Ekeland’s variational principle in complete quasi-G-metric spaces

E. Hashemi\textsuperscript{a}, M. B. Ghaemi\textsuperscript{b,}* \\
\textsuperscript{a}Department of Mathematics, College of Basic Sciences, Karaj Branch, Islamic Azad University, Alborz, Iran. \\
\textsuperscript{b}Department of Mathematics, Iran University of Science and Technology, Tehran, Iran.

Abstract

In this paper, by concept of $\Gamma$-function which is define on q-G-m (quasi-G-metric) space, we establish a generalized Ekeland’s variational principle in the setting of lower semicontinuous from above. As application we prove generalized flower petal theorem in q-G-m.

Keywords: $\Gamma$-Function, q-G-m space, generalized EVP, lower semicontinuous from above function, generalized Caristi’s (common) fixed point theorem, nonconvex minimax theorem, generalized flower petal theorem.

2010 MSC: 54H25, 54C60.

1. Introduction

EVP was first studied in 1972. Many equivalents have been found by scholars over the years for primitive EVP\cite{10, 11}, see \cite{2–7, 17, 18, 20, 22}. Interesting applications in various fields of applied mathematics are found. A number of generalized of these results have been reviewed by other researchers \cite{1–4, 8, 12–16, 23–30}.

2. Ekeland’s variational principle

In this paper, $\theta : (-\infty, \infty) \rightarrow (0, \infty)$ is a nondecreasing function, a function $g : U \rightarrow (-\infty, \infty)$ is said to be lower semicontinuous from above (shortly Lsca) at $r_0$, when for each sequence $\{r_n\}$ in $U$ such that $r_n \rightarrow r_0$ and $g(r_1) \geq g(r_2) \geq \cdots \geq g(r_n) \geq \cdots$, we have $g(r_0) \leq \lim_{n \rightarrow \infty} g(r_n)$. The function $g$ is said to be Lsca on $U$, when $g$ is Lsca at every point of $U$, $g$ is proper when $h \not\equiv \infty$.

Theorem 2.1 (\cite{9, Ekeland theorem}). Let $U$ be a complete metric space with meter $d$, $g : U \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, semicontinuous, and bounded below function. Then there exists $v \in U$ such that $g(v) \leq g(u)$, $d(u, v) \leq 1$, and $g(w) > g(v) - \epsilon d(v, w)$ for all $v \neq w$.

*Corresponding author

Email addresses: ehhagh_hashemi@yahoo.com (E. Hashemi), mghaemi@iust.ac.ir (M. B. Ghaemi)

doi: 10.22436/jnsa.012.03.06

Received: 2018-01-19  Revised: 2018-09-28  Accepted: 2018-10-25
Definition 2.2 ([19]). Assume that $U$ is a nonempty set and mapping

$$G : U \times U \times U \longrightarrow [0, \infty)$$

is satisfying the following conditions:

(i) $G(r, s, t) = 0$ if $r = s = t$;
(ii) $G(r, r, s) > 0$ for all $r, s \in U$, where $r \neq s$;
(iii) $G(r, r, t) \leq G(r, s, t)$ for all $r, s, t \in U$ with $r \neq t$;
(iv) $G(r, s, t) = G(p(r, s, t))$ such that $p$ is a permutation of $r, s, t$;
(v) $G(r, s, t) \leq G(r, \alpha, \alpha) + G(\alpha, s, t)$ for all $r, s, t, \alpha$ in $U$.

Then $G$ is said to be $G$-metric and pair $(U, G)$ is said to be $G$-metric space.

Definition 2.3 ([19]). Let $(U, G)$ be a $G$-metric space. A sequence $\{r_n\}$ in $U$ is said to be

(a) $G$-Cauchy sequence if for all $\epsilon > 0$, there exists $q_0 \in \mathbb{N}$ such that for every $p, q, l \in \mathbb{N}$ and $p, q, l \geq q_0$ then $G(r_{q_0}, r_{q+p}, r_{q+l}) < \epsilon$;
(b) $G$-convergent to $r \in U$ if for all $\epsilon > 0$, there exists natural number $q_0$ such that for all $p, q \geq q_0$, then $G(r_{q_0}, r_{p+q}, r_p) < \epsilon$.

Proposition 2.4 ([19]). Assume that $(U, G)$ is a $G$-metric space, then the following statements are equivalent:

(a) $\{r_n\}$ is a $G$-causly sequence;
(b) for each $\epsilon > 0$, there exists natural number $q_0$ such that for all $p, q \geq q_0$, then $G(r_{q_0}, r_{q+p}, r_p) < \epsilon$.

Definition 2.5. A function $\sigma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is subadditive when $\sigma(r + s) \leq \sigma(r) + \sigma(s)$, and $\sigma(\epsilon r) = \epsilon \sigma(r)$ for every $\epsilon > 0$.

Definition 2.6. Let $U$ be a nonempty set. A function

$$G : U \times U \times U \longrightarrow [0, \infty)$$

is said to be quasi-$G$-metric ($q$-$G$-m) if the following conditions be satisfied

1. $G(r, s, t) = 0$ if $r = s = t$;
2. $G(r, r, s) > 0$ for all $r, s \in U$, $r \neq s$;
3. $G(r, r, t) \leq G(r, s, t)$ for all $r, s, t \in U$, $t \neq s$;
4. $G(r, s, t) \leq G(r, \epsilon, \epsilon) + G(\epsilon, s, t)$ for all $r, s, t, \epsilon \in U$.

$(U, G)$ is said to be $q$-$G$-m space when $U$ is a nonempty set and $G$ is a $q$-$G$-m. The concept of Cauchy sequence, convergence, and complete space are defined as $G$-metric space.

Definition 2.7. Let $(U, G)$ be a $q$-$G$-m space. A function $\Gamma : U \times U \times U \longrightarrow [0, \infty)$ is said to be $\Gamma$-function when

1. $\Gamma(r, s, t) \leq \Gamma(r, \epsilon, \epsilon) + \Gamma(\epsilon, s, t)$ for all $r, s, t, \epsilon \in U$;
2. if $r \in U$, $(s_n)_{n \in \mathbb{N}}$ be a sequence in $U$ which is convergent to $s$ in $U$ and $\Gamma(r, s, s_n) \leq M$, then $\Gamma(r, s, s_n) \leq M$;
3. for every $\epsilon > 0$, there exists $\delta > 0$ such that $\Gamma(r, \epsilon, \epsilon) \leq \delta$ and $\Gamma(\epsilon, s, t) \leq \delta$ imply $G(r, s, t) \leq \epsilon$.

Example 2.8 ([21]). Let $(U, d)$ be a metric space and $G : U^3 \longrightarrow [0, \infty)$ defined by $G(r, s, t) = \max\{d(r, s), d(r, t), d(s, t)\}$ for all $r, s, t \in U$. Then $\Gamma = G$ is a $\Gamma$-function on $U$.

Example 2.9. Assume that

$$G : U^3 \longrightarrow [0, \infty), \quad G(r, s, t) = \frac{1}{3}(|t - r| + |r - s|)$$

is a function, then $G$ is a $q$-$G$-m but isn’t $G$ metric.
Proof. q-G-m is obvious. We show that \( G(r, s, t) \neq G[p(r, s, t)] \) (p is a permutation of \( r, s, t \)). Since
\[
G(3, 5, 2) = \frac{1}{3}(|2 - 3| + |3 - 5|) = 1, \quad G(2, 3, 5) = \frac{1}{3}(|3 - 2| + |5 - 2|) = \frac{4}{3},
\]
then \( G \) is not a G-metric.

Example 2.10. Let \( G(r, s, t) \) be the same as in the previous example. Then \( \Gamma = G \) is a \( \Gamma \)-function.

Proof. (a) and (b) are obvious. Let \( \epsilon > 0 \) be given, put \( \delta = \frac{\epsilon}{2} \) if \( \Gamma(r, \epsilon, \epsilon) = \frac{1}{2}(|t - \epsilon| + |\epsilon - s|) < \frac{\epsilon}{2} \), then
\[
G(r, s, t) = \frac{1}{3}(|t - s| + |r - s|) \leq \frac{1}{3}(|t - \epsilon| + |\epsilon - s| + |r - \epsilon| + |\epsilon - s|) < \epsilon.
\]

So, (c) is established.

Lemma 2.11 ([21]). Assume that \( (U, G) \) is a G-metric space and \( \Gamma \) is a \( \Gamma \)-function on \( U \). Let \( \{u_n\} \) and \( \{v_n\} \) be two sequences in \( U \), \( \{\rho_n\} \) and \( \{\phi_n\} \) be in \([0, \infty), \) which are convergent to zero. Let \( u, v, w, \epsilon \in U \), then

1) if \( \Gamma(v, u_n, u_n) \leq \rho_n \) and \( \Gamma(u_n, v, w) \leq \phi_n \) for all \( n \in \mathbb{N} \), then \( G(v, w) < \epsilon \) and hence \( w = v; \)
2) if \( \Gamma(u_m, u_n, u_n) \leq \rho_n \) and \( \Gamma(u_n, u_m, w) \leq \phi_n \) for every \( m > n \), then \( G(v_n, v_m, w) \) is convergent to zero and hence \( v_n \rightarrow w; \)
3) if \( \Gamma(u_m, u_n, u_l) \leq \rho_n \) for all \( m, n, l \in \mathbb{N} \) with \( n \leq m \leq l \), then \( \{u_n\} \) is a G-Cauchy sequence;
4) if \( \Gamma(u_n, \epsilon, \epsilon) \leq \rho_n \) for all \( n \in \mathbb{N} \), then \( \{u_n\} \) is a G-Cauchy sequence.

Lemma 2.12. Let \( \Gamma \) be a \( \Gamma \)-function on \( U \times U \times U \). If sequence \( \{r_n\} \) be in \( U \) that \( \limsup_{n \rightarrow \infty} \Gamma(r_n, r_m, r_l) = 0 \) if \( n \leq m \leq l \), then \( \{r_n\} \) will be a G-Cauchy sequence in \( U \).

Proof. Assume \( \rho_n = \sup\{\Gamma(r_n, r_m, r_l)\}, n \leq m \leq l \), then \( \lim_{n \rightarrow \infty} \rho_n = 0 \). By Lemma 2.11 (3), \( \{r_n\} \) is a G-Cauchy sequence.

Lemma 2.13. Let \( g : U \rightarrow [-\infty, \infty] \) be a function and \( \Gamma \) be a \( \Gamma \)-function on \( U \times U \times U \). The set \( P(r) \) is defined by
\[
P(r) = \{s \in U; s \neq r, \Gamma(r, s, s) \leq 0(g(r))(g(r) - g(s))\}.
\]
If \( P(r) \) be nonempty, then for every \( s \in P(r) \), we will have\[P(s) \subseteq P(r) \text{ and } g(s) \leq g(r).
\]

Proof. Let \( s \in P(r) \). So \( s \neq r \) and \( \Gamma(r, s, s) \leq 0(g(r))(g(r) - g(s)) \). Since \( \Gamma(r, s, s) \geq 0 \) and \( 0 \) is nondecreasing and positive function, then \( g(r) \geq g(s) \). If \( P(s) = \emptyset \) then \( P(s) \subseteq P(r) \). Therefore \( t \neq s \) and \( \Gamma(s, t, t) \leq 0(g(s))(g(s) - g(t)) \) as above \( g(s) \geq g(t) \). Since \( \Gamma \) be a \( \Gamma \)-function, then
\[
\Gamma(r, t, t) \leq \Gamma(r, s, s) + \Gamma(s, t, t) \leq 0(g(r))(g(r) - g(t)).
\]
We claim that \( t \neq r \). Assume that \( t = r \) so \( \Gamma(r, t, t) = 0 \). On the other hand
\[
\Gamma(r, s, s) \leq 0(g(r))(g(r) - g(s)) \leq 0(g(r))(g(r) - g(t)) = 0 \Rightarrow \Gamma(r, s, s) = 0,
\]
then \( \Gamma(r, s, s) = 0 \). For every \( \epsilon > 0 \), we have \( \Gamma(r, t, t) = 0 < \epsilon \) and \( \Gamma(t, s, s) = 0 < \epsilon \) then by definition \( \Gamma \)-function, we have \( G(t, s, s) < \epsilon \), so \( G(t, s, s) = 0 \) and \( t = s \). This is a contradiction, therefore \( t \in P(r) \) and \( P(s) \subseteq P(r) \).

Proposition 2.14. Assume that \( (U, G) \) is a complete q-G-m space and \( g : U \rightarrow [-\infty, \infty] \) is a proper and bounded below function, \( \Gamma \) is a \( \Gamma \)-function on \( U \times U \times U \). Let
\[
P(r) = \{s \in U; s \neq r, \Gamma(r, s, s) \leq 0(g(r))(g(r) - g(s))\}.
\]
Let \( \{r_n\} \) be a sequence in \( U \) such that \( P(r_n) \) be nonempty and for all \( n \in \mathbb{N}, r_{n+1} \in P(r_n) \). Then, there exists...
\(r_0 \in U\) such that \(r_n \longrightarrow r_0\) and \(r_0 \in \bigcap_{n=1}^{\infty} P(r_n)\). Also if for every \(n \in \mathbb{N}\), we have \(g(r_{n+1}) \leq \inf_{t \in P(r_n)} g(t) + \frac{1}{n}\), then \(\bigcap_{n=1}^{\infty} P(r_n)\) will only has one member.

**Proof.** At first we prove that \(\{r_n\}\) is a Cauchy sequence by Lemma 2.13, \(g(r_n) \geq g(r_{n+1})\) for all \(n \in \mathbb{N}\). Therefore \(\{g(r_n)\}\) is nonincreasing. On the other hand \(g\) is bounded below then \(\lim_{n \to \infty} g(r_n) = u\), and \(g(r_n) \geq u\) for all \(n \in \mathbb{N}\). We claim that

\[
\limsup_{n \to \infty} \Gamma(r_n, r_m, r_m) : m > n = 0.
\]

We have

\[
\Gamma(r_n, r_m, r_m) \leq \Gamma(r_n, r_{n+1}, r_{n+1}) + \Gamma(r_{n+1}, r_m, r_m)
\]

\[
\leq \Gamma(r_n, r_{n+1}, r_{n+1}) + \Gamma(r_{n+1}, r_{n+2}, r_{n+2}) + \cdots + \Gamma(r_{m-1}, r_m, r_m),
\]

then

\[
\Gamma(r_n, r_m, r_m) \leq \sum_{j=n}^{m-1} \Gamma(r_n, r_j, r_{j+1}) \leq \theta(g(r_n))(g(r_n) - g(r_0))
\]

for all \(m, n \in \mathbb{N}\) with \(m > n\).

Put \(\rho_n = \theta(g(r_n))(g(r_n) - u)\), then \(\sup_{m > n} \Gamma(r_n, r_m, r_m) \leq \rho_n\) for all \(n \in \mathbb{N}\). Since \(\lim_{n \to \infty} g(r_n) = u\), we result

\[
\limsup_{n \to \infty} \Gamma(r_n, r_m, r_m) : m > n = 0
\]

and \(\lim_{n \to \infty} \rho_n = 0\). By Lemma 2.12, \(\{u_n\}\) is a G-Cauchy sequence. Then, there exists \(r_0 \in U\) such that \(r_n \to u_0\). We show that \(r_0 \in \bigcap_{n=1}^{\infty} P(r_n)\). Since \(g\) is Lsca, then \(g(r_0) \leq \lim_{n \to \infty} g(r_n) = u \leq g(r_k)\).

Let \(n \in \mathbb{N}\), we have

\[
\Gamma(r_n, r_m, r_m) \leq \sum_{j=n}^{m-1} \Gamma(r_j, r_{j+1}, r_{j+1}) \leq \theta(g(r_n))(g(r_n) - g(r_0))
\]

for all \(m \in \mathbb{N}\) with \(m > n\). By Definition 2.7 (2), we have

\[
\Gamma(r_n, r_0, r_0) \leq \theta(g(r_n))(g(r_n) - g(r_0))
\]

for all \(n \in \mathbb{N}\). Also \(r_0 \neq r\) for all \(n \in \mathbb{N}\), suppose it is not, then there exists \(j \in \mathbb{N}\) such that \(r_0 = r_j\). Since

\[
\Gamma(r_j, r_{j+1}, r_{j+1}) = \theta(g(r_j))(g(r_j) - g(r_{j+1})) \leq \theta(g(r_j))(g(r_j) - g(r_0)) = 0,
\]

then we have \(\Gamma(r_j, r_{j+1}, r_{j+1}) = 0\) and in the same way

\[
\Gamma(r_{j+1}, r_{j+2}, r_{j+2}) = 0.
\]

Now assume \(\epsilon > 0\), \(\Gamma(r_j, r_{j+1}, r_{j+1}) = 0 < \delta\), and \(\Gamma(r_{j+1}, r_{j+2}, r_{j+2}) = 0 < \delta\). Therefor by Definition 2.7 (3) we get it \(G(r_j, r_{j+2}, r_{j+2}) < \epsilon\). Then \(r_j = r_{j+2}\) that is a contradiction because of \(r_j \neq r_{j+2}\). Since \(r_{j+1} \in P(r_j)\), then \(P(r_{j+1}) \subseteq P(r_j)\) and \(r_{j+2} \in P(r_{j+1})\). So \(r_{j+2} \in P(r_j)\). We suppose \(r_{j+2} \neq r_j\) for all \(n \in \mathbb{N}\).

We have \(r_0 \in \bigcap_{n=1}^{\infty} P(r_n)\), then \(\bigcap_{n=1}^{\infty} P(r_n) \neq \emptyset\). Let \(g(r_{n+1}) \leq \inf_{t \in P(r_n)} g(t) + \frac{1}{n}\) for all \(r_0 \neq r_n\). We show that

\[
\bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}.\]

Assume that \(w \in \bigcap_{n=1}^{\infty} P(r_n)\), then

\[
\Gamma(r_n, w, w) \leq \theta(g(r_n))(g(r_n) - g(w)) \leq \theta(g(r_1))(g(r_1) - g(r_n)) + \frac{1}{n} \leq \theta(g(r_1))(g(r_1) - g(r_{n+1}) + \frac{1}{n}).
\]
Let 
\[
\varphi_n = \theta(g(r_1))(g(r_n) - g(r_{n+1}) + \frac{1}{n})
\]
for all \(n \in \mathbb{N}\), then \(\lim_{n \to \infty} \varphi_n = 0\), we get it \(\lim_{n \to \infty} \Gamma(r_n, w, w) = 0\). On the other hand \(\{r_m\}\) is a G-Cauchy sequence. Then \(\lim_{n \to \infty} \Gamma(r_m, r_m, r_n) = 0\) and we get it \(r_n \to \infty\), by uniqueness \(w = r_0\). Then 
\[
\bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}.
\]
□

**Theorem 2.15** (Generalized Ekeland’s variational principle). Assume that \((U, G)\) is a complete \(q\)-G-m space and \(g : U \to (-\infty, \infty)\) be a proper, bounded below and Lsca function. \(\Gamma\) is a \(\Gamma\)-function on \(U \times U \times U\), then there exists \(r \in U\) such that 
\[
\Gamma(v, r, r) > \theta(g(r))(g(r) - g(v))
\]
for all \(r \in U\) with \(v \neq r\).

**Proof.** Suppose it isn’t true. Then for every \(r \in U\), there exists \(s \in U, s \neq r\) such that \(\Gamma(r, s, s) \leq \theta(g(r))(g(r) - g(s))\). That is \(P(r) \neq \emptyset\). We define the sequence \(\{r_n\}\) as follows. Put \(r_1 = \varepsilon\), we choose \(r_2 \in P(r_1)\) such that \(g(r_2) \leq \inf_{r \in P(r_1)} g(r) + 1\). In the same way suppose that \(r_n \in U\) is given. We choose \(r_{n+1} \in P(r_n)\) such that \(g(r_{n+1}) \leq \inf_{r \in P(r_n)} g(r) + \frac{1}{n}\). By proposition 2.14, there exists \(r_0 \in U\) such that 
\[
\bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}.
\]
By lemma 2.13, we have \(P(r_0) \subseteq \bigcap_{n=1}^{\infty} P(r_n) = \{r_0\}\) then \(P(r_0) = \{r_0\}\). This is a contradiction. Therefore there exists \(v \in U\) such that 
\[
\Gamma(v, r, r) > \theta(g(v))(g(v) - g(r)).
\]

□

**Theorem 2.16** (Generalized Caristi’s common fixed point theorem for a family of multivalued maps). Assume that \((U, G)\) is a complete \(q\)-G-m space and \(g : U \to (-\infty, \infty)\) be a proper, bounded below and Lsca function. \(\Gamma\) is a \(\Gamma\)-function on \(U \times U \times U\). Let \(I\) be any index set and for each \(j \in J\), suppose \(T_j : U \to 2^U\) is multivalued map such that for each \(r \in U\), there is \(s = s(r, j) \in T_j(r)\) with 
\[
\Gamma(r, s, s) \leq \theta(g(r))(g(r) - g(s)).
\]
(2.1)

Then there is \(w \in U\) such that \(w \in \bigcap_{j \in J} T_j(w)\), and \(\Gamma(w, w, w) = 0\).

**Proof.** By Theorem 2.15, there exists \(w \in U\) such that \(\Gamma(w, r, r) > \theta(g(w))(g(w) - g(r))\) for all \(r \in U\) with \(r \neq w\). Now we show that \(w \in \bigcap_{j \in J} T_j(w)\) and \(\Gamma(w, w, w) = 0\). According to the assumption, there exists \(r(t, j) \in T_j(w)\) such that \(\Gamma(w, t, t) \leq \theta(g(t))(g(t) - g(t(w, j)))\). We show that \(t(w, j) = w\) for all \(j \in J\). On the contrary, let \(t(w, j_0) \neq w\) for some \(j_0 \in J\), then 
\[
\Gamma(w, t, t) \leq \theta(g(w))(g(w) - g(t)) < \Gamma(w, t, t),
\]
which is a contradiction. Therefore \(w = t(w, j) \in T_j(w)\) for all \(j \in T\).

Since \(\Gamma(w, w, w) \leq \theta(g(w))(g(w) - g(w)) = 0\), we obtain \(\Gamma(w, w, w) = 0\).

□

**Remark 2.17.** We conclude that Theorem 2.16 concludes Theorem 2.15.

On the contrary, for each \(r \in U\), there exists \(s \in U\) with \(s \neq r\) such that 
\[
\Gamma(r, s, s) \leq \theta(g(r))(g(r) - g(s)).
\]
Put $T : U \rightarrow 2^U \setminus \emptyset$ by

$$T(r) = \{s \in U : s \neq r, \Gamma(r, s, s) \leq \theta(g(r))(g(r) - fg(s))\}.$$ 

By Theorem 2.16, $T$ has a fixed point $w \in U$, this means, $w \in T(w)$. This is a contradiction, because $w \notin T(w)$.

**Theorem 2.18** (Nonconvex maximal element theorem for a family of multivalued maps). Assume that $(U, G)$ is a complete $g$-G-m space and $g : U \rightarrow (-\infty, \infty]$ be a proper, bounded below and Lsca function. $\Gamma$ is a $\Gamma$-function on $U \times U \times U$, and $J$ be any index set. For each $j \in J$, let $T_j : U \rightarrow 2^U$ be a multivalued map. Suppose that for each $(r, j) \in U \times J$ with $T_j(r) \neq \emptyset$, there exists $s = s(r, j) \in U$ with $s \neq r$ such that (2.1) holds. Then there exists $w \in U$ such that $T_j(W) = \emptyset$ for each $j \in J$.

**Proof.** By Theorem 2.15, there exists $w \in U$, such that $\Gamma(w, r, r) > \theta(g(w))(g(w) - f(r))$ for all $r \in U$ with $r \neq w$. We prove that $T_j(w) = \emptyset$ for each $j \in J$. Indeed, if $T_j(w) \neq \emptyset$, for some $j_0 \in J$, according to the assumption, there exists $t = t(w, j_0) \in U$ with $t \neq w$ such that $\theta(w, t, t) \leq \theta(g(w))(g(w) - g(t))$. Also $\Gamma(w, t, t) > \theta(g(w))(g(w) - g(t))$, which is a contradiction. \hfill \Box

**Remark 2.19.** We conclude that Theorem 2.18 concludes Theorem 2.15.

On the contrary, thus for each $r \in U$, there exists $s \in U$ with $s \neq r$ such that

$$\Gamma(r, s, s) \leq \theta(g(r))(g(r) - g(s)).$$

For each $r \in U$, we define $T(r) = \{s \in U : s \neq r, \Gamma(r, s, s) \leq \theta(g(r))(g(r) - g(s))\}$. Then $T(r) \neq \emptyset$ for all $r \in U$. But by Theorem 2.18, there exists $w \in U$ such that $T(w) = \emptyset$, which is a contradiction.

3. **Nonconvex optimization and minimax theorems**

**Theorem 3.1** (Generalized Takahashi’s nonconvex minimization theorem). Assume that $(U, G)$ is a complete $g$-G-m space and $g : U \rightarrow (-\infty, \infty]$ be a proper, bounded below and Lsca function. $\Gamma$ is a $\Gamma$-function on $U \times U \times U$. Suppose that for any $r \in U$ with $g(r) > \inf_{w \in U} fg(w)$ there exists $s \in U$ with $s \neq r$ such that (2.1) holds. Then there exists $w \in U$ such that $g(w) = \inf_{t \in U} g(t)$.

**Proof.** By Theorem 2.15, there exists $w \in U$ such that $\Gamma(w, r, r) > \theta(g(w))(g(w) - g(r))$ for all $r \in U$, $r \neq w$. Now we prove that $g(w) = \inf_{t \in U} g(t)$.

On the contrary, then $g(w) > \inf_{t \in U} g(t)$. According to the assumption, there exists $s \in U \setminus \{w\}$, with $s \neq w$ such that $\Gamma(w, s, s) \leq \theta(g(w))(g(w) - g(s))$. Then we have $\Gamma(w, s, s) \leq \theta(g(w))(g(w) - g(s)) < \Gamma(w, s, s)$, which is a contradiction. \hfill \Box

**Remark 3.2.** Using Theorem 3.1, we can conclude Theorem 2.15.

On the contrary, then for each $r \in U$, there exists $s \in U$ with $s \neq r$ such that $\Gamma(r, s, s) \leq \theta(g(r))(g(r) - g(s))$. By Theorem 3.1, there exists $w \in U$ such that $g(w) = \inf_{t \in U} g(t)$. According to the assumption, there exists $z \in U$ with $z \neq r$, such that $\Gamma(z, z, z) \leq \theta(g(w))(g(w) - g(z)) \leq 0$. Then $\Gamma(w, z, z) = 0$ and $g(w) = g(z) = \inf_{t \in U} g(t)$. There exists $t \in U$ with $t \neq z$ such that $\Gamma(z, t, t) \leq \theta(g(z))(g(z) - g(t)) \leq 0$. Then we have $\Gamma(z, t, t) = 0$ and $g(w) = g(z) = g(t) = \inf_{t \in U} g(r)$. Since $\Gamma(w, z, z) \leq \Gamma(w, z, z) + \Gamma(z, t, t)$, then $\Gamma(w, t, t) = 0$. For $e > 0$ we have $\Gamma(w, z, z) = 0 < \delta$, $\Gamma(z, t, t) = 0 < \delta$ then $G(w, t, t) < e$, that is, $w = t$. Also for $e > 0$ we have $\Gamma(z, w, w) = 0 < \delta$, $\Gamma(w, t, t) = 0 < \delta$, then $G(z, t, t) < e$ that is, $z = t$, which is a contradiction.

**Theorem 3.3** (Nonconvex minimax theorem). Assume that $(U, G)$ is a complete $g$-G-m space and $\Gamma$ is a $\Gamma$-function on $U \times U \times U$. Let $F : U \times U \rightarrow (-\infty, \infty]$ be a proper Lsca and bounded below function in the first argument. Suppose that for each $r \in U$ with $\{x \in U : F(r, x) > \inf_{a \in U} F(a, x)\} \neq \emptyset$, there exists $s = s(r) \in U$ with $s \neq r$ such that

$$\Gamma(r, s, s) \leq \theta(F(r, w))(F(r, w) - F(s, w)) \leq \theta(g(r))(g(r) - fg(s))$$

(3.1)

for all $w \in \{x \in U : F(r, x) > \inf_{a \in U} F(a, x)\}$. Then $\inf_{r \in U} \sup_{s \in U} F(u, s) = \sup_{s \in U} \inf_{r \in U} F(r, s)$. 

Proof. By Theorem 3.1, for every \( s \in U \), there exists \( r(s) \in U \) such that \( F(r(s), s) = \inf_{r \in U} F(r, s) \). Then \( \sup_{s \in U} F(r(s), s) = \sup_{s \in U} \inf_{r \in U} F(r, s) \).

By displacement of \( r(s) \) with an arbitrary \( r \in U \) and then getting \( \inf \), we obtain \( \inf_{r \in U} \sup_{s \in U} F(r, s) = \sup_{s \in U} \inf_{r \in U} F(r, s) \).

\( \square \)

**Theorem 3.4** (Nonconvex equilibrium theorem). Assume that \((U, G)\) is a complete q-G-m space and \( \Gamma \) is a \( \Gamma \)-function on \( U \times U \times U \). Let \( F \) and \( \theta \) be the same as in Theorem 3.3. Let, for each \( r \in U \) with \( \{x \in U : F(r, x) < 0\} \neq \emptyset \), there exists \( s = s(r) \in U \) with \( s \neq r \) such that (3.1) holds for all \( t \in U \). Then there exists \( y \in U \) such that \( F(y, s) \geq 0 \) for all \( s \in U \).

**Proof.** From Theorem 2.15 for each \( t \in U \), there exists \( y(t) \in U \) such that \( \Gamma(y(t), t, r) \geq \theta(F(y(t), t))(F(y(t), t) - F(r, t)) \) for all \( r \in U \) with \( r \neq y(t) \). We show that there exists \( y \in U \) such that \( F(y, s) \geq 0 \) for all \( s \in U \). On the contrary, for each \( r \in U \) there exists \( s \in U \) such that \( F(r, s) < 0 \). Then for each \( r \in U \), \( \{x \in U : F(r, x) < 0\} \neq \emptyset \). According to the assumption, there exists \( s = s(y(t)) \), \( y \neq y(t) \) such that \( \Gamma(y(t), s, s) \leq \theta(F(y(t), t))(F(y(t), t) - F(s, t)) \), which is a contradiction. \( \square \)

**Example 3.5.** Let \( U = [0, 1] \) and \( G(r, s, t) = \max\{|r - s|,|r - t|,|s - t|\} \). Then \((X, G)\) is a complete q-G-m space. Suppose that \( a, b \) be positive real numbers with \( a > b \). Suppose \( H : U \times U \rightarrow \mathbb{R} \) with \( H(r, s) = \frac{s^3}{r^3} - \frac{s}{r} \). Therefore, function \( r \rightarrow H(r, s) \) is proper, lower semicontinuous and bounded below, and \( H(1, s) \geq 0 \) for every \( s \in U \). Also \( H(r, s) \geq 0 \) for every \( r \in [\frac{b}{a}, 1] \) and for every \( s \in U \). In fact, for every \( r \in [0, \frac{b}{a}] \), \( H(r, s) = ar - bs < 0 \) when \( s \in [\frac{a}{b}, 1] \). Then set \( \{x \in U : H(r, x) < 0\} \neq \emptyset \) for every \( r \in [0, \frac{b}{a}] \). Let \( s, r, s, r \in U \), \( r \geq s \), we have \( r - s = \frac{3}{2}(\frac{s}{r} - \frac{s}{r}) - \frac{3}{2}(\frac{s}{r} - \frac{s}{r}) \) for every \( x \in U \). Let \( \theta : [0, \infty) \rightarrow [0, \infty) \) with \( \theta(t) = \frac{2}{3} \) be defined. Therefore \( G(r, s, t) \leq \theta(H(r, x))(H(r, x) - H(s, x)) \) for every \( r \geq s \), and \( r, s, x \in U \). By Theorem 3.4 there exists \( y \in U \) such that \( H(y, s) \geq 0 \) for every \( s \in U \).

4. Applications

**Definition 4.1.** Let \((U, G)\) be a q-G-m space and \( a, b \in U \). Suppose that \( \lambda : U \rightarrow (0, \infty) \) be a function and \( \Gamma \) be a \( \Gamma \)-function on \( U \). Define \( \Gamma_\varepsilon(a, b, \lambda) = \{r \in U : \varepsilon \Gamma(a, b, r, \lambda) \leq \lambda(a) \} \Gamma(b, a, \lambda) - \Gamma(b, r, r) \}

such that \( \varepsilon \in (0, \infty) \) and \( a, b \in U \).

**Lemma 4.2.** Assume that \((U, G)\) is a complete q-G-m space and \( g : U \rightarrow (-\infty, \infty] \) be a proper, bounded below and Lipschitz function and \( \Gamma \) is a \( \Gamma \)-function on \( U \times U \times U \). Let \( \varepsilon > 0 \). Suppose that there exists \( x \in U \) such that \( g(x) < \infty \) and \( \Gamma(x, x, x) = 0 \). Then there exists \( t \in U \) such that

(i) \( \varepsilon \Gamma(x, t, t) \leq \theta(g(x))(g(x) - g(t)) \);

(ii) \( \Gamma(t, r, r) > \theta(g(t))(g(t) - g(r)) \) for all \( r \in U \) with \( r \neq t \).

**Proof.** Let \( x \in U \), \( g(x) < +\infty \) and \( \Gamma(x, x, x) = 0 \). Put \( S = \{r \in U : \varepsilon \Gamma(x, r, r) \leq \theta(g(x))(g(x) - g(r)) \} \).

Therefore \((S, G)\) is a nonempty complete q-G-m space. By Theorem 2.15, there exists \( t \in S \) such that \( \varepsilon \Gamma(x, t, r) > \theta(g(t))(g(t) - g(r)) \) for all \( r \in S \) with \( r \neq t \). For any \( r \in U \setminus S \), since \( \varepsilon \Gamma(x, t, t) + \Gamma(t, r, r) \geq \varepsilon \Gamma(x, r, r) > \theta(g(x))(g(x) - g(r)) \geq \varepsilon \Gamma(x, t, t) + \theta(g(t))(g(t) - g(r)) \), therefore \( \varepsilon \Gamma(t, r, r) > \theta(g(t))(g(t) - g(r)) \) for all \( r \in U \setminus S \). Then \( \varepsilon \Gamma(t, r, r) > \theta(g(t))(g(t) - g(r)) \) for all \( r \in U \) with \( r \neq t \). \( \square \)

**Theorem 4.3** (Generalized flower petal theorem). Suppose that \( P \) be a proper complete subset of a q-G-m space \( U \) and \( a \in P \). Let \( \Gamma \) be a \( \Gamma \)-function on \( U \) with \( \Gamma(a, a, a) = 0 \). Let \( b \in P \cap \Gamma(b, P, P) = \inf_{r \in P} \Gamma(b, r, r) \geq u \) and \( \Gamma(b, a, a) = u > 0 \) and there exists a function \( \lambda \) from \( U \) into \((0, \infty)\) satisfying \( \lambda(r) = \theta(F(b, r, r)) \) for some nondecreasing function \( \theta \) from \((-\infty, \infty)\) into \((0, \infty)\). Then for each \( \varepsilon > 0 \), there exists \( t \in P \cap \Gamma_\varepsilon(a, b, \lambda) \) such that \( \Gamma_\varepsilon(t, b, \lambda) \Gamma(P \setminus t) = \emptyset \) and \( (a, t, t) \leq e^{-1}\lambda(a)(s - r) \).

**Proof.** \((P, G)\) is a complete q-G-m space. Consider \( g : P \rightarrow (-\infty, \infty], g(r) = \Gamma(b, r, r) \). Since \( g(a) = \)
\( \Gamma(b, a, a) = s < \infty \) and \( \Gamma(b, P, P) = \inf_{r \in P} \Gamma(b, r, r) \geq u \) then \( g \) is a proper lower semicontinuous and bounded below function. By Lemma 4.2, there exists \( t \in P \) such that

(i) \( \epsilon \Gamma(a, t, t) \leq \lambda(a) |(g(a) - g(t))| \);

(ii) \( \epsilon \Gamma(t, r, r) > \lambda(t)(g(t) - g(t)) \) for all \( r \in P \) with \( r \neq t \).

Applying (i), we have \( t \in P \bigcap \Gamma(a, b, \lambda) \). Also, applying (i) again, we have \( \Gamma(a, t, t) \leq e^{-1}\lambda(a)(\Gamma(b, a, a) - \Gamma(b, t, t)) \leq e^{-1}\lambda(a)(s - r) \). By (ii), we obtain \( \epsilon(t, r, r) > \lambda(t)(\Gamma(b, t, t) - \Gamma(b, r, r)) \) for all \( r \in P \) with \( r \neq t \). Therefore \( u \notin \Gamma_{e}(t, b, \lambda) \) for all \( r \in P \setminus \{t\} \) or \( \Gamma_{e}(t, b, \lambda) \cap \{P \setminus \{t\}\} = \emptyset \). \( \Box \)

References


