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Oscillation conditions of the second-order noncanonical difference equations

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Abstract

We derive new oscillatory conditions for the second-order noncanonical difference equations of the type

$$\Delta(r(\nu)\Delta x(\nu)) + q(\nu)x(\nu+\sigma) = 0, \quad \nu \geqslant \nu_0,$$

by creating monotonical properties of nonoscillatory solutions. Our oscillatory outcomes are effectively an extension of the previous ones. We provide several examples to demonstrate the efficacy of the new criteria.

Keywords: Oscillation, nonoscillation, second-order, canonical, noncanonical, delay, difference equations.

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1. Introduction

There has recently been a surge in interest in deriving sufficient criteria for the oscillatory and non-oscillatory properties of solutions to various classes of difference equations; see, for example, the monographs [2, 4, 9] and the references cited therein. Many authors were concerned with the oscillatory conditions for first-order and higher-order difference equations; for example, [1, 3, 12, 13, 15, 18, 20]. Many applications of delay and advanced difference equations in various fields of science have attracted attention; one can refer to [7, 10, 14, 17].

In this paper, we discussed the following second-order noncanonical difference equations with a variable coefficient of the form

$$\Delta(r(\nu)\Delta x(\nu)) + q(\nu)x(\nu+\sigma) = 0, \quad \nu \geqslant \nu_0. \tag{1.1}$$

The forward difference operator Δ is defined by $\Delta x(\nu) = x(\nu+1) - x(\nu)$. The three constraints are presumed to be satisfied throughout the paper:

(C_1) $\{q(\nu)\}_{\nu=\nu_0}^{\infty}$ is a positive real sequence;

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(C_2) $\{r(\nu)\}_{\nu=\nu_0}^{\infty}$ is a positive real sequence;

(C_3) σ is an integer.

"Let ν_0 be a nonnegative integer that is fixed. A real sequence $\{x(\nu)\}$ defined for $\nu \geqslant \min\{\nu_0, \nu_0+\sigma\}$ and satisfying the equation (1.1) for $\nu \geqslant \nu_0$ is called a solution of (1.1). An oscillatory solution $\{x(\nu)\}$ of (1.1) is a solution of (1.1) if for every positive integer $N > 0$, there exists an integer $\nu \geqslant N$ with the property that $x(\nu)x(\nu+1) \leqslant 0$, otherwise $\{x(\nu)\}$ is said to be nonoscillatory. If every solution of the equation (1.1) is oscillatory, then the equation (1.1) is called oscillatory" [16].

We say that (1.1) is in the noncanonical form if

$$\theta(\nu) := \sum_{s=\nu}^{\infty} \frac{1}{r(s)} < \infty.$$

The goal of this paper is to present new difference inequalities that lead to new solution monotonicity properties, which can then be used to derive new oscillatory conditions for the delay and advanced difference equations. Arul et al. [5] discussed the difference equation

$$\Delta(p(\nu)\phi(\Delta x(\nu))) + f(\nu, x(\nu+1)) = 0, \quad \nu = 0,1,2,3,\dots, \tag{1.2}$$

and obtained oscillatory criteria for positive solutions of (1.2). Saker [19] investigated the equation

$$\Delta(p(\nu)\Delta x(\nu)) + q(\nu)f(x(\nu-\sigma)) = 0, \quad \nu = 0,1,2,3,\dots, \tag{1.3}$$

and established some sufficient conditions for oscillatory of every solution of (1.3). Li [11] examined the second-order non-linear difference equation

$$\Delta(p(\nu)g(\Delta x(\nu))) + q(\nu+1)f(x(\nu+1)) = 0, \quad \nu = 0,1,2,\dots, \tag{1.4}$$

and established oscillatory criteria for the solutions of (1.4). Zhang et al. [21] used the same methods as in [5] and obtained oscillation conditions for the equation.

$$\Delta(p(\nu)(\Delta x(\nu))^{\alpha}) + q(\nu+1)f(x(\nu+1)) = 0, \quad \nu = 0,1,2,\dots,$$

where σ is a positive quotient of odd integers. Grace et al. [8] established new oscillation conditions for all solutions of the following nonlinear second-order neutral difference equations

$$\Delta(a(\nu)(\Delta u(\nu))^{\alpha}) + b(\nu)y^{\gamma}(\nu-\tau+1) + c(\nu)y^{\mu}(\nu+\sigma+1) = 0$$

and

$$\Delta(a(\nu)(\Delta u(\nu))^{\alpha}) = b(\nu)y^{\gamma}(\nu-\tau+1) + c(\nu)y^{\mu}(\nu+\sigma+1),$$

where $u(\nu) = y(\nu) + q_1(\nu)y^{\beta}(\nu-k) - q_2(\nu)y^{\delta}(\nu-k)$.

This paper presents new difference inequalities that lead to new solution monotonicity properties, which are then used to derive new oscillatory conditions for the delay and advanced difference equations. We aim to find oscillatory conditions for all solutions of (1.1) to oscillate. Our derived results are the discrete analogues of the sharp results of [6].

In the following sections, for the sake of convenience, a functional inequality holds for all sufficiently large positive integer ν, when written without defining its domain of validity.

2. Some useful lemmas

We can easily see that the set of positive solutions to (1.1) has the following structure.

Lemma 2.1. *If $\{x(\nu)\}$ is an eventually positive solution of* (1.1)*, then $\{x(\nu)\}$ satisfies one of the following criteria:*

(A_1) $r(\nu)\Delta x(\nu) > 0$, $\Delta(r(\nu)\Delta x(\nu)) < 0$;
(A_2) $r(\nu)\Delta x(\nu) < 0$, $\Delta(r(\nu)\Delta x(\nu)) < 0$ *for* $\nu \geqslant \nu_1 \geqslant \nu_0$.

The following assertions will demonstrate that (A_2) is the most important class.

Lemma 2.2. *Assume that*

$$\sum_{s=\nu_0}^{\infty} \theta(s+1)q(s) = \infty \tag{2.1}$$

holds. Then every eventually positive solution $\{x(\nu)\}$ *of* (1.1) *satisfies* (A_2) *and, moreover*

(i) $\lim_{\nu\to\infty} x(\nu) = 0$;
(ii) $x(\nu) + r(\nu)\Delta x(\nu)\theta(x) \geqslant 0$;
(iii) $\left\{\frac{x(\nu)}{\theta(\nu)}\right\}$ *is increasing.*

Proof. On the contrary, assume that $\{x(\nu)\}$ is an eventually positive solution of (1.1) satisfying the condition (A_1) for $\nu \geqslant \nu_1 \geqslant \nu_0$. Summing (1.1) from ν_1 to ∞, we get

$$r(\nu_1)\Delta y(\nu_1) \geqslant \sum_{s=\nu_1}^{\infty} q(s)x(s+\sigma).$$

Since $\{x(\nu)\}$ is positive and increasing, there exists a positive constant k such that $x(\nu) \geqslant k$ and $x(\nu+\sigma) \geqslant k$ eventually. Then we have

$$r(\nu_1)\Delta x(\nu_1) \geqslant k\sum_{s=\nu_1}^{\infty} q(s) \geqslant \sum_{s=\nu_1}^{\infty} \theta(s+1)q(s),$$

which contradicts the equation (2.1) and we conclude that (A_2) is satisfying by $\{x(\nu)\}$. Consequently, we have $\lim_{\nu\to\infty} x(\nu) = l \geqslant 0$. We claim that l=0. If not, then $x(\nu) \geqslant l > 0$. A sum of (1.1) from ν_1 to $\nu-1$ yields

$$-r(\nu)\Delta x(\nu) \geqslant l\sum_{s=\nu_1}^{\nu-1} q(s).$$

Summing once more from ν_1+1 to ∞, we get

$$x(\nu_1+1) \geqslant l\sum_{u=\nu_1+1}^{\infty} \frac{1}{r(u)}\sum_{s=\nu_1}^{u-1} q(s) = l\sum_{s=\nu_1}^{\infty} \theta(s+1)q(s) = \infty,$$

which is a contradiction. Thus we conclude that $l = 0$.

We proceed as follows to prove part (ii). By the decreasing nature of $\{r(\nu)\Delta x(\nu)\}$, we have

$$x(\nu) \geqslant \sum_{s=\nu}^{\infty} \frac{-r(s)\Delta x(s)}{r(s)} \geqslant -r(\nu)\Delta x(\nu)\sum_{s=\nu}^{\infty} \frac{1}{r(s)} = -r(\nu)\Delta x(\nu)\theta(\nu),$$

which implies that part (iii) holds true. This completes the proof. □

We have not mentioned whether the equation (1.1) is a delay or an advanced difference equation in the previous results. However, in the following sections, we define oscillatory criteria for the delay and advanced difference equations individually.

3. Delay difference equation

We will assume throughout this section that (1.1) is a delay equation. That is

$$\sigma \leqslant -1. \tag{3.1}$$

For solutions of (1.1) from the class (A_2), we're about to create some new monotonic properties.

Lemma 3.1. *Assume that* (2.1) *and* (3.1) *hold. Assume further that there exists a* $\delta_0 > 0$ *such that*

$$q(\nu)\theta(\nu+1)\theta(\nu)r(\nu) \geqslant \delta_0 \tag{3.2}$$

eventually. If $\{x(\nu)\}$ *is a positive solution of* (1.1), *then*

(i) $\left\{\frac{x(\nu)}{\theta^{\delta_0}(\nu)}\right\}$ *is decreasing;*
(ii) $\lim_{\nu\to\infty} \frac{x(\nu)}{\theta^{\delta_0}(\nu)} = 0$;
(iii) $\left\{\frac{x(\nu)}{\theta^{1-\delta_0}(\nu)}\right\}$ *is increasing.*

Proof. Suppose that the equation (1.1) has an eventually positive solution $\{x(\nu)\}$. Then (2.1) ensures that $\{x(\nu)\}$ and $\{x(\nu+\sigma)\}$ satisfying the condition (A_2) for $\nu \geqslant \nu_1 \geqslant \nu_0$. A summation of (1.1) from ν_1 to $\nu - 1$ yields:

$$-r(\nu)\Delta x(\nu) = -r(\nu_1)\Delta x(\nu_1) + \sum_{s=\nu_1}^{\nu-1} q(s)x(s+\sigma) \geqslant -r(\nu_1)\Delta x(\nu_1) + x(\nu)\sum_{s=\nu_1}^{\nu-1} q(s),$$

which, because of (3.2), leads to

$$\begin{aligned} -r(\nu)\Delta x(\nu) &\geqslant -r(\nu_1)\Delta x(\nu_1) + \delta_0 x(\nu)\sum_{s=\nu_1}^{\nu-1} \frac{1}{\theta(s+1)\theta(s)r(s)} \\ &\geqslant -r(\nu_1)\Delta x(\nu_1) - \delta_0 x(\nu)\sum_{s=\nu_1}^{\nu-1} \frac{\Delta\theta(s)}{\theta(s+1)\theta(s)}. \end{aligned} \tag{3.3}$$

Now,

$$\sum_{s=\nu_1}^{\nu-1} \frac{\Delta\theta(s)}{\theta(s+1)\theta(s)} = -\frac{1}{\theta(\nu)} + \frac{1}{\theta(\nu_1)}. \tag{3.4}$$

Using the above inequality (3.4) in (3.3), we get

$$\begin{aligned} -r(\nu)\Delta x(\nu) &\geqslant -r(\nu_1)\Delta x(\nu_1) - \delta_0 x(\nu)\left(-\frac{1}{\theta(\nu)} + \frac{1}{\theta(\nu_1)}\right) \\ &= -r(\nu_1)\Delta x(\nu_1) + \delta_0\frac{x(\nu)}{\theta(\nu)} - \delta_0\frac{x(\nu)}{\theta(\nu_1)} \geqslant \delta_0\frac{x(\nu)}{\theta(\nu)}, \end{aligned} \tag{3.5}$$

where we have used that $x(\nu) \to 0$ as $\nu \to \infty$. Consequently,

$$\Delta\left(\frac{x(\nu)}{\theta^{\delta_0}(\nu)}\right) \leqslant \frac{\theta^{\delta_0-1}(\nu)\left[r(\nu)\Delta x(\nu)\theta(\nu) + \delta_0 x(\nu)\right]}{r(\nu)\theta^{\delta_0}(\nu)\theta^{\delta_0}(\nu+1)} \leqslant 0.$$

So, $\left\{\frac{x(\nu)}{\theta^{\delta_0}(\nu)}\right\}$ is decreasing and there exists $\lim_{\nu\to\infty}\frac{x(\nu)}{\theta^{\delta_0}(\nu)} = l \geqslant 0$. We assert that $l = 0$. If not, then $\frac{x(\nu)}{\theta^{\delta_0}(\nu)} \geqslant l > 0$ eventually. Let us introduce the sequence $\{z(\nu)\}$ defined by

$$z(\nu) = (r(\nu)\Delta x(\nu)\theta(\nu) + x(\nu))\theta^{-\delta_0}(\nu).$$

By Lemma 2.2 (ii), we have $z(\nu) > 0$ and

$$\Delta z(\nu) = \Delta(r(\nu)\Delta x(\nu))\theta^{1-\delta_0}(\nu) + \frac{\delta_0\Delta x(\nu)}{\theta^{\delta_0}(\nu+1)} + \frac{\delta_0 x(\nu)}{r(\nu)\theta(\nu)\theta^{\delta_0}(\nu+1)}$$

$$
\begin{aligned}
&= -q(\nu)x(\nu+\sigma)\theta^{1-\delta_0}(\nu+1) + \frac{\delta_0 \Delta x(\nu)}{\theta^{\delta_0}(\nu+1)} + \frac{\delta_0 x(\nu)}{r(\nu)\theta(\nu)\theta^{\delta_0}(\nu+1)} \\
&\leqslant \frac{-\delta_0 x(\nu+\sigma)\theta^{1-\delta_0}(\nu+1)}{\theta(\nu+1)\theta(\nu)r(\nu)} + \frac{\delta_0 \Delta x(\nu)}{\theta^{\delta_0}(\nu+1)} + \frac{\delta_0 x(\nu)}{r(\nu)\theta(\nu)\theta^{\delta_0}(\nu+1)} \\
&\leqslant -\frac{\delta_0 x(\nu+\sigma)}{r(\nu)\theta(\nu)\theta^{\delta_0}(\nu+1)} + \frac{\delta_0 \Delta x(\nu)}{\theta^{\delta_0}(\nu+1)} + \frac{\delta_0 x(\nu)}{r(\nu)\theta(\nu)\theta^{\delta_0}(\nu+1)} \\
&\leqslant \frac{\delta_0 \Delta x(\nu)}{\theta^{\delta_0}(\nu+1)}.
\end{aligned}
$$

Employing (3.5) and the fact that $x(\nu) \geqslant l\theta^{\delta_0}(\nu)$, we get that

$$\Delta z(\nu) \leqslant -\frac{{\delta_0}^2 l}{r(\nu)\theta(\nu)} < 0.$$

Summing the above inequalities from ν_1 to $\nu-1$, we have,

$$z(\nu_1) \geqslant \delta_0 l \ln \tfrac{\theta(\nu_1)}{\theta(\nu)} \to \infty \text{ as } \nu \to \infty,$$

which is a contradiction and we conclude that $\lim_{\nu\to\infty} \frac{x(\nu)}{\theta^{\delta_0}(\nu)} = 0$.

Finally, we shall show that $\left\{\frac{x(\nu)}{\theta^{1-\delta_0}(\nu)}\right\}$ is increasing. We can rewrite equation (1.1) in equivalent form

$$\Delta\left(r(\nu)\Delta x(\nu)\theta(\nu) + x(\nu)\right) + \theta(\nu+1)q(\nu)x(\nu+\sigma) = 0. \tag{3.6}$$

By Lemma 2.2 (iii), we see that $\left\{\frac{x(\nu)}{\theta(\nu)}\right\}$ is an increasing sequence. Summing the equation (3.6) from ν to ∞, we have

$$
\begin{aligned}
r(\nu)\Delta x(\nu)\theta(\nu) + x(\nu) &\geqslant \sum_{s=\nu}^{\infty} \theta(s+1)q(s)x(s+\sigma) \\
&\geqslant \sum_{s=\nu}^{\infty} \theta(s+1)q(s)x(s) \\
&\geqslant \sum_{s=\nu}^{\infty} \theta(s+1)\theta(s)q(s)\frac{x(s)}{\theta(s)} \\
&\geqslant \frac{x(\nu)}{\theta(\nu)} \sum_{s=\nu}^{\infty} \theta(s+1)\theta(s)q(s) \geqslant \frac{x(\nu)}{\theta(\nu)}\delta_0 \sum_{s=\nu}^{\infty} \frac{1}{r(\nu)} = x(\nu)\delta_0.
\end{aligned}
$$

From the last inequality, we get

$$\Delta\left(\frac{x(\nu)}{\theta^{1-\delta_0}(\nu)}\right) = \frac{\theta^{\delta_0}(\nu)[r(\nu)\Delta x(\nu)\theta(\nu) + x(\nu)(1-\delta_0)]}{r(\nu)\theta(\nu)\theta(\nu+1)} \geqslant 0.$$

This completes the proof. □

Lemma 3.1 provides

$$\left\{\tfrac{x(\nu)}{\theta^{\delta_0}(\nu)}\right\} \downarrow \text{ and } \left\{\tfrac{x(\nu)}{\theta^{1-\delta_0}(\nu)}\right\} \uparrow,$$

which ensures the oscillatory criterion that follows.

Theorem 3.2. *Assume that* (2.1), (3.1), *and* (3.2) *hold. If*

$$\delta_0 > \frac{1}{2},$$

then every solution of (1.1) *is oscillatory.*

If $\delta_0 \leqslant \frac{1}{2}$, then the results presented in Lemma 3.1 can be improved. Since $\{\theta(\nu)\}$ is a decreasing sequence, there is a constant $\lambda \geqslant 1$ that satisfies

$$\frac{\theta(\nu+\sigma)}{\theta(\nu)} \geqslant \lambda. \tag{3.7}$$

We introduce the constant $\delta_1 > \delta_0$ as follows

$$\delta_1 = \frac{\lambda^{\delta_0}\delta_0}{1-\delta_0}.$$

Lemma 3.3. *Assume that (2.1), (3.1), and (3.2) hold. If $\{x(\nu)\}$ is a positive solution of (1.1), then*

(i) $\left\{\frac{x(\nu)}{\theta^{\delta_1}(\nu)}\right\}$ *is decreasing;*

(ii) $\lim_{\nu\to\infty} \frac{x(\nu)}{\theta^{\delta_1}(\nu)} = 0$;

(iii) $\left\{\frac{x(\nu)}{\theta^{1-\delta_1}(\nu)}\right\}$ *is increasing.*

Proof. Let us suppose that the equation (1.1) has an eventually positive solution $\{x(\nu)\}$ satisfying the condition (A_2) for $\nu \geqslant \nu_1 \geqslant \nu_0$. Summing (1.1) from ν_1 to $\nu-1$ and using the decreasing nature of $\left\{\frac{x(\nu)}{\theta^{\delta_0}(\nu)}\right\}$, we get

$$\begin{aligned}-r(\nu)\Delta x(\nu) &\geqslant -r(\nu_1)\Delta x(\nu_1) + \sum_{s=\nu_1}^{\nu-1} \frac{q(s)x(s)\theta^{\delta_0}(s+\sigma)}{\theta^{\delta_0}(s)} \\ &\geqslant -r(\nu_1)\Delta x(\nu_1) + \frac{x(\nu)}{\theta^{\delta_0}(\nu)} \sum_{s=\nu_1}^{\nu-1} q(s)\theta^{\delta_0}(s+\sigma),\end{aligned}$$

which, because of (3.7), implies

$$-r(\nu)\Delta x(\nu) \geqslant -r(\nu_1)\Delta x(\nu_1) + \frac{\lambda^{\delta_0}\delta_0 x(\nu)}{\theta^{\delta_0}(\nu)} \sum_{s=\nu_1}^{\nu-1} \frac{\theta^{\delta_0-2}(s)}{r(s)}.$$

Evaluating the sum, we see that

$$-r(\nu)\Delta x(\nu) \geqslant -r(\nu_1)\Delta x(\nu_1) - \delta_1\theta^{\delta_0-1}(\nu_1)\frac{x(\nu)}{\theta^{\delta_0}(\nu)} + \delta_1\frac{x(\nu)}{\theta(\nu)}.$$

Since $\frac{x(\nu)}{\theta^{\delta_0}(\nu)} \to 0$ as $\nu \to \infty$, we see that

$$-r(\nu)\Delta x(\nu) \geqslant \delta_1\frac{x(\nu)}{\theta(\nu)}.$$

By using the same procedure as we used in the proof of Lemma 3.1, we obtain $\left\{\frac{x(\nu)}{\theta^{\delta_1}(\nu)}\right\}$ is decreasing. We can verify the remaining assertions by following the same steps as in the proof of Lemma 3.1. □

If $\delta_1 < 1$, repeating the above procedure and introducing $\delta_2 > \delta_1$ follows

$$\delta_2 = \delta_0\frac{\lambda^{\delta_1}}{1-\delta_1}.$$

In general, as follows as $\delta_j < 1$ for $j = 1, 2, \dots, k-1$, we can define

$$\delta_k = \delta_0 \frac{\lambda^{\delta_{k-1}}}{1-\delta_{k-1}}, \tag{3.8}$$

provided that $\delta_k < 1$. We can verify that by following the steps in the proof of Lemma 3.3, $\left\{\frac{x(\nu)}{\theta^{\delta_k}(\nu)}\right\} \downarrow$ and $\left\{\frac{x(\nu)}{\theta^{1-\delta_k}(\nu)}\right\} \uparrow$. Consequently, the following theorem is obvious.

Theorem 3.4. *Assume that* (2.1), (3.1), (3.2), *and* (3.8) *hold. If there exists a* $k \in \mathbb{N}$ *such that*

$$\liminf_{\nu\to\infty} \sum_{s=\nu+\sigma}^{\nu-1} \frac{q(s)\theta(s+1)}{1-\delta_k} > \left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}, \tag{3.9}$$

then every solution of (1.1) *is oscillatory.*

Proof. Assume, on the contrary, that (1.1) possesses an eventually positive solution $\{x(\nu)\}$. Condition (2.1) guarantees that $\{x(\nu)\}$ satisfies the condition (A_2). Define a sequence $\{\delta_k\}$ by (3.8). Consider the sequence $\{w(\nu)\}$ given by

$$w(\nu) = r(\nu)\Delta x(\nu)\theta(\nu) + x(\nu).$$

From Lemma 2.2 (ii), we have $w(\nu) > 0$ and, moreover,

$$\Delta w(\nu) = r(\nu)\Delta x(\nu)\Delta\theta(\nu) + \theta(\nu+1)\Delta(r(\nu)\Delta x(\nu)) + \Delta x(\nu) = -\theta(\nu+1)q(\nu)x(\nu+\sigma). \tag{3.10}$$

Since $\left\{\frac{x(\nu)}{\theta^{\delta_k}(\nu)}\right\}$ is decreasing, then we attain $r(\nu)\Delta x(\nu)\theta(\nu) + \delta_k x(\nu) \leqslant 0$ and so

$$w(\nu) \leqslant (1-\delta_k)x(\nu).$$

Putting the last inequality into (3.10), it is clear that $\{w(\nu)\}$ is a positive solution of the following delay difference inequality of first order

$$\Delta w(\nu) + \frac{q(\nu)\theta(\nu+1)}{1-\delta_k} w(\nu+\sigma) \leqslant 0. \tag{3.11}$$

According to the Theorem in [9], conditions (3.9) guarantees that the difference inequality (3.11) has no positive solution. This contradiction leads to the completion of the proof. □

4. Advanced difference equation

The method described above can be adapted to work with advanced difference equations, such as when

$$\sigma \geqslant 1. \tag{4.1}$$

The key constant δ_0 is slightly changed into λ_0 as follows.

Lemma 4.1. *Assume that* (2.1) *and* (4.1) *hold. Assume further that there exists a* $\lambda_0 > 0$ *with the property that*

$$q(\nu)\theta(\nu+1)\theta(\nu+\sigma)r(\nu) \geqslant \lambda_0, \tag{4.2}$$

eventually. If $\{x(\nu)\}$ *is a positive solution of* (1.1), *then*

(i) $\left\{\frac{x(\nu)}{\theta^{\lambda_0}(\nu)}\right\}$ *is decreasing;*

(ii) $\lim_{\nu\to\infty} \frac{x(\nu)}{\theta^{\lambda_0}(\nu)} = 0;$

(iii) $\left\{\frac{x(\nu)}{\theta^{1-\lambda_0}(\nu)}\right\}$ *is increasing.*

Proof. Assume that $\{x(\nu)\}$ is an eventually positive solution of (1.1). Then (2.1) ensures that $\{x(\nu)\}$ satisfies the condition (A_2) for $\nu \geqslant \nu_1 \geqslant \nu_0$. By Lemma 2.2 (iii),

$$\frac{x(\nu+\sigma)}{\theta(\nu+\sigma)} \geqslant \frac{x(\nu)}{\theta(\nu)}.$$

A summation of (1.1) from ν_1 to $\nu-1$ yields

$$\begin{aligned}-r(\nu)\Delta x(\nu) &\geqslant -r(\nu_1)\Delta x(\nu_1) + \lambda_0 x(\nu)\sum_{s=\nu_1}^{\nu-1}\frac{1}{\theta^2(s)r(s)}\\ &= -r(\nu_1)\Delta x(\nu_1) + \lambda_0\frac{x(\nu)}{\theta(\nu)} - \lambda_0\frac{x(\nu)}{\theta(\nu_1)} \geqslant \lambda_0\frac{x(\nu)}{\theta(\nu)},\end{aligned}$$

where we have used that $x(\nu) \to 0$ as $\nu \to \infty$. Therefore,

$$\Delta\left(\frac{x(\nu)}{\theta^{\lambda_0}(\nu)}\right) \leqslant 0.$$

Applying the same procedure as we followed in the proof of the Lemma 3.1, the parts (ii) and (iii) can be proved. Hence the proof. □

Assuming that $\lambda_0 < 1$ we can introduce the constant $\lambda_1 > \lambda_0$ as follows. Since $\{\theta(\nu)\}$ is decreasing, there exists a constants $\eta \geqslant 1$ such that

$$\frac{\theta(\nu)}{\theta(\nu+\sigma)} \geqslant \eta$$

and hence

$$\lambda_1 = \lambda_0\frac{\eta^{\lambda_0}}{1-\lambda_0}.$$

In general, as far as $\lambda_{k-1} < 1$, we can define

$$\lambda_k = \lambda_0\frac{\eta^{\lambda_{k-1}}}{1-\lambda_{k-1}} \tag{4.3}$$

and verify that

$$\left\{\frac{x(\nu)}{\theta^{\lambda_k}(\nu)}\right\}\downarrow \text{ and } \left\{\frac{x(\nu)}{\theta^{1-\lambda_k}(\nu)}\right\}\uparrow.$$

We can establish the following oscillatory criteria for advanced difference equations in the same way as we did in the "delay" section.

Theorem 4.2. *Assume that* (2.1), (4.1), (4.2), *and* (4.3) *hold. If there exists an integer* $k \in \mathbb{N}$ *such that*

$$\lambda_k > \frac{1}{2},$$

then every solution of (1.1) *is oscillatory.*

Theorem 4.3. *Assume that (2.1),(4.1),(4.2) and (4.3) holds. If there exists a* $k \in \mathbb{N}$ *such that*

$$\liminf_{\nu\to\infty}\sum_{s=\nu+1}^{\nu+\sigma}\frac{q(s)\theta(s+1)}{1-\lambda_k} > \left(\frac{\sigma-1}{\sigma}\right)^{\sigma}$$

then every solution of (1.1) is oscillatory.

5. Ordinary difference equation

The results proved above can be applied for ordinary difference equation $(\sigma = 0)$,

$$\Delta(r(\nu)\Delta x(\nu)) + q(\nu)x(\nu) = 0. \tag{5.1}$$

Now the sequences $\{\delta_k\}$ and $\{\lambda_k\}$ are identical and defined by

$$\delta_k = \frac{\delta_0}{1 - \delta_{k-1}} \tag{5.2}$$

with δ_0 adjusted in (3.2). Theorems 3.4 and 4.2 can be reduced to the following.

Theorem 5.1. *Assume that* (2.1) *and* (5.2) *hold. If there exists an integer* $k \in \mathbb{N}$ *such that*

$$\delta_k > \frac{1}{2},$$

then every solution of (5.1) *is oscillatory.*

6. Some examples

Example 6.1. Consider the following second-order noncanonical delay difference equation of the form

$$\Delta\left(a^\nu \Delta x(\nu)\right) + a^{\nu-1}x(\nu-1) = 0; \quad \nu = 1, 2, \ldots, \tag{6.1}$$

where $a > 1$.

We have $r(\nu) = a^\nu$, $q(\nu) = a^{\nu-1}$, $\sigma = -1$, and

$$\theta(\nu) := \frac{1}{a^{\nu-1}(a-1)}.$$

Choose $\delta_0 = \frac{1}{(a-1)^2}$. Then $0 < \delta_0 < \frac{1}{2}$ for $1 < a < 1 + \sqrt{2}$. Hence all the conditions of the Theorem 3.2 are satisfied for $1 < a < 1 + \sqrt{2}$. Thus, all the solutions of the equation (6.1) are oscillatory for $1 < a < 1 + \sqrt{2}$.

Example 6.2. Let us consider the following second-order noncanonical delay difference equation

$$\Delta\left(2^\nu \Delta x(\nu)\right) + 2^{\nu-3}x(\nu-1) = 0; \quad \nu = 1, 2, 3, \ldots. \tag{6.2}$$

Here $r(\nu) = 2^\nu$, $q(\nu) = 2^{\nu-3}$, $\sigma = -1$, and $\theta(\nu) = \frac{1}{2^{\nu-1}}$. Choose $\delta_0 = \frac{1}{4}$. Then $\lambda = 2$ and $\lambda_1 = \delta_0 \frac{\lambda^{\delta_0}}{1-\delta_0} = \frac{2^{\frac{1}{4}}}{3}$. Clearly

$$\sum_{s=\nu+\sigma}^{\nu-1} \frac{q(s)\theta(s+1)}{1-\delta_1} = 0.414183205,$$

which is greater than $\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma} = 0.25$. Then all the constraints of the Theorem 3.4 are verified and hence the equation (6.2) is oscillatory.

Example 6.3. Let us consider the following second-order noncanonical advanced difference equation of the form

$$\Delta\left(2^\nu \Delta x(\nu)\right) + \frac{2^\nu}{3}x(\nu+1) = 0; \quad \nu = 0, 1, 2, \ldots. \tag{6.3}$$

Here, $r(\nu) = 2^\nu$, $q(\nu) = \frac{2^\nu}{3}$, $\sigma = 1$. We can easily show that $\theta(\nu) = \frac{1}{2^{\nu-1}}$. Choose $\lambda_0 = \frac{1}{3}$. Then $\eta = 2$. Now, $\lambda_1 = \frac{\lambda_0 \eta^{\lambda_0}}{1-\lambda_0} = 0.629961 > \frac{1}{2}$. Clearly $\frac{1}{2} < \lambda_1 < 1$. Thus, all the constraints of the Theorem 4.2 are verified. Hence the equation (6.3) is oscillatory.

Example 6.4. Let us consider the following second-order ordinary difference equation

$$\Delta\left(2^\nu \Delta x(\nu)\right) + \frac{2^\nu}{5}x(\nu) = 0; \quad \nu = 0, 1, 2, \ldots. \tag{6.4}$$

Here, $r(\nu) = 2^\nu$, $q(\nu) = \frac{2^\nu}{5}$ and $\delta_0 = \frac{2}{5}$. Clearly $\delta_1 = \frac{\delta_0}{1-\delta_0} = \frac{2}{3} > \frac{1}{2}$. Hence all the constraints of the Theorem 5.1 are verified. Thus the equation (6.4) is oscillatory.

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