



Using differentiation matrices for pseudospectral method solve Duffing Oscillator



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Abstract

This article presents an approximate numerical solution for nonlinear Duffing Oscillators by pseudospectral (PS) method to compare boundary conditions on the interval $[-1, 1]$. In the PS method, we have been used differentiation matrix for Chebyshev points to calculate numerical results for nonlinear Duffing Oscillators. The results of the comparison show that this solution had the high degree of accuracy and very small errors. The software used for the calculations in this study was Mathematica V.10.4.

Keywords: Duffing oscillator, pseudospectral methods, differential matrix, Duffing system, Chebyshev points.

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1. Introduction

In science and engineering, the Duffing Oscillator was a common model for nonlinear phenomena. The most general forced form of the Duffing equation is:

$$\frac{\partial^2}{\partial t^2}x(t) + \alpha \frac{\partial}{\partial t}x(t) + \beta x(t)^3 + \gamma x(t) = \delta \cos(\theta t), \quad -1 \leq t \leq 1, \quad x(-1) = 0, x(1) = 0, \quad (1.1)$$

where α , β , γ , δ , θ are parameters: α controls the amount of damping; β controls the amount of non-linearity in the restoring force; γ controls the linear stiffness; δ is the amplitude of the periodic driving force; θ is the angular frequency of the periodic driving force.

Equation (1.1) depends on the different γ, β , we had some special cases: $\gamma > 0, \beta > 0$: Hard Spring Duffing Oscillator; $\gamma > 0, \beta < 0$: Soft Spring Duffing Oscillator; $\gamma < 0, \beta > 0$: Inverted Duffing Oscillator;

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$\gamma = 0, \beta > 0$: Nonharmonic Duffing Oscillator. These special cases had been extensively studied in the literature [7].

Several approaches have been studied so far dealing with the nonlinear Duffing Oscillators such as The differential transform method [12]; The Jacobi elliptic function cn [16]; The analysis method [1, 6, 8]; The Taylor Expansion [5]; The Legendre pseudospectral method [14, 15]; A Chebyshev collocation algorithm [13]; The Enhanced Cubication Method [4]; The Improved Taylor Matrix Method [2]; The Postverification Method [10], the energy balance method [9].

2. Pseudospectral method using differential matrix for Chebyshev points

Let $p(x)$ a polynomial of degree n , and we know that it is valued at the points $p(x_0), p(x_1), \dots, p(x_n)$, then the first and second derivatives $p'(x)$ at the same points are expressed in matrix form:

$$p'(x_j) = Dp(x_j), p''(x_j) = D^2p(x_j), \quad j = 0, 1, \dots, n, \quad (2.1)$$

where $D = \{d_{ij}\}$ is the so-called differentiation matrix [11]. In case when the Chebyshev-Gauss-Lobatto points are chose as the collocation points $y_k = \cos(k\pi/n)$, [3]

$$D_{i,j} = \begin{cases} \frac{1+2n^2}{6} & i = j = 0 \\ \frac{c_i}{2c_j} \frac{(-1)^{i+j}}{\sin[(i+j)\pi/(2n)] \sin[(i-j)\pi/(2n)]} & i \neq j \\ \frac{\cos(j\pi/n)}{2\sin(j\pi/n)} & 0 < i = j < n \\ -\frac{1+2n^2}{6} & i = j = n \end{cases} \quad (2.2)$$

here $c_k = 1$ when $k = 1, 2, \dots, n-1$ and $c_k = 2$ when $k = 0, n$. The application of differential algebra in ordinary differential equations can also extend to nonlinear differential equations, so we transformed the matrix D into matrices [11]:

$$\begin{aligned} E^{(1)} &= \{d_{ij}\}, & 1 \leq i, j \leq n-1 \\ e_0^{(1)} &= \{d_{i0}\}, e_n^{(1)} = \{d_{in}\}, & 0 < i < n \end{aligned} \quad (2.3)$$

for a first-order differential element, the form $u'(x_i) = E^{(1)}u(x_i)$.

With a second-order differential element, we use $D^2 = d_{ij}^2$ and define the matrices:

$$\begin{aligned} E^{(2)} &= \{d_{ij}^2\}, & 1 \leq i, j \leq n-1 \\ e_0^{(2)} &= \{d_{i0}^2\}, e_n^{(2)} = \{d_{in}^2\}, & 0 < i < n \end{aligned} \quad (2.4)$$

has the form $u''(x_i) = E^{(2)}u(x_i)$.

3. The approximation of the nonlinear Duffing Oscillators

If $\alpha = 0$, then (1.1) become the form:

$$\frac{\partial^2}{\partial t^2} x(t) = f(x(t)), \quad (3.1)$$

with boundary conditions $x(-1) = a, x(1) = b$, then follow [11] we have

$$E^{(2)}x(t_j) = f(x(t_j)), \quad j = \overline{1, n-1}, \quad (3.2)$$

here $f(x(t_j))$ denotes the vector with elements $\{f(x_n(t_j))\}$. To find the solution of the equation (3.1), we will be proceed with an iterative procedure with the forming equation:

$$E^{(2)}x^{(k)} = f(x^{(k)}), \quad k = 1, 2, \dots \quad (3.3)$$

It is important to determine the iterative equation (3.3). The iterative procedure is simple, we assumed $u^{(0)} = \text{const}$, then found $u^{(1)}, u^{(2)}, \dots$, stop it until the error $\varepsilon = |u^{(k)} - u^{(k-1)}| < \varepsilon_0$.

Example 3.1. We consider the following nonlinear oscillator:

$$x'' + x + \beta x^3 = 0, \quad -1 \leq t \leq 1, \quad x(-1) = a, x(1) = b. \tag{3.4}$$

We will have an iterative procedure:

$$(E^{(2)} + R)x^{(k)} = \beta(x^{(k-1)})^3 - be_0^{(2)} - ae_n^{(2)} \tag{3.5}$$

in which R is a square identity matrix.

In the case of the more general equation (1.1), we transformed into the form:

$$E^{(2)}x(t_j) + \alpha E^{(1)}x(t_j) + \gamma x(t_j) = \delta \cos(\theta t_j) - \beta x^{(*)}(t_j)^3. \tag{3.6}$$

Therefore it can then be written in iterative matrix notation as

$$(E^{(2)} + QE^{(1)} + R)x^{(k)} = \delta \cos(\theta t_j) - \beta(x^{(k-1)})^3 \tag{3.7}$$

in where Q, R is a square matrix with the elements α, γ on the main diagonal. Equation (3.7) is repeated until the error $\varepsilon < \varepsilon_0$.

4. Results

a. The Hard Spring Duffing Oscillator case

We gave the error $\varepsilon_0 = 10^{-8}$, the points $n = 64$. And $\alpha = 0.9, \beta = 0.5, \gamma = 0.7, \delta = 0.65, \theta = 4$. Table 1 shows a comparison of numerical results and error with Mathematica calculations for Hard Spring Duffing Oscillator case. In the Fig. 1, the results were calculated base on the program by the pseudospectral method, and then the solid line shows that the result calculated by Mathematica v.10.4.

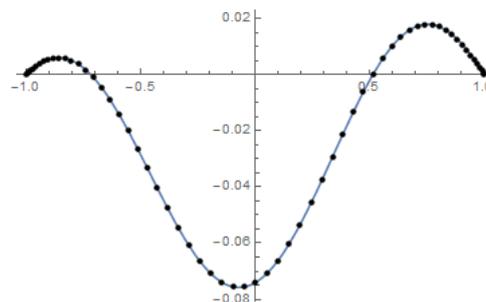


Figure 1: In case of The Hard Spring Duffing Oscillator with $\alpha = 0.9, \beta = 0.5, \gamma = 0.7, \delta = 0.65, \theta = 4$.

Table 1: Comparison of numerical results and error with Mathematica calculations depend for Hard Spring Duffing Oscillator.

j	$x(t_j)$	PS method	Mathematica 10.4	Error
1	0.998795	0.000156882	0.000156814	6.84933×10^{-8}
10	0.881921	0.126629	0.126628	7.41368×10^{-8}
20	0.55557	0.00564902	0.00564893	9.18076×10^{-8}
30	0.0980171	-0.0662731	-0.662732	1.12573×10^{-7}
40	-0.382683	-0.0476281	-0.0476282	1.23559×10^{-7}
50	-0.77301	0.00378497	0.00378492	5.64273×10^{-8}
60	-0.980785	0.00148411	0.0014861	6.74252×10^{-8}

b. The Soft Spring Duffing Oscillator case

We gave the error $\varepsilon_0 = 10^{-8}$, the points $n = 256$. And $\alpha = 1, \beta = -0.7, \gamma = 0.5, \delta = 0.1, \theta = 2\pi$. Table 2 shows a comparison of numerical results and error with Mathematica calculations for Soft Spring Duffing Oscillator case. In the Fig. 2, the results were calculated base on the program by the pseudospectral method, and then the solid line shows that the result calculated by Mathematica v.10.4.

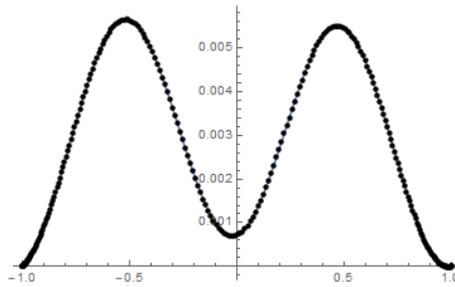


Figure 2: In case of The Soft Spring Duffing Oscillator with $\alpha = 1, \beta = -0.7, \gamma = 0.5, \delta = 0.1, \theta = 2\pi$.

Table 2: Comparison of numerical results and error with Mathematica calculations for Soft Spring Duffing Oscillator.

j	$x(t_j)$	PS method	Mathematica 10.4	Error
1	0.999925	-1.15572×10^{-7}	-9.13587×10^{-8}	2.42141×10^{-8}
50	0.817585	0.00127932	0.00127935	2.90011×10^{-8}
100	0.33689	0.00470433	0.00470434	1.32052×10^{-8}
150	-0.266713	0.00308706	0.00308708	2.28701×10^{-8}
200	-0.77301	0.00291083	0.00291084	9.47306×10^{-9}
250	-0.99729	1.25929×10^{-5}	1.25938×10^{-5}	9.03146×10^{-10}

c. The Inverted Duffing Oscillator case

We gave the error $\epsilon_0 = 10^{-8}$, the points $n = 64$. And $\alpha = 2, \beta = 0.7, \gamma = -1, \delta = 0.5, \theta = 2$. In the Table 3, we show competition the numerical results and error with Mathematica’s calculations depend on the Inverted Duffing Oscillator. In the Fig. 3, the results were calculated base on the program by the pseudospectral method, and then the solid line shows that the result calculated by Mathematica v.10.4.

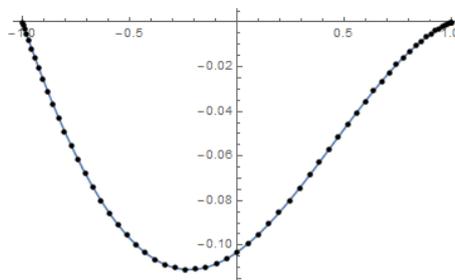


Figure 3: In case of The Inverted Duffing Oscillator with $\alpha = 2, \beta = 0.7, \gamma = -1, \delta = 0.5, \theta = 2$.

Table 3: Comparison of numerical results and error with Mathematica calculations for Inverted Duffing Oscillator.

j	$x(t_j)$	PS method	Mathematica 10.4	Error
1	0.998795	-4.87939×10^{-5}	-4.88112×10^{-5}	1.72549×10^{-8}
10	0.881921	-0.00668666	-0.00668668	5.56609×10^{-8}
20	0.55557	-0.0405467	-0.0405467	1.92109×10^{-8}
30	0.0980171	-0.0951851	-0.0951851	3.53075×10^{-9}
40	-0.382683	-0.106307	-0.106307	5.53807×10^{-9}
50	-0.77301	-0.0553363	-0.0553363	2.6213×10^{-9}
60	-0.980785	-0.00539904	-0.00539904	4.57523×10^{-9}

d. The Nonharmonic Duffing Oscillator case

We gave the error $\epsilon_0 = 10^{-8}$, the points $n = 128$. And $\alpha = 5, \beta = 0.9, \gamma = 0, \delta = 0.9, \theta = 5$. Table 4 shows a comparison of numerical results and error with Mathematica calculations for Nonharmonic Duff-

ing Oscillator case. In the Fig. 4, the results were calculated base on the program by the pseudospectral method, and then the solid line shows that the result calculated by Mathematica v.10.4.

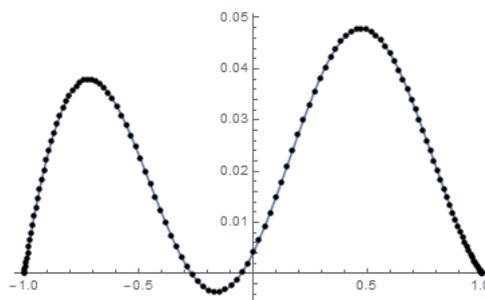


Figure 4: In case of The Nonharmonic Duffing Oscillator with $\alpha = 5, \beta = 0.9, \gamma = 0, \delta = 0.9, \theta = 5$.

Table 4: Comparison of numerical results and error with Mathematica calculations depend for Nonharmonic Duffing Oscillator.

j	$x(t_j)$	PS method	Mathematica 10.4	Error
1	0.999699	1.83279×10^{-5}	1.83161×10^{-5}	1.18298×10^{-8}
20	0.881921	0.0105519	0.0105518	1.04707×10^{-9}
40	0.55557	0.0455841	0.0455841	1.24329×10^{-8}
60	0.0980171	0.0148238	0.0148237	1.94355×10^{-8}
80	-0.382683	0.00989231	0.0098923	1.12557×10^{-8}
100	-0.77301	0.0366845	0.0366845	1.39201×10^{-8}
125	-0.99729	0.000766572	0.000766565	7.55144×10^{-9}

5. Conclusion

We used the pseudospectral method that used differential matrix for Chebyshev points to solve 4 special cases of the Duffing oscillator. The numerical results demonstrate the efficiency and the reliability method for solving this problem.

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