The endpoint Fefferman-Stein inequality for the strong maximal function with respect to nondoubling measure

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Abstract

Let $d\mu(x_1, \ldots, x_n) = d\mu_1(x_1) \cdots d\mu_n(x_n)$ be a product measure which is not necessarily doubling in $\mathbb{R}^n$ (only assuming $d\mu_i$ is doubling on $\mathbb{R}$ for $i = 2, \ldots, n$), and $M^n_{d\mu}$ be the strong maximal function defined by

$$M^n_{d\mu} f(x) = \sup_{x \in R \in \mathcal{R}} \frac{1}{\mu(R)} \int_R |f(y)| d\mu(y),$$

where $\mathcal{R}$ is the collection of rectangles with sides parallel to the coordinate axes in $\mathbb{R}^n$, and $\omega, \nu$ are two nonnegative functions. We give a sufficient condition on $\omega, \nu$ for which the operator $M^n_{d\mu}$ is bounded from $L^1(\nu d\mu)$ to $L^{1, \infty}(\omega d\mu)$. By interpolation, $M^n_{d\mu}$ is bounded from $L^p(\nu d\mu)$ to $L^{p}(\omega d\mu)$, $1 < p < \infty$.

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1. Introduction

Since the classical theory of harmonic analysis may be described as centering around the Hardy-Littlewood maximal operator and its relationship with certain singular integral operators, the maximal function have attracted the attention of a lot of researchers, such as [1, 3–5, 8–10, 13, 16, 22, 30].

Let $\mathcal{B}_x$ be a collection of bounded sets containing $x \in \mathbb{R}^n$, and $\nu$ be a positive measure. Given a locally integrable function $f$, denote

$$Mf(x) = \sup_{R \in \mathcal{B}_x} \frac{1}{\nu(R)} \int_R |f(y)| d\nu(y).$$

If $\mathcal{B}_x$ is the collection of all the cubes containing $x \in \mathbb{R}^n$ and whose sides parallel to the coordinate axes, $d\nu(x) = dx$, then we obtain the usual Hardy-Littlewood maximal function $Mf(x)$. When $\mathcal{B}_x$ denotes the
collection of all rectangles $R$ containing $x \in \mathbb{R}^n$ whose sides parallel to the coordinate axes, $M \triangleq M_{d\nu}^n$ is the strong maximal operator with respect to measure $d\nu$. If $d\nu(x) = dx$, denote $M^n = M_{d\nu}^n$.

For every non-negative, locally integrable weight $\omega$, Fefferman and Stein in [11] proved the following well known inequality

$$\int_{\mathbb{R}^n} (Mf)^p(x)\omega(x)dx \lesssim \int_{\mathbb{R}^n} (|f(x)|^p M\omega(x))dx, \ 1 < p < \infty.$$  

Inequalities of this type are important, for example, they can be used to derive the boundedness of vector-valued maximal operators. More details can be seen in [11–13]. Similar inequalities were also obtained for singular integral operators in [5]. In this situation, $M\omega(x)$ in the right hand was replaced by $M(\omega^r)^{1/r}$. The above inequality is also true for the strong maximal function $M^n$ if $\omega \in A_\infty^n$ [19, 21].

For the usual Hardy-Littlewood maximal function $Mf(x)$, the form of the endpoint Fefferman-Stein inequality is the following

$$\omega(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| M\omega(x)dx, \ \lambda > 0.$$  

For strong maximal function $M^nf(x)$, it is more complicated. When $n = 2$, Mitsis in [21] has obtained that

$$\omega(\{x : M^2f(x) > \lambda\}) \lesssim \int |f(x)| \left(1 + \log^+ \frac{|f(x)|}{\lambda}\right) M^2\omega(x)dx, \ \lambda > 0,$$

if $\omega \in A_\infty^2$ for some $1 < p < \infty$. Recently, Luque and Parissis in [19] improved this result to any dimension $n \geq 2$ provided only $\omega \in A_\infty^n$. We want to point out that there is no assumption on the weight to establish Fefferman-Stein inequality for the Hardy-Littlewood maximal function.

The classical theory of one-parameter harmonic analysis for maximal functions and singular integrals on $(\mathbb{R}^n; \mu)$ has been developed under the assumption that the underlying measure $\mu$ satisfies the doubling property, i.e., there exists a constant $C > 0$ such that $\mu(B(x; 2r)) \leq C\mu(B(x; r))$ for every $x \in \mathbb{R}^n$ and $r > 0$. However, some recent results [20, 23, 25, 31] show that it is possible to dispense with the doubling condition for most of the classical theory. It is well known that the use of doubling measure has two main advantages. One is that we can work with nested property. Another one is that the faces of the cubes have measure zero. As in the paper [20, 25], we will only maintain the last property. If $\mu$ is a nonnegative Radon measure without mass-points, one can choose an orthonormal system in $\mathbb{R}^n$ so that any cube $Q$ with sides parallel to the coordinate axes satisfies the property $\mu(\partial Q) = 0$ [20, Theorem 2]). The profit of this property is the continuity of the measure $\mu$ on cubes which can ensure that there is a Calderón-Zygmund decomposition [20, 25]. For the development of multi-parameter harmonic analysis, we refer the readers to the works in [2, 9, 14, 15, 17].

Therefore, there is a nature question: can the Fefferman-Stein inequality be established with a general measure $\nu$ for the strong maximal operator $M_{d\nu}^n$?

Let $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ be a product measure, where $\mu_i$, $i = 1, \ldots, n$ are all nonnegative Radon measures without mass-points and complete. The assumption that $\mu_i$ are complete is just a technical requirement to allow exchange integral order. For a rectangle $R \subseteq \mathbb{R}^n$, we mean a rectangle whose sides parallel to the coordinate axes. Under this kind of product measure, in [7], we investigated the $L^p(\omega d\mu)$ boundedness of strong maximal functions $M^n_{\omega d\mu}$ and $M^n_{d\mu}$ when $\omega \in A_{\infty}^n$ defined by the following.

**Definition 1.1.** Let $1 < p < \infty$ and $p' = p/(p-1)$. We say that a weight $\omega$ satisfies the $A_{\infty}^n(\mu)$ condition if

$$[\omega]_{A_p^n(\mu)} = \sup_{R \in \mathcal{R}} \left(\frac{1}{\mu(R)} \int _R \omega d\mu\right) \left(\frac{1}{\mu(R)} \int _R \omega^{1-p'} d\mu\right)^{p-1} < \infty,$$

where $\mathcal{R}$ is a collection of all rectangles $R$ whose sides parallel to the coordinate axes.
We say $\omega \in A^n_1(\mu)$ if there exists a constant $C > 0$ such that

$$M^n_{d\mu} \omega(x) \leq C \omega(x)$$

for almost every $x \in \mathbb{R}^n$.

Define $A^n_\infty(\mu)$ by

$$A^n_\infty(\mu) = \bigcup_{1 \leq p < \infty} A^p_n(\mu).$$

Notice that $A^p_n(\mu) \subseteq A^q_n(\mu)$ whenever $r \geq q$, and if $\omega \in A^\infty_n(\mu)$, then $\omega \in A^p_n(\mu)$ for some $1 < p < \infty$. It is easy to see that if $\omega \in A^p_n(\mu)$ for some $1 < p < \infty$, $\omega(x_1, \ldots, x_{i-1}, \gamma x_{i+1}, \ldots, x_n) \in A^1_n(\mu_i)$ uniformly with respect to $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. It has been proved in [7] that the behavior of $A^\infty_n(\mu)$ and the relationship between $A^p_n(\mu)$ weights and the strong maximal function $M^n_{d\mu}$ are very similar to the classical case when we add some conditions to the product measure $\mu$.

There are also some interesting results about two weights. We refer the reader to the work in [6, 24, 26–28]. In [26, 27], Pérez provided a sufficient condition on weights $\omega, \nu$ to ensure the boundedness of the general maximal functions $M_n^p$ including the boundedness of $M^n_{d\mu}$ from $L^p(\omega)$ to $L^q(\nu)$. More precisely, if the couple of weights $(\omega, \nu)$ satisfies the following condition: there are constants $0 < \lambda < 1$, $0 < c = c(\lambda) < \infty$ such that for all measurable sets $E$

$$\omega(\{x : M^n_{\nu}(x) > \lambda\}) \leq c \omega(E),$$

which is weaker than the $A^\infty_n$ condition, and

$$\sup_B \frac{1}{|B|} \int_B \omega(y) dy \left( \sup_B \frac{1}{|B|} \int_B \nu(y)^{(1-p')/p} dy \right)^{p-1} < \infty$$

for some $1 < r < \infty$, then $M^n_{\omega}$ is bounded from $L^p(\omega)$ to $L^q(\nu)$.

This idea can be extended to our strong maximal function $M^n_{d\mu}$. A couple of weights $(\omega, \nu)$ is said to be satisfied condition (A), if

$$\left( \frac{1}{|R|} \right) \int_R \omega(x) d\mu(x) \cdot \sup_{x \in \mathbb{R}} \nu^{-1}(x) \leq C$$

for all rectangles $R$ in $\mathbb{R}^n$. Then the main result of the current paper is the following.

**Theorem 1.2.** Assume that $\mu(x) = \mu_1(x_1) \cdot \mu_2(x_2) \cdots \mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete. Assume also that each $\mu_i$ for $2 \leq i \leq n$ is doubling on $\mathbb{R}$. If $(\omega, \nu)$ is a couple of weights such that $\omega \in A^\infty_n = A^\infty_\infty(\mu)$ and that the condition (A) holds, then

$$\omega(\{x : M^n_{d\mu}(f)(x) > \lambda\}) \leq \int \frac{|f(x)|}{\lambda} \left( 1 + \log^+ \frac{|f(x)|}{\lambda} \right)^{n-1} \nu(x) d\mu(x),$$

(1.1)

where $\omega(E)$ denotes $\int_E \omega(x) d\mu(x)$ for every $\mu$-measurable set $E$.

By interpolation, the above endpoint two weights Fefferman-Stein inequality implies the strong two weights Fefferman-Stein inequality.

**Theorem 1.3.** Assume that $\mu(x) = \mu_1(x_1) \cdot \mu_2(x_2) \cdots \mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete. Assume also that each $\mu_i$ for $2 \leq i \leq n$ is doubling on $\mathbb{R}$. If $(\omega, \nu)$ is a couple of weights such that $\omega \in A^\infty_n$ and that the condition (A) holds, then

$$\|M^n_{d\mu}(f)\|_{L^p(\omega d\mu)} \lesssim \|M^n_{d\mu}(f)\|_{L^p(\nu d\mu)},$$

$1 < p < \infty$.

It is easy to check that $(\omega, M^n_{d\mu} \omega)$ satisfies condition (A), and then we have.
Corollary 1.4. Assume that $\mu(x) = \mu_1(x_1) \cdot \mu_2(x_2) \cdots \mu_n(x_n)$ is a product measure where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete. Assume also that each $\mu_i$ for $2 \leq i \leq n$ is doubling on $\mathbb{R}$, and $\omega \in A^n_{\infty}$, then
\[
\omega([x : M^n_{\mu}(f)(x) > \lambda]) \lesssim \int \frac{|f(x)|}{\lambda} \left( 1 + (\log^+ |f(x)|)^{n-1} \right) M^n_{\mu} \omega(x) d\mu(x),
\]
and
\[
\|M^n_{\mu}(f)\|_{L^p(\omega d\mu)} \lesssim \|M^n_{\mu}(f)\|_{L^p(M^n_{\mu} \omega d\mu)}, \quad 1 < p < \infty.
\]

Let $d\mu = dx$, then the above corollary is the main theorem of [19]. By changing variables, in the above results, the product measure $\mu$ can be assumed that $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures in $\mathbb{R}$ without mass-points and complete, permitted only one direction with non-doubling condition.

The organization of the paper is as follows. Section 2 gives some auxiliary lemmas, such as reverse Hölder’s inequality of weights $A^n_{\infty}(\mu)$, and the asymptotic estimate of the $L^p(d\mu)$ norm of $M^n_{\omega d\mu}$ as $p \to 1^+$. In last section, we give the proof of Theorem 1.2.

Finally, we make some conventions. Throughout the paper, $c$ denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants with subscript, such as $c_1$, do not change in different occurrences. We denote $f \lesssim cg$ by $f \gtrsim g$. If $f \lesssim g \lesssim f$, we write $f \approx g$. In order to indicate the dependence of the constant on some parameter $n$ (say), we write $\Lambda \lesssim_n B$.

2. Auxiliary lemmas

In this section, firstly we give some lemmas about weights $A^n_{\infty}(\mu)$ obtained in [7].

Lemma 2.1. Let $\mu$ be a nonnegative Radon measure. If $\omega \in A^n_{\infty}(\mu)$, then for $0 < \alpha < 1$, there is a positive constant $\beta < 1$ such that whenever $F$ is a measurable set of a rectangle $R$, we have
\[
\frac{\mu(F)}{\mu(R)} \leq \alpha \quad \text{implies} \quad \frac{\omega(F)}{\omega(R)} \leq \beta,
\]
which is equivalent to say that for $0 < \alpha' < 1$, there is a positive constant $\beta' < 1$ such that whenever $F$ is a measurable set of a rectangle $R$,
\[
\frac{\mu(F)}{\mu(R)} \geq \alpha' \quad \text{implies} \quad \frac{\omega(F)}{\omega(R)} \geq \beta'.
\]

Lemma 2.2. Assume that $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ is a product measure, where $\mu_i, i = 1, \ldots, n$ are all nonnegative Radon measures without mass-points and complete. If $\omega \in A^n_{\infty}(\mu)$, then $\omega$ satisfies a reverse Hölder’s inequality, that is, there exist two positive constants $c$ and $\delta$ such that for every rectangle $R$
\[
\left( \frac{1}{\mu(R)} \int_R \omega^{1+\delta} d\mu \right)^{1/(1+\delta)} \leq \frac{c}{\mu(R)} \int_R \omega d\mu,
\]
and $c$ may be taken as close to 1 as $\delta \to 0^+$.

All the proofs of above lemmas can be seen in [7], and we omit it. If $\omega \in A^n_p(\mu)$, $p > 1$, then $\omega^{1-p'} \in A^n_{p'}(\mu)$, where $1/p + 1/p' = 1$. Consequently, by Lemma 2.2, it is easy to deduce the following result.

Lemma 2.3. Let $p > 1$, and $\omega \in A^n_p(\mu)$, then there is an $\epsilon > 0$ such that $\omega \in A^n_{p-\epsilon}(\mu)$. 

Lemma 2.4. Assume that \( \mu(x) = \mu_1(x_1)\mu_2(x_2)\cdots\mu_n(x_n) \) is a product measure where \( \mu_i, i = 1, \ldots, n \) are all nonnegative Radon measures in \( \mathbb{R} \). Let \( \{R_k\} \) be a sequence of rectangles in \( \mathbb{R}^n \) satisfying property \( P_2 \), and \( S^n_k \) be the slice of \( R_k \) at \( x_n \). Then \( \{S^n_k\} \) satisfies property \( P_1 \) for any \( x_n \), that is

\[
\mu'(S^n_k \cap \bigcup_{i < k} S^n_i) \leq \epsilon \mu'(S^n_k).
\]

Proof. Since \( R_k \) is a rectangle in \( \mathbb{R}^n \), we may assume \( R_k = I_k \times J_k \) where \( J_k \) is the one-dimensional projection to the \( x_n \) axes and \( I_k \) is a rectangle in \( \mathbb{R}^{n-1} \). The conclusion is obvious if \( x_n \notin J_k \). When \( x_n \in J_k \), setting \( J = \{i < k, S^n_i \cap S^n_k \neq \emptyset\} \), one has

\[
R_k \cap \bigcup_{i \in J} I_i \subseteq R_k \cap \bigcup_{i < k} I_i.
\]

By the assumption that the side lengths of the \( x_n \) direction are decreasing in \( \{R_k\} \), one has

\[
R_k \cap \bigcup_{i \in J} I_i = (\bigcup_{i \in J} S^n_i \cap S^n_k) \times J_k.
\]

Hence by the property \( P_2 \), one has

\[
\mu'(\bigcup_{i \in J} S^n_i \cap S^n_k)\mu_n(J_k) = \mu(R_k \cap \bigcup_{i \in J} I_i) \leq \mu(R_k \cap \bigcup_{i < k} I_i) \leq \epsilon \mu(R_k),
\]

which yields our desired result immediately.

Lemma 2.5. Assume that \( \mu(x) = \mu_1(x_1)\mu_2(x_2)\cdots\mu_n(x_n) \) is a product measure where \( \mu_i, i = 1, \ldots, n \) are all nonnegative Radon measures in \( \mathbb{R} \) without mass-points and complete. Assume also that each \( \mu_i \) for \( 2 \leq i \leq n \) is doubling on \( \mathbb{R} \) and \( \omega \in A^\infty_n \). Let \( \{R_k\} \) be a sequence of rectangles in \( \mathbb{R}^n \) satisfying property \( P_1 \). Then if \( M^p_\omega d\mu \) is \( L^p(\omega d\mu) \) bounded with norm at most \( O((p-1)^{-r}) \), \( 1 < p \leq 2 \), for some \( r > 0 \), one has

\[
\| \sum_{R_k} \chi_{R_k} \|_{L^p(\omega d\mu)} \leq C(p')^{r+1} \omega(R_k)^{1/p'}, \quad 1 < p \leq 2.
\]

Proof. Setting \( E_k = R_k \setminus \bigcup_{i < k} R_i \), one has \( \mu(E_k) \geq (1 - \epsilon)\mu(R_k) \) by property \( P_1 \) and the fact that the sets \( \{E_k\} \) are pairwise disjoint. Since \( \omega \in A^\infty_n(\mu) \), one has \( \omega(E_k) \geq \beta \omega(R_k) \) for some \( 0 < \beta < 1 \). Arguing by duality we assume that \( \varphi \) is a function satisfying \( \|\varphi\|_{L^p(\omega d\mu)} = 1 \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), then one has

\[
\int \sum_{R_k} \chi_{R_k} \varphi \omega d\mu = \sum_k \int_{R_k} \varphi \omega d\mu = \sum_k \frac{1}{\omega(R_k)} \int_{R_k} \varphi \omega d\mu \omega(R_k).
\]
\[ \sum_{\omega(E_k)} \omega(E_k) \inf_{x \in R_k} M^n_{\omega d\mu}(\varphi)(x) \leq \int_{\bigcup E_k} M^n_{\omega d\mu}(\varphi)(x) d\mu(x) \leq \|M^n_{\omega d\mu}(\varphi)\|_{L^p(\omega d\mu)} \omega(\cup R_k)^{1/p'} \leq O((p'-1)^r) \omega(\cup R_k)^{1/p'} \]

by the assumption of the \(L^p(\omega d\mu)\) norm of \(M^n_{\omega d\mu}\).

**Lemma 2.6.** Assume that \(\mu(x) = \mu_1(x_1) \mu_2(x_2) \cdots \mu_n(x_n)\) is a product measure where \(\mu_i, i = 1, \ldots, n\) are all nonnegative Radon measures in \(\mathbb{R}\) without mass-points and complete. Assume also that each \(\mu_i\) for \(2 \leq i \leq n\) is doubling on \(\mathbb{R}\) and \(\omega \in A_\infty^n\). Let \(\{R_k\}\) be a sequence of rectangles in \(\mathbb{R}^n\) satisfying property \(P_2\). Suppose also that \(M^{n-1}_{\omega d\mu'}\) is bounded on \(L^p(\mathbb{R}^{n-1}, \omega' d\mu')\) with norm at most \(O((p-1)^{-1})\), \(1 < p \leq 2\), uniformly in a.e. \(x_n\) for some \(r > 0\). Then there exists a constant \(C\) independent of \(\{R_k\}\) such that

\[ \| \sum R_k \|_{L^p(\omega d\mu')} \leq C(p')^{r+1} \omega(\cup R_k)^{1/p'}, \quad 1 < p \leq 2. \]

**Proof.** Let \(S_{k}^{n}\) denote the slice of \(R_k\) by a hyperplane perpendicular to the \(x^n\)-axis, at height \(x_n\). Using Lemma 2.4, we have

\[ \mu'(S_{k}^{n} \cap \bigcup_{i < k} S_{i}^{n}) \leq \frac{1}{2} \mu'(S_{k}^{n}). \]

Since \(\omega \in A_\infty^n\), one has \(\omega' \in A_\infty^{n-1}\) uniformly in a.e. \(x_n\). Then by Lemma 2.5,

\[ \| \sum X S_{k}^{n} \|_{L^p(\omega' d\mu')} \leq C(p')^{r+1} \omega'(\cup S_{k}^{n})^{1/p'}, \quad 1 < p \leq 2, \]

uniformly in a.e. \(x_n\), which follows the desired result by taking the \(p'\)-th power on both sides of above inequality and integrate in \(x_n\).

**Lemma 2.7.** Assume that \(\mu(x) = \mu_1(x_1) \cdot \mu_2(x_2) \cdots \mu_n(x_n)\) is a product measure where \(\mu_i, i = 1, \ldots, n\) are all nonnegative Radon measures in \(\mathbb{R}\) without mass-points and complete. Assume also that each \(\mu_i\) for \(2 \leq i \leq n\) is doubling on \(\mathbb{R}\) and \(\omega \in A_\infty^n\). Then \(M^n_{\omega d\mu}\) is \(L^p(\omega d\mu)\) bounded with norm at most \(O((p-1)^{-n})\), \(1 < p \leq 2\).

**Remark 2.8.** The asymptotic estimate of \(M^n\) was obtained by Long and Shen in [18].

**Proof.** The proof is by induction on \(n\). For \(n = 1\), \(M^n_{\omega d\mu}\) is the classical Hardy-Littlewood maximal operator \(M_{\omega d\mu}\) with respect to measure \(\omega d\mu\). Let \(\lambda > 0\) and \(E_\lambda = \{x : M_{\omega d\mu}(f) > \lambda\}\), then the main result of [29] gives

\[ \omega(E_\lambda) \leq 5\lambda^{-1} \int |f(x)| \omega(x) d\mu(x). \]

By interpolation since \(M_{\omega d\mu}\) is \(L^\infty(\omega d\mu)\) to \(L^\infty(\omega d\mu)\) with norm 1, we obtain

\[ \|M_{\omega d\mu}(f)\|_{L^p(\omega d\mu)} \leq c(p-1)^{-1}. \]

Suppose that \(n > 1\) and the lemma holds for \(n-1\). Since \(\omega \in A_\infty^n\), one has \(\omega' \in A_\infty^{n-1}\) uniformly in a.e. \(x_n\). By the inductive hypothesis, \(M^{n-1}_{\omega' d\mu'}\) is \(L^p(\omega' d\mu')\) bounded with norm at most \(O((p-1)^{-1-n})\), \(1 < p \leq 2\).

Let \(\lambda > 0\) and \(\{R_k\}\) be a cover of \(E_\lambda = \{x : M^n_{\omega d\mu}(f) > \lambda\}\) such that

\[ \frac{1}{\omega(R_k)} \int_{R_k} |f(x)| \omega(x) d\mu(x) > \lambda. \]
With no loss of generality, we may assume that \( \{R_k\} \) is a finite sequence, and that \( R_k \) are arranged so that the side length in \( x_n \) direction is decreasing. We now follow a well-known selecting procedure argument. We choose \( R_1^* = R_1 \), and assume \( R_1^*, \ldots, R_k^* \) have been selected. We obtain \( R_{k+1}^* \) as the first rectangle on the list of \( R_i \) after \( R_k^* \) such that

\[
\mu(R \cap \bigcup_{i \leq k} \hat{R}_i^*) < \frac{1}{2} \mu(R).
\]

That is \( \{R_k^*\} \) satisfies the property \( P_2 \). By Lemma 2.6, we obtain

\[
\| \sum \chi_{R_k} \|_{L^p(\omega d\mu)} \leq C(p')^{n} \omega(\bigcup R_k^*)^{1/p'}, \quad 1 < p \leq 2.
\] (2.1)

Moreover, arguing as in the proof of Theorem 1.2, one has

\[
\omega(\bigcup R_k^*) \lesssim \omega(\bigcup R_k^*),
\] (2.2)

by the assumption that each \( \mu_i \) for \( 2 \leq i \leq n \) is doubling on \( \mathbb{R} \) and \( \omega \in A^n_{\infty} \). Finally, by (2.1),

\[
\omega(\bigcup R_k^*) \lesssim \sum \omega(R_k^*) \lesssim \sum \frac{1}{\lambda} \int_{R_k^*} |f(x)| \omega(x) d\mu(x)
\]

\[
\leq \frac{1}{\lambda} \| \sum \chi_{R_k} \|_{L^p(\omega d\mu)} \|f\|_{L^p(\omega d\mu)} \leq C(p')^{n-1} \omega(\bigcup R_k^*)^{1/p'} \frac{1}{\lambda} \|f\|_{L^p(\omega d\mu)},
\]

from which it follows that

\[
\omega(\bigcup R_k^*) \lesssim \left( C(p')^{n} \frac{1}{\lambda} \|f\|_{L^p(\omega d\mu)} \right)^p.
\]

Hence, by (2.2),

\[
\omega(E_\lambda) \lesssim \left( C(p')^{n} \frac{1}{\lambda} \|f\|_{L^p(\omega d\mu)} \right)^p.
\] (2.3)

From Lemma 2.3, since \( \omega \in A^n_{\infty}(\mu) \), there is an \( \varepsilon > 0 \) such that \( \omega \in A^n_{p-\varepsilon}(\mu) \). Then (2.3) holds for \( p - \varepsilon \).

It is also known that \( \omega \in A^n_{p+\varepsilon}(\mu) \). By interpolation, we complete the proof.

As a direct corollary of Lemma 2.6 and Lemma 2.7, we can obtain the following result.

**Corollary 2.9.** Assume that \( \omega \in A^n_{\infty}(\mu) \) and that \( \{R_k\} \) is a sequence of rectangles in \( \mathbb{R}^n \) satisfying property \( P_2 \). Then if \( p \) is big enough,

\[
\| \sum \chi_{R_k} \|_{L^p(\omega d\mu)} \lesssim p^n \omega(\bigcup R_k)^{1/p}.
\]

### 3. Endpoint Fefferman-Stein inequality

**Proof of Theorem 1.2.** It suffices to prove the theorem for \( \lambda = 1 \). Denote \( E = \{x: M_{d\mu}^n(f)(x) > 1\} \). Let \( \{R_k\} \) be a cover of \( E \) such that

\[
\frac{1}{\mu(R_k)} \int_{R_k} |f(x)| d\mu(x) > 1.
\] (3.1)

Since we only need to prove (1.1) for any compact subset \( K \) of \( E \), without loss of generality, we may assume \( \{R_k\} \) is a finite sequence, and \( R_k \) are arranged so that the side length in \( x_n \) direction is decreasing.

We now choose a subset \( \{R_k^*\} \) of \( \{R_k\} \) such that \( \{R_k^*\} \) satisfies property \( P_2 \) and

\[
\omega(\bigcup R_k^*) \lesssim \omega(\bigcup R_k^*).\] (3.2)

Let \( R_1^* = R_1 \), and assume that \( R_1^*, \ldots, R_k^* \) have been selected. We obtain \( R_{k+1}^* \) as the first rectangle on the list of \( R_i \) after \( R_k^* \) such that

\[
\mu(R \cap \bigcup_{i \leq k} \hat{R}_i^*) < \frac{1}{2} \mu(R).
\]
This selection process will be end after a finite steps. It is obvious that \( \{R^*_k\} \) satisfies the property \( P_2 \). Now assume that some \( R \in \{R_k\} \) was not selected, then we can find some positive integer \( k \) such that

\[
\mu(R \cap \bigcup_{i \leq k} \hat{R}_i^*) \geq \frac{1}{2} \mu(R),
\]

which implies that for all \( x \in R \),

\[
M_{d \mu}^n(x \cup \hat{R}_i^*) \geq \frac{1}{2}.
\]

Hence

\[
\cup R_k \subseteq \{x : M_{d \mu}^n(x \cup R_i^*) \geq \frac{1}{2}\}.
\]

Since \( \omega \in A_{n_\infty}^n \), then \( \omega \in A_{n_\infty}^n(\mu) \) for some \( 1 < p < \infty \). By the result of \( M_{d \mu}^n \) being bounded on \( L^p(\omega d \mu) \) ([7, Theorem 1.6]), we conclude that

\[
\omega(\cup R_k) \lesssim \omega(\cup \hat{R}_i^*). \tag{3.3}
\]

Let \( S^x_k \) denote the slice of \( R^*_k \) at \( x \) and then \( R^*_k = S^x_k \times J_k, \hat{R}^*_k = S^x_k \times \hat{J}_k \) if \( x \in J_k \). Using property \( P_2 \) of \( \{R^*_k\} \), and Lemma 2.4 we have

\[
\mu'(S^x_k \cap \bigcup_{i < k} S_i^{x_n} \leq \frac{1}{2} \mu'(S^x_k).
\]

Since \( \omega \in A_{n_\infty}^n \), one has \( \omega' \in A_{n_\infty}^{n-1} \) uniformly in a.e. \( x_n \). Then

\[
\omega'(S^x_k \cap \bigcup_{i < k} S_i^{x_n}) \leq \beta \omega'(S^x_k),
\]

uniformly in a.e. \( x_n \), for some \( 0 < \beta < 1 \). Denote \( F_k = S^x_k \setminus \bigcup_{i < k} S_i^{x_n} \). It is obvious that \( \omega'(F_k) \geq (1 - \beta) \omega'(S^x_k) \). By classical result, when \( \mu_n \) is doubling, \( \omega(x', x_n) d \mu_n \) is also doubling uniformly for \( x' \in \mathbb{R}^{n-1} \). Therefore using (3.3)

\[
\omega(\cup R_k) \lesssim \sum_k \omega(\hat{R}_i^*) = \sum_k \int_{S^x_k} \left( \int_{J_k} \omega(x', x_n) d \mu_n(x_n) \right) d \mu'(x') \lesssim \sum_k \int_{J_k} \left( \int_{F_k} \omega(x_1, x_2) d \mu'(x') \right) d \mu_n = \int_{\bigcup J_k \times F_k} \omega(x) d \mu \lesssim \omega(\cup_k R^*_k),
\]

which gives (3.2).

Observe that property \( P_2 \) of \( \{R^*_k\} \) also implies that

\[
\mu(R^*_k \cap \bigcup_{i < k} R^*_i) \leq \frac{1}{2} \mu(R^*_k).
\]

It follows that \( \omega(R^*_k \cap \bigcup_{i < k} R^*_i) \leq \beta \omega(R^*_k) \) for some \( 0 < \beta < 1 \), since \( \omega \in A_{n_\infty}^n \). Then setting \( E_k = R^*_k \setminus \bigcup_{i < k} R^*_i \), we have

\[
\omega(R^*_k) \geq \omega(E_k) \geq (1 - \beta) \omega(R^*_k), \quad \mu(R^*_k) \geq \mu(E_k) \geq \frac{1}{2} \omega(R^*_k).
\]

Using (3.1) and (3.2), we obtain

\[
\omega(E) \lesssim \omega(\cup \hat{R}_i^*) \leq \sum \omega(R^*_k) \leq \sum \frac{\omega(R^*_k)}{\mu(R^*_k)} \int_{R^*_k} |f(y)| d \mu(y) = \int |f(y)| \sum_k \frac{\omega(R^*_k)}{\mu(R^*_k)} X_{R^*_k}(y) d \mu(y).
\]
For locally integrable functions $f$ and $g$, define the linear operators

$$Tf(x) = \sum_k \frac{1}{\mu(R_k)} \int_{R_k} f(y) d\mu(y) \chi_{E_k}(x), \quad T^*f(x) = \sum_k \frac{1}{\mu(R_k^*)} \int_{E_k} f(y) d\mu(y) \chi_{R_k^*}(x).$$

It is easy to check that

$$\int Tf(x) g(x) d\mu(x) = \int T^*g(x) f(x) d\mu(x),$$

$$T1(x) = \sum_k \chi_{E_k}(x), \quad T^*1(x) = \sum_k \frac{\mu(E_k)}{\mu(R_k)} \chi_{R_k^*}(x) \approx \sum_k \chi_{R_k^*}(x),$$

and

$$T^*\omega(x) = \sum_k \frac{\omega(E_k)}{\mu(R_k^*)} \chi_{R_k^*}(x) \approx \sum_k \frac{\omega(R_k)}{\mu(R_k^*)} \chi_{R_k^*}(x).$$

Hence

$$\omega(\cup R_k^*) \lesssim \int |f(y)| T^*\omega(y) d\mu(y) = \left( \int_{\{y : T^*\omega(y) \leq \nu(y)\}} |f(y)| T^*\omega(y) d\mu(y) \right) + \left( \int_{\{y : T^*\omega(y) > \nu(y)\}} |f(y)| T^*\omega(y) d\mu(y) \right) \leq \int f(y) \nu(y) d\mu(y) + \int_{\{y : T^*\omega(y) > \nu(y)\}} |f(y)| T^*\omega(y) \nu(y) d\mu(y).$$

Recall a known result from [1]: For any $\theta > 0$, there exists a constant $c_0 > 0$ such that for all $s, t > 0$ we have

$$st \leq c_0 s[1 + (\log^+ s)^{n-1}] + \exp(\theta t^{1/(n-1)}) - 1, \quad n \geq 2.$$

Applying the pointwise estimate above we get for any $\theta > 0$,

$$\int_{\{y : T^*\omega(y) > \nu(y)\}} |f(y)| \frac{T^*\omega(y)}{\nu(y)} \nu(y) d\mu(y) \leq c_0 \int_{\{y : T^*\omega(y) > \nu(y)\}} |f(y)| [1 + (\log^+ |f(y)|)^{n-1}] \nu(y) d\mu(y)$$

$$+ \int_{\{y : T^*\omega(y) > \nu(y)\}} \left( \exp \left( \theta \left( \frac{T^*\omega(y)}{\nu(y)} \right)^{1/(n-1)} \right) - 1 \right) \nu(y) d\mu(y).$$

Therefore

$$\omega(\cup R_k^*) \lesssim (1 + c_0) \int |f(y)| [1 + (\log^+ |f(y)|)^{n-1}] \nu(y) d\mu(y) + I,$$

where

$$I = \int_{\{y : T^*\omega(y) > \nu(y)\}} \left( \exp \left( \theta \left( \frac{T^*\omega(y)}{\nu(y)} \right)^{1/(n-1)} \right) - 1 \right) \nu(y) d\mu(y).$$

Using the Taylor expansion of $e^t$ we can write

$$I = \sum_{j=1}^{\infty} \frac{\theta^j}{j!} \int_{\{y : T^*\omega(y) > \nu(y)\}} \left( \frac{T^*\omega(y)}{\nu(y)} \right)^j \nu(y) d\mu(y) = \sum_{1 \leq j \leq n-1} + \sum_{j > n-1} = I_1 + I_2.$$

For $I_1$, one can easily get

$$\left( \frac{T^*\omega(y)}{\nu(y)} \right)^j \leq \frac{T^*\omega(y)}{\nu(y)} \left( \frac{T^*\omega(y)}{\nu(y)} \right)^{j-1} \leq T^*\omega(y) \frac{T^*\omega(y)}{\nu(y)}.$$
Therefore
\[
\int_1 \leq \sum_{1 \leq j \leq n-1} \frac{\theta^j}{j!} \int \theta^j \omega(y) \, d\mu(y) \lesssim_n \theta \int T_1(x) \omega(y) \, d\mu(y) = \theta \sum_k \int E_k \omega(y) \, d\mu = \theta \omega(\cup R^*_k)
\]
from the definition of \(T\) and \(E_k\) provided \(\theta < 1\).

For item \(I_2\), since weights \(\omega, \nu\) satisfy condition (A), one has
\[
\frac{1}{\mu(R^*_k)} \omega(R^*_k) \leq c \inf_{x \in R^*_k} \nu(x),
\]
then
\[
T^* \omega(x) \approx \sum_k \frac{\omega(R^*_k)}{\mu(R^*_k)} \chi_{R^*_k}(x) \leq \sum_k \nu(x) \chi_{R^*_k}(x) = \nu(x) T^* 1(x).
\]
Therefore
\[
I_2 = \sum_{j > n-1} \frac{\theta^j}{j!} \int \frac{T^* \omega(y)}{\nu(y)} \frac{T^* \omega(y)}{\nu(y)} \nu(y) \, d\mu(y)
\]
\[
\lesssim \sum_{j > n-1} \frac{\theta^j}{j!} \int (T^* 1(y))^{j/(n-1)} T^* \omega(y) \, d\mu(y)
\]
\[
\lesssim \sum_{j > n-1} \frac{\theta^j}{j!} \int (T^* 1(y))^{j/(n-1)} T^* \omega(y) \, d\mu(y)
\]
\[
= \sum_{j > n-1} \frac{\theta^j}{j!} \int \cup R^*_k T (T^* 1(y))^{j/(n-1)} \omega(y) \, d\mu(y).
\]
Observing that \(T f(x) \lesssim \sum_k \chi_{E_k}(x) \inf_{y \in R^*_k} M^N_{d\mu}(f)(y) \leq M^N_{d\mu}(f)(x)\), so we have
\[
\|T f\|_{L^p_0(\omega \, d\mu)} \lesssim \|f\|_{L^p_0(\omega \, d\mu)}
\]
for some \(1 < p_0 < \infty\), since \(\omega \in A_\infty\). This together with Corollary 2.9 and Hölder’s inequality yield
\[
\int_{\cup R^*_k} T (T^* 1(y))^{j/(n-1)} \omega(y) \, d\mu(y) \lesssim \omega(\cup R^*_k) \frac{1}{p_0} \|T^* 1(y))^{j/(n-1)}\|_{L^p_0(\omega \, d\mu)}
\]
\[
= \omega(\cup R^*_k) \frac{1}{p_0} \left( \int (T^* 1(y))^{j p_0/(n-1)} \omega(y) \, d\mu(y) \right)^{1/p_0} \lesssim \left( \frac{j p_0}{n-1} \right)^{j} \omega(\cup R^*_k),
\]
which follows that
\[
I_2 \lesssim \sum_{j > n-1} \frac{\theta^j}{j!} \left( \frac{j p_0}{n-1} \right)^{j} \omega(\cup R^*_k) \lesssim_n \sum_{j > n-1} \frac{(e \theta p_0/(n-1))^j}{\sqrt{j}} \omega(\cup R^*_k) \lesssim_n \frac{(e \theta p_0/(n-1))^n}{\sqrt{n}} \omega(\cup R^*_k),
\]
by choosing \(\theta\) small enough such that \(e \theta p_0/(n-1) < 1\).

At last, we obtain that
\[
\omega(\cup R^*_k) \lesssim \omega_n(1 + c_0) \int |f(y)|(1 + (\log^+ |f(y)|)^{n-1}) \nu(y) \, d\mu(y) + (\theta + (e \theta p_0/(n-1))^n) \omega(\cup R^*_k),
\]
which yields that
\[
\omega(\cup R^*_k) \lesssim \int |f(y)|(1 + (\log^+ |f(y)|)^{n-1}) \nu(y) \, d\mu(y),
\]
by letting \(\theta\) sufficiently small.

Thus we complete the proof using (3.2).
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References