractional differential equation

$$
D^{1/2}\left(D^{1/2} y\right) = \ln x, \quad x > 0. \tag{4.1}
$$

One can easily show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are two linearly independent solutions of the corresponding homogeneous equation

$$D^{1/2}\left(D^{1/2}y\right) = 0. \tag{4.2}$$

Let us find the particular solution $y_p(x)$ of equation (4.1) using the method of variation of parameters, precisely, we want to apply the formula

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 f(x)}{W^\alpha[y_1, y_2]x^{1-\alpha}}\,dx + y_2 \int \frac{y_1 f(x)}{W^\alpha[y_1, y_2]x^{1-\alpha}}\,dx \\ &= -y_1 \int \frac{y_2 f(x)}{W[y_1, y_2]x^{2-2\alpha}}\,dx + y_2 \int \frac{y_1 f(x)}{W[y_1, y_2]x^{2-2\alpha}}\,dx. \end{aligned}$$

Since $y_1 = 1$, $y_2 = \sqrt{x}$, $f(x) = \ln x$ and $W[y_1, y_2] = \dfrac{1}{2\sqrt{x}}$, we obtain

$$\begin{aligned} y_p(x) &= -\int \frac{\sqrt{x}\ln x}{\frac{1}{2\sqrt{x}}\,x}\,dx + \sqrt{x}\int \frac{\ln x}{\frac{1}{2\sqrt{x}}\,x}\,dx \\ &= -2\int \ln x\,dx + 2\sqrt{x}\int \frac{\ln x}{\sqrt{x}}\,dx \\ &= 2x\ln x - 6x\,. \end{aligned}$$

Thus, the general solution $y(x)$ can be written as

$$y(x) = c_1 + c_2\sqrt{x} + 2x\ln x - 6x\,.$$

Example 4.2. Consider the following fractional differential equation

$$xD^{1/2}\left(D^{1/2}y\right) - \frac{1}{2}y = x^{3/2} + x, \quad x > 0. \tag{4.3}$$

One can easily show that $y_1 = x$ and $y_2 = x^{-1/2}$ are two linearly independent solutions of the corresponding homogeneous equation

$$xD^{1/2}\left(D^{1/2}y\right) - \frac{1}{2}y = 0. \tag{4.4}$$

Let us find the particular solution $y_p(x)$ of equation (4.3) using the formula

$$y_p(x) = -y_1 \int \frac{y_2 f(x)}{W[y_1, y_2]x^{2-2\alpha}}\,dx + y_2 \int \frac{y_1 f(x)}{W[y_1, y_2]x^{2-2\alpha}}\,dx.$$

Substitute $y_1 = x$, $y_2 = x^{-1/2}$, $f(x) = x^{1/2} + 1$ and $W[y_1, y_2] = -\frac{3}{2}x^{-1/2}$ in the above equation, we get

$$\begin{aligned} y_p(x) &= -x\int \frac{x^{-1/2}(x^{1/2}+1)}{-\frac{3}{2}x^{-1/2}\,x}\,dx + x^{-1/2}\int \frac{x(x^{1/2}+1)}{-\frac{3}{2}x^{-1/2}\,x}\,dx \\ &= x^{3/2} + \frac{2}{3}\,x\ln x - \frac{4}{9}\,x. \end{aligned}$$

Thus the general solution of equation (4.3) can be written as

$$y(x) = c_1 x + c_2 x^{-1/2} + x^{3/2} + \frac{2}{3}\,x\ln x - \frac{4}{9}\,x.$$

Example 4.3. Let us solve the following fractional differential equation

$$D^{1/2}\left(D^{1/2}\left(D^{1/2}y\right)\right) = x^{5/2}. \tag{4.5}$$

One can verify that $y_1 = 1,\ y_2 = x$ and $y_3 = x^{1/2}$ are linearly independent solutions of the homogeneous equation

$$D^{1/2}\left(D^{1/2}\left(D^{1/2}y\right)\right) = 0.$$

By simple calculations, we can obtain

$$W^{1/2} = -\frac{1}{4}, \quad W_1^{1/2} = -\frac{1}{2}x, \quad W_2^{1/2} = -\frac{1}{2} \quad \text{and} \quad W_3^{1/2} = x^{1/2}.$$

Now, see (3.7),

$$\begin{aligned} y_p(x) &= y_1 \int \frac{W_1^{1/2} f(x)}{W^{1/2} x^{1/2}}\, dx + y_2 \int \frac{W_2^{1/2} f(x)}{W^{1/2} x^{1/2}}\, dx + y_3 \int \frac{W_3^{1/2} f(x)}{W^{1/2} x^{1/2}}\, dx \\ &= \int \frac{-\frac{1}{2} x x^{5/2}}{-\frac{1}{4} x^{1/2}}\, dx + x \int \frac{-\frac{1}{2} x^{5/2}}{-\frac{1}{4} x^{1/2}}\, dx + x^{1/2} \int \frac{x^{1/2} x^{5/2}}{-\frac{1}{4} x^{1/2}}\, dx \\ &= 2 \int x^3\, dx + 2x \int x^2\, dx - 4x^{1/2} \int x^{5/2}\, dx \\ &= \frac{1}{42} x^4. \end{aligned}$$

Thus the general solution of equation (4.5) is given by

$$y(x) = c_1 + c_2 x + c_3 x^{1/2} + \frac{1}{42} x^4.$$

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