Hitting probabilities for non-convex lattice

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Abstract

In this paper, we consider three lattices with cells represented in Figures 1, 3, and 5 and we determine the probability that a random segment of constant length intersects a side of the lattice considered.

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1. Introduction

Caristi and Stoka \cite{7} and \cite{8} introduced in the Buffon-Laplace type problems so-called obstacles. They considered two lattices with axial symmetry and in a first moment \cite{7} they study with eight triangular and circular sector obstacles and in the second moment \cite{8} they analyze twelve obstacles. Several other authors considered different lattices with different types of obstacles and studied the probability that a random body test intersect the fundamental cell \cite{2, 5}, and \cite{4}. In particular, in \cite{1}, the authors studied a Laplace type problem with obstacles for two Delone hexagonal lattices and in \cite{6} for a regular lattice of Dirichlet-Voronoi. In this study, starting from the results obtained by Duma and Stoka \cite{9} for Buffon type problems with a non-convex lattice we consider a Laplace type problem for three lattices with triangular obstacles, circular sector obstacles and triangular and sectors circular together. We study the probability that a random segment of constant length intersects the fundamental cells in Figures 1, 3, and 5.

2. Obstacles triangular

Let $\mathcal{R}_1(a, b; m)$ be the lattice with the fundamental cell $C_1$ represented in Figure 1, where $a < b$ and $m < a/2$. From Figure 1 we have

$$\text{area} C_1 = 3ab - \frac{5}{2}m^2. \quad (2.1)$$

We compute the probability that a random segment $s$ of constant length $l < \frac{a}{2} - m$ intersects a side of lattice $\mathcal{R}_1$, i.e., the probability $P_{\text{int}}^{(1)}$ that the segment $s$ intersects a side of fundamental cell $C_1$.

The position of segment $s$ is determined by its center and by the angle $\varphi$ that it formed with the side BC (or AF) of the cell $C_1$. 

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To compute \( P_{\text{int}} \) we consider the limiting positions of segment \( s \), for a fixed angle of \( \varphi \), in the cell \( C_1 \).

We obtain the Figure 2

![Figure 1](image1.png)

![Figure 2](image2.png)

and the formula

\[
\text{area}\hat{C}_1 (\varphi) = \text{area}C_1 - \sum_{i=1}^{11} \text{area}a_i (\varphi). \tag{2.2}
\]

**Theorem 2.1.** We have

\[
P_{\text{int}}^{(1)} = \frac{2 \left[ 2(a + b)1 - \frac{l^2}{2} - \frac{\pi}{2}m^2 \right]}{\pi \left( 3ab - \frac{\pi}{2}m^2 \right)}.
\]

**Proof.** By Figure 2 we have

\[
\text{area}A_1A_3 = \frac{ml}{2} \cos \varphi, \quad \text{area}a_i (\varphi) = \text{area}a_5 (\varphi) = \frac{ml}{2} \cos \varphi - \frac{m^2}{2},
\]

\[
\text{area}a_2 (\varphi) = (b - ml) \cos \varphi, \quad \text{area}a_{11} (\varphi) = \frac{a_1}{2} \sin \varphi - \frac{ml}{2} \sin \varphi - \frac{l^2}{4} \sin 2\varphi,
\]

\[
\text{area}a_6 (\varphi) = \frac{bl}{2} \cos \varphi - ml \cos \varphi, \quad \text{area}a_3 (\varphi) = \text{area}a_7 (\varphi) = \text{area}a_{10} (\varphi) = \frac{ml}{2} (\sin \varphi + \cos \varphi),
\]

\[
\text{area}a_4 (\varphi) = a_1 \sin \frac{ml}{2} \sin \varphi - \frac{l^2}{4} \sin 2\varphi, \quad \text{area}a_8 (\varphi) = \frac{a_1}{2} \sin \varphi - \frac{ml}{2} \sin \varphi,
\]

\[
\text{area}a_9 (\varphi) = \frac{bl}{2} \cos \varphi - \frac{ml}{2} \cos \varphi.
\]

We can write that

\[
A_1 (\varphi) = \sum_{i=1}^{11} \text{area}a_i (\varphi) = 2a_1 \sin \varphi + 2bl \cos \varphi - \frac{l^2}{2} \sin 2\varphi - m^2. \tag{2.3}
\]

Replacing this relation in formula (2.2) follows

\[
\text{area}C_1 (\varphi) = \text{area}C_1 - A_1 (\varphi). \tag{2.4}
\]

We denote with \( M_1 \), the set of segments \( s \) that they have center in the cell \( C_1 \), and with \( N_1 \) the set of segments \( s \) entirely contained in the cell \( C_1 \), so we have [11],

\[
P_{\text{int}}^{(1)} = 1 - \frac{\mu (N_1)}{\mu (M_1)}, \tag{2.5}
\]

where \( \mu \) is the Lebesgue measure in the euclidean plane.

To compute the measure \( \mu (M_1) \) and \( \mu (N_1) \) we use the kinematic measure of Poincarè [10]:

\[
dk = dx \wedge dy \wedge d\varphi,
\]

where \( x, y \) are the coordinate of center of \( s \) and \( \varphi \) the fixed angle.

For \( \varphi \in \left[ 0, \frac{\pi}{2} \right] \) we have

\[
\mu (M_1) = \int_0^{\frac{\pi}{2}} d\varphi \int_{(x,y) \in \hat{C}_1} dx dy = \int_0^{\frac{\pi}{2}} \left( \text{area}C_1 \right) d\varphi = \frac{\pi}{2} \text{area}C_1. \tag{2.6}
\]

In the same way, considering formula (2.4) we can write
\( \mu(N_1) = \int_0^\frac{\pi}{2} d\varphi \int_{\{(x,y)\in C_1(\varphi)\}} dx dy = \int_0^\frac{\pi}{2} \left[ \text{area} C_1(\varphi) \right] d\varphi \)

\( = \int_0^\frac{\pi}{2} \left[ \text{area} C_1 - A_1(\varphi) \right] d\varphi = \frac{\pi}{2} \text{area} C_1 - \int_0^\frac{\pi}{2} [A_1(\varphi)] d\varphi. \)  

Replacing in the (2.5) the relations (2.6) and (2.7) we obtain

\[ p^{(1)}_{\text{int}} = \frac{2}{\pi \text{area} C_1} \int_0^\frac{\pi}{2} [A_1(\varphi)] d\varphi. \]  

Considering formula (2.3) we have

\[ \int_0^\frac{\pi}{2} [A_1(\varphi)] d\varphi = 2(a + b) l - \frac{l^2}{2} - \frac{\pi}{2} m^2. \]  

The relations (2.1), (2.8) and (2.9) give us the result. \( \square \)

**Remark 2.2.** For \( m = 0 \) the obstacles become points and the probability \( p^{(1)}_{\text{int}} \) becomes:

\[ p^{(1)} = \frac{4(a + b) l - l^2}{3\pi ab}. \]  

So, we find a result of a previous paper [3].

### 3. Obstacles circular sectors

We consider the lattice \( \mathcal{R}_2(a, b; m) \) with the fundamental cell \( C_2 \) represented in Figure 3.

By this figure we have that the formula (2.2) is valid for the cell \( C_2 \). Then we have

\[ \text{area} C_2(\varphi) = 3ab - \frac{5\pi}{4} m^2. \]

As in the paragraph 1, we compute the probability \( p^{(2)}_{\text{int}} \) that a segment \( s \) intersects a side of fundamental cell \( C_2 \).

Considering the limiting positions of segment \( s \), for a fixed angle \( \varphi \), in the cell \( C_2 \). We obtain the Figure 4

![Figure 3](image1)

![Figure 4](image2)

and the formula

\[ \text{area} \tilde{C}_2(\varphi) = \text{area} C_2 - \sum_{i=1}^{13} \text{area} b_i(\varphi). \]

**Theorem 3.1.** We have

\[ p^{(2)}_{\text{int}} = \frac{2 \left[ 2(a + b) l - l^2 - \frac{\pi m^2 (5\pi - 6)}{8} \right]}{\pi (3ab - \frac{5\pi}{4} m^2)}. \]

**Remark 2.2.** For \( m = 0 \) the obstacles become points and the probability \( p^{(2)}_{\text{int}} \) become:

\[ p^{(2)} = \frac{4(a + b) l - l^2}{3\pi ab}. \]  

### 4. Obstacles triangular and circular sectors

We consider the lattice \( \mathcal{R}_3(a, b; m) \) with the fundamental cell \( C_3 \) represented in Figure 5.
From Figure 5 we have
\[ \text{area} C_3 = 3ab - m^2 - \frac{3\pi m^2}{4}. \]

As in the previous paragraphs, we compute the probability \( P_{\text{int}}^{(3)} \) that a segment \( s \) intersects a side of fundamental cell \( C_3 \).

Considering the limiting positions of segment \( s \), for a fixed angle \( \varphi \), in the cell \( C_2 \). We obtain Figure 6

\[ \text{area} \widehat{C}_3 (\varphi) = \text{area} C_3 - \sum_{i=1}^{11} \text{area} c_i (\varphi). \]

**Theorem 4.1.** We have
\[ P_{\text{int}}^{(3)} = \frac{4(a + b) l - l^2}{\pi(3ab - m^2 - \frac{3\pi m^2}{4})}. \]

**Remark 4.2.** If \( m = 0 \), the obstacles become points and the probability \( P_{\text{int}} \) becomes
\[ P^{(3)} = \frac{4(a + b) l - l^2}{3\pi ab}, \quad (4.1) \]

**Remark 4.3.** The relation (2.10), (3.1) and (4.1) give us the evident formula
\[ P^{(1)} = P^{(2)} = P^{(3)}. \]

**References**