Simultaneous iteration for variational inequalities over common solutions for finite families of nonlinear problems

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Abstract

In this paper, we apply Theorem 3.2 of [G. M. Lee, L.-J. Lin, J. Nonlinear Convex Anal., 18 (2017), 1781–1800] to study the variational inequality over split equality fixed point problems for three finite families of strongly quasi-nonexpansive mappings. Then we use this result to study variational inequalities over split equality for three various finite families of nonlinear mappings. We give a unified method to study split equality for three various finite families of nonlinear problems. Our results contain many results on split equality fixed point problems and multiple sets split feasibility problems as special cases. Our results can treat large scale of nonlinear problems by group these problems into finite families of nonlinear problems, then we use simultaneous iteration to find the solutions of these problems. Our results will give a simple and quick method to study large scale of nonlinear problems and will have many applications to study large scale of nonlinear problems.

Keywords: Split equality fixed point problem, split fixed point problem, quasi-pseudocontractive mapping, demicontractive mapping, pseudo-contractive mapping.

2010 MSC: 47H06, 47H09, 47H10, 47J25, 65K15.

1. Introduction

Let $T : H_1 \to H_1$, and let $\text{Fix}(T) = \{x \in H_1 : x = Tx\}$ denote the fixed point set of $T$. For each $i \in \{1, 2, 3\}$, let $H_i$ be a real Hilbert space. Let $C$ and $Q$ be nonempty closed convex subsets of $H_1$ and $H_2$, respectively and $A : H_1 \to H_2$ be a bounded linear operator.

The split feasibility problem (SFP) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [6] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction.

The split feasibility problem (SFP) is the problem:

Find $\bar{x} \in H_1$ such that $\bar{x} \in C$ and $A\bar{x} \in Q$.

Let $F : C \to H_1$ be an operator. The variational inequality problem $\text{VIP}(F, C)$ is the following problem:

Find $\bar{x} \in C$ such that $\langle F\bar{x}, u - \bar{x} \rangle \geq 0$ for all $u \in C$. 

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The solution set of the variational inequality problem is denoted by $VI(F, C)$. The variational inequality problem $VIP(F, C)$ has many applications in engineering, optimization, and signal recovery problem, see for example, Chuang et al. [11] and references therein.

Let $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operators, the split equality problem (SEFP) which was first introduced by Moudafi [18] is the problem:

$$\text{Find } \bar{x} \in C, \bar{y} \in Q \text{ such that } A\bar{x} = B\bar{y}. $$

The split equality problem has many applications such as decomposition method for PDE, application in image science, game theory, and intensity-modulated radiation [18]. It is easy to see that when $B = I$, and $H_2 = H_3$, then (SEFP) is reduced to (SFP). Moudafi [18] introduced an iteration process to establish a weak convergence theorem for split equality problem under suitable assumptions.

Let $T : H_1 \to H_1, S : H_2 \to H_2$ be firmly quasi-nonexpansive mappings such that $\text{Fix}(T) \neq \emptyset, \text{Fix}(S) \neq \emptyset$, and let $A : H_1 \to H_3, B : H_2 \to H_3$ be bounded linear operators. Moudafi and Al-Shemas [19] introduced an iteration process and established a weak convergence theorem for split equality fixed point problem (SEFPP):

$$\text{Find } \bar{x} \in \text{Fix}(T), \bar{y} \in \text{Fix}(S) \text{ such that } A\bar{x} = B\bar{y}. $$

When $B = I$, and $H_2 = H_3$, then (SEFPP) is reduced to the split common fixed point problem (SCFPP) [7, 17]:

$$\text{Find } \bar{x} \in H_1 \text{ such that } \bar{x} \in \text{Fix}(U) \text{ and } A\bar{x} \in \text{Fix}(W). $$

Recently, many results on split equality fixed point problem have been found and one is referred to [8, 10, 23, 24, 26, 27] and references therein.

Recently, Lee and Lin [14], studied variational inequality problem over split equality fixed point sets of strongly quasi-nonexpansive mappings with applications to variational inequality problem over split equality fixed point for the same type of $m$ nonlinear operators.

In this paper, we apply Lee and Lin [14, Theorem 3.2] to study the variational inequality over split equality fixed point problems for three finite families of strongly quasi-nonexpansive mappings. Then we use this result to study variational inequalities over split equality for three various finite families of nonlinear mappings. We give a unified method to study split equality for three various finite families of nonlinear problems. Our results contain many results on split equality fixed point problems and multiple sets split feasibility problems as special cases. Our results can treat large scale of nonlinear problems by group these problems into finite families of nonlinear problems, then we use simultaneous iteration to find the solutions of these problems. Our results will give a simple and quick method to study large scale of nonlinear problems and will have many applications to study large scale of nonlinear problems.

2. Preliminaries

For each $i \in \{1, 2, 3, 4\}$, let $H_i$ be a (real) Hilbert space with inner products $\langle \cdot, \cdot \rangle$ and norms $\| \cdot \|$, and let $I_i : H_i \to H_i$ be the identity mapping on $H_i$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $H_i$ and $x \in H_i$, we denote the strongly convergence and the weak convergence of $\{x_n\}_{n \in \mathbb{N}}$ to $x \in H_i$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Throughout this paper, we use these notations unless specified otherwise. Let $C$ be a nonempty subset of a real Hilbert space $H_i$, and let $T : C \to H_i$. Then $T$ is

(1) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(2) quasi-nonexpansive if $\text{Fix}(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and for all $y \in \text{Fix}(T)$;
(3) $\rho$-strongly quasi-nonexpansive (in short $\rho$- SQNE), where $\rho \geq 0$, if $\text{Fix}(T) \neq \emptyset$ and

$$\|Tx - y\|^2 \leq \|x - y\|^2 - \rho \|Tx - x\|^2$$

for all $x \in C, y \in \text{Fix}(T)$.
(4) monotone if \((x - y, Tx - Ty) \geq 0\) for all \(x, y \in C\);
(5) \(\gamma\)-strongly monotone if there exists \(\gamma > 0\) such that \((x - y, Tx - Ty) \geq \gamma \|x - y\|^2\) for all \(x, y \in C\);
(6) pseudocontractive if \(|Tx - Ty|^2 \leq \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2\) for all \(x, y \in C\);
(7) \(k\)-demicontractive if \(\text{Fix}(T) \neq \emptyset\) and there exists \(-\infty < k < 1\) such that \(|Tx - y|^2 \leq \|x - y\|^2 + k \|Tx - x\|^2\) for all \(x \in C\) and for all \(y \in \text{Fix}(T)\);
(8) \(k\)-strictly pseudononspreading \([20]\) if there exists \(k \in (0, 1)\) such that \(|Tx - Ty|^2 \leq \|x - y\|^2 + k \|x - Tx - (y - Ty)\|^2 + \langle x - Tx, y - Ty\rangle\) for all \(x, y \in C\);
(9) firmly nonexpansive if \(|Tx - Ty|^2 + \|(I_1 - T)x - (I_1 - T)y\|^2 \leq \|x - y\|^2\) for all \(x, y \in C\);
(10) directed if \(\text{Fix}(T) \neq \emptyset\), and \(\langle Tx - y, Tx - x\rangle \leq 0\) for all \(x \in C\) and for all \(y \in \text{Fix}(T)\);
(11) demiclosed if for each sequence \(\{x_n\}\) and \(x \in C\) with \(x_n \rightharpoonup x\) and \((1 - T)x_n \rightarrow 0\) implies that \((1 - T)x = 0\);
(12) \(\alpha\)-averaged if there exist \(\alpha \in (0, 1)\) and a nonexpansive mapping \(S : C \rightarrow H_1\) such that \(T = (1 - \alpha)I + \alpha S\);
(13) hemicontinuous if, for all \(x, y \in C\), the mapping \(g : [0, 1] \rightarrow H_1\), defined by \(g(t) = T(tx + (1 - t)y)\) is continuous with respect to weak topology on \(H_1\);
(14) quasi-pseudocontractive if \(\text{Fix}(T) \neq \emptyset\) and \(|Tx - y|^2 \leq \|x - y\|^2 + |Tx - x|^2\) for all \(x \in C\) and for all \(y \in \text{Fix}(T)\);
(15) \(\alpha\)-inverse-strongly monotone (in short \(\alpha\)-ism) if \(\langle x - y, Tx - Ty\rangle \geq \alpha \|Tx - Ty\|^2\) for all \(x, y \in C\) and \(\alpha > 0\).

**Lemma 2.1** ([3]). Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H_1\). Let \(T : C \rightarrow H_1\) be a nonexpansive mapping, and let \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in \(C\). If \(x_n \rightharpoonup w\) and \(\lim_{n \to \infty} \|x_n - Tx_n\| = 0\), then \(Tw = w\).

Let \(f : H_1 \rightarrow (-\infty, \infty)\) be a proper, lower-semicontinuous, and convex function. Then the subdifferential \(\partial f\) of \(f\) is defined by

\[
\partial f(x) = \{u \in H_1 : f(y) \geq f(x) + \langle y - x, u\rangle \text{ for all } y \in H_1\}.
\]

Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H_1\). For each \(x \in H_1\), there is a unique element \(u \in C\) such that \(u = \arg \min_{y \in C} \|x - y\|\). The mapping \(P_C : H_1 \rightarrow C\) which is defined by \(P_C x = \arg \min_{y \in C} \|x - y\|\) for \(x \in H_1\) is called the metric projection from \(H_1\) onto \(C\).

**Proposition 2.2** ([1]). Let \(C\) be a nonempty subset of a Hilbert space \(H_1\), and let \(T : C \rightarrow H_1\) be nonexpansive, and \(\alpha \in (0, 1)\). Then the following are equivalent:

(i) \(T\) is \(\alpha\)-averaged;
(ii) \((\forall x \in C)(\forall y \in C), \|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1 - \alpha}{\alpha} \|(I_1 - T)x - (I_1 - T)y\|^2.\)

**Lemma 2.3** ([15]). Let \(T : H_1 \rightarrow H_1\) be a \(k\)-demicontractive operator with \(k < 1\). Denote \(T_\lambda = (1 - \lambda)I_1 + \lambda T\) for \(\lambda \in (0, 1 - k)\). Then for any \(x \in H_1, z \in \text{Fix}(T)\),

\[
\|T_\lambda x - z\|^2 \leq \|x - z\|^2 - (1 - k - \lambda)\|T_\lambda x - x\|^2.
\]

**Lemma 2.4** ([20]). Let \(C\) be a nonempty closed convex subset of \(H_1\) and \(T : C \rightarrow C\) be a \(k\)-strictly pseudononspreading mapping with \(\text{Fix}(T) \neq \emptyset\). Set \(T_\lambda = \lambda I_1 + (1 - \lambda)T, \lambda \in [k, 1)\). Then the following hold:

(i) \(\text{Fix}(T_\lambda) = \text{Fix}(T)\);
(ii) \(T_\lambda\) is demiclosed;
(iii) \(|T_\lambda x - T_\lambda y|^2 \leq |x - y|^2 + \frac{2}{1 - k}(x - T_\lambda x, y - T_\lambda y) - (\lambda - k)\|x - T_\lambda x - (y - T_\lambda y)\|^2.\)

The equilibrium problem \((EP)\) [2] is the problem:

Find \(z \in C\) such that \(g(z, y) \geq 0\) for each \(y \in C\),
where $g : C \times C \to \mathbb{R}$ is a bifunction. The solution set of equilibrium problem (EP) is denoted by $\text{EP}(C, g)$. We say that $g : C \times C \to \mathbb{R}$ satisfies the following conditions (A1)-(A4) if the following conditions hold:

(A1) $g(x, x) = 0$ for each $x \in C$;
(A2) $g$ is monotone, i.e., $g(x, y) + g(y, x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, lim sup $g(tz + (1 - t)x, y) \leq g(x, y)$; 
(A4) for each $x \in C$, the scalar function $y \to g(x, y)$ is convex and lower semicontinuous.

**Theorem 2.5 ([12]).** Let $g : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). For $r > 0$, define $T^g_r : H_1 \to C$ by

$$T^g_r x = \left\{ z \in C : g(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(i) $T^g_r$ is single-valued;
(ii) $T^g_r$ is firmly nonexpansive, that is, $\|T^g_r x - T^g_r y\|^2 \leq \langle x - y, T^g_r x - T^g_r y \rangle$ for all $x, y \in H$;
(iii) $\{x \in H : T^g_r x = x\} = \{x \in C : g(x, y) \geq 0, \forall y \in C\}$;
(iv) $\{x \in C : g(x, y) \geq 0, \forall y \in C\}$ is a closed and convex subset of $C$.

Here, $T^g_r$ is called the resolvent of $g$ for $r > 0$.

**Theorem 2.6 ([14]).** Let $M : C \to H_1$ be a hemi-continuous and monotone mapping. Suppose that $M$ is locally bounded on $C$. Then, for $r > 0$ and $x \in H_1$, define $T_r : H_1 \to C$ by

$$T_r x = \left\{ z \in C : \langle y - z, Mz \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(i) $T_r$ is single-valued;
(ii) $T_r$ is firmly nonexpansive, that is, $\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y \rangle$ for all $x, y \in H$;
(iii) $\{x \in H : T_r x = x\} = \text{VI}(M, C)$;
(iv) $\text{VI}(M, C)$ is a closed and convex subset of $C$.

**Theorem 2.7 ([14]).** Let $T : C \to H_1$ be a hemi-continuous and pseudocontractive mapping. Suppose that $T$ is locally bounded on $C$. Then, for each $r > 0$ and each $x \in H_1$, define $F_r : H_1 \to C$ by

$$F_r x = \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(i) $F_r$ is single-valued;
(ii) $F_r$ is firmly nonexpansive;
(iii) $\text{Fix}(F_r) = \text{Fix}(T)$;
(iv) $\text{Fix}(T)$ is a closed and convex subset of $C$.

**Proposition 2.8 ([5]).** Let $A : H_1 \to H_2$ be a bounded linear operator with $\|A\| > 0$ and $T : H_2 \to H_2$ be an operator satisfying $TAW = Aw$ for some $w \in H_1$. Further let $V = I_1 - \frac{1}{\|A\|^2} A^*(I_2 - T)A$. If $T$ is an $\alpha$-SQNE operator for some $\alpha > 0$, then

(i) $\text{Fix}(V) = A^{-1} \text{Fix}(T)$;
(ii) $V$ is $\alpha$-SQNE.
If $T$ is demiclosed, then $V$ is demiclosed.

Define $L = \{1, 2, \ldots, m\}$ and $\Delta_m = (\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in \mathbb{R}^m, \omega_i \geq 0, i \in L,$ and $\sum_{i=1}^{m} \omega_i = 1)$.  

**Proposition 2.9** ([5]). For each $i \in L$, let $S_i : H_1 \to H_1$ be demiclosed and $p_i$-SQNE Suppose that $\cap_{i \in I} \text{Fix}(S_i) \neq \emptyset$. Let $S = \sum_{i=1}^{m} \omega_i S_i$, where $\omega \in \Delta_m$. Then $S$ is a $p$-SQNE operator with $p = \sum_{i=1}^{m} (\omega_i \rho^{-1})^{-1} - 1$ and $S$ is demiclosed.

**Proposition 2.10** ([11]). Let $C$ be a nonempty subset of $H_1$, let $\{T_i\}_{i \in I}$ be a finite family of quasi-nonexpansive operators from $C$ to $H_1$ such that $\cap_{i \in I} \text{Fix}(T_i) \neq \emptyset$, let $\{\omega_i : i \in I\}$ be strict positive numbers such that $\sum_{i \in I} \omega_i = 1$. Then $\text{Fix}(\sum_{i \in I} \omega_i T_i) = \cap_{i \in I} \text{Fix}(T_i)$.

**Lemma 2.11** ([9]). Let $T : H_1 \to H_1$ be a $L_1$-Lipschitz continuous mapping with $L_1 > 0$. Denote $K = (1 - \xi)I_1 + \xi T(I_1 + T)$. If $0 < \eta < \xi < \frac{1}{1 + \sqrt{1 + L_1^2}}$, then

(i) $\text{Fix}(T) \subseteq \text{Fix}(K)$;
(ii) if $T$ is demiclosed, then $K$ is also demiclosed;
(iii) in addition, if $T : H_1 \to H_1$ is quasi-pseudocontractive, then $K$ is quasi-nonexpansive.

Let $C$ be a nonempty closed convex subset of $H_1$, and let the indicate function $\iota_C : H_1 \to [0, \infty)$ be defined by

$$\iota_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then $\iota_C$ is a proper, lower semicontinuous, and convex function and $\iota_C^\alpha = P_C$.

Let $g \in \Gamma_0(H_1)$ and $\lambda \in (0, \infty)$. The proximal operator of $g \in \Gamma_0(H_1)$ of order $\lambda \in (0, \infty)$ is

$$\text{prox}_{\lambda g} x = \arg\min_{v \in H_1} \{g(v) + \frac{1}{2\lambda} \|v - x\|^2\}, x \in H_1.$$

**Lemma 2.12** ([11]). Let $g \in \Gamma_0(H_1)$ and $\lambda \in (0, \infty)$. Then

(i) $\text{prox}_{\lambda g} = (I_1 + \lambda \iota_C)^{-1} = I_1^\lambda g$;
(ii) $\text{prox}_{\lambda g}$ is firmly nonexpansive;
(iii) if $C$ is a nonempty closed convex subset of $H_1$ and $g = \iota_C$, then $\text{prox}_{\lambda g} = P_C$ for all $\lambda \in (0, \infty)$.

**Lemma 2.13** ([16]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $T : C \to C$ be a $k$-strictly pseudocontractive mapping. Then $T$ is demiclosed.

3. Variational inequalities over split equality fixed point for finite families of nonlinear mappings

For each $i \in \{1, 2, 3, 4\}$, let $H_i$ be a Hilbert space, $I_i$ be the identity mapping on $H_i$, $V_i : H_i \to H_i$ be $L_i$-Lipschitz continuous, $F_i : H_i \to H_i$ be $\kappa_i$-Lipschitz continuous and $\eta_i$-strongly monotone with $\kappa_i > 0$, and $\eta_i > 0$. Let

$$L = \max_{1 \leq i \leq 3} L_i, \kappa = \max_{1 \leq i \leq 3} \kappa_i, \eta = \min_{1 \leq i \leq 3} \eta_i, \mu \in (0, \frac{2\eta}{\kappa^2}) \text{ and } \gamma \in (0, \frac{\tau}{\lambda}),$$

where $\tau = \mu(\eta - \frac{1}{2}\mu \kappa^2)$. For each $i \in \{1, 2, 3\}$, let $A_i : H_i \to H_4$ be a bounded linear operator with adjoint $A_i^*$. Suppose that $\|A_i\| > 0, 0 < \xi < \frac{1}{\sum_{i=1}^{3} \|A_i\|^2}$. Let $A_1 : H_1 \to H_2$ be a bounded linear operator with adjoint $A_1^*$ and let $B_2 : H_2 \to H_3$ be a bounded linear operator with adjoint $B_2^*$. Suppose that $\|B_1\| > 0$ and $\|B_2\| > 0$. The product $\bigotimes_{i=1}^{3} H_i = H_1 \times H_2 \times H_3$ is a Hilbert space with inner product and norm given by

$$\langle x, y \rangle = \sum_{i=1}^{3} \langle x_i, y_i \rangle \text{ and } \|x\|^2 = \sum_{i=1}^{3} \|x_i\|^2$$
for any \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \bigotimes_{1 \leq i \leq 3} H_i \). For any \( x = (x_1, x_2, x_3) \in \bigotimes_{1 \leq i \leq 3} H_i \), let \( V, F, I = I_1 \times I_2 \times I_3 : \bigotimes_{1 \leq i \leq 3} H_i \to \bigotimes_{1 \leq i \leq 3} H_i \) be defined by

\[
\begin{align*}
I(x) &= (x_1, x_2, x_3), \\
V(x) &= (V_1(x_1), V_2(x_2), V_3(x_3)), \\
F(x) &= (F_1(x_1), F_2(x_2), F_3(x_3)).
\end{align*}
\]

Let \( \{\alpha_n\}_{n=0}^{\infty} \) be a sequence in \((0, 1)\) such that \( \lim_{n \to \infty} \alpha_n = 0 \), and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). In the following, we use these notations and assumptions unless specified otherwise.

**Theorem 3.1** ([14]). For each \( i \in \{1, 2, 3\} \), let \( \rho_i > 0 \) and let \( T_i : H_i \to H_i \) be a demiclosed \( \rho_i\)-strongly quasi-nonexpansive mapping. Suppose that

\[
\Lambda = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \text{Fix}(T_1), y \in \text{Fix}(T_2), z \in \text{Fix}(T_3), A_1(x) = A_2(y) = A_3(z) \neq \emptyset\}.
\]

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3, \) and let the sequences \( \{x_n, y_n, z_n\}_{n \in \mathbb{N}} \) be defined by

(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n F_1) T_1(x_n - \frac{\delta}{\lambda} A_1^* (2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);

(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (1 - \mu \alpha_n F_2) T_2(y_n - \frac{\delta}{\lambda} A_2^* (2A_2(y_n) - A_1(x_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);

(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (1 - \mu \alpha_n F_3) T_3(z_n - \frac{\delta}{\lambda} A_3^* (2A_3(z_n) - A_1(x_n) - A_2(y_n))) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in \mathcal{V}(\mu F - \gamma V, \Lambda) \).

**Remark 3.2.** The assumption “" \( \rho_i > 0 \) ” is needed in [14, Theorem 3.2].

The following theorem and corollary are essential tools in this paper.

**Theorem 3.3.** For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\} \), let \( \sigma_i > 0, \tau_j > 0 \) and \( \delta_k > 0 \), and let

(i) \( M_i : H_i \to H_i \) be a demiclosed \( \sigma_i\)-strongly quasi-nonexpansive mapping;

(ii) \( Q_j : H_2 \to H_2 \) be a demiclosed \( \tau_j\)-strongly quasi-nonexpansive mapping;

(iii) \( G_k : H_3 \to H_3 \) be a demiclosed \( \delta_k\)-strongly quasi-nonexpansive mapping.

Let \( (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s \). Suppose that

\[
\Lambda = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^{m} \text{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), z \in \bigcap_{k=1}^{s} \text{Fix}(G_k), A_1(x) = A_2(y) = A_3(z) \neq \emptyset\}.
\]

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3, \) and let the sequences \( \{x_n, y_n, z_n\}_{n \in \mathbb{N}} \) be defined by

(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n F_1) \sum_{i=1}^{m} \zeta_i M_i (x_n - \frac{\delta}{\lambda} A_1^* (2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);

(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (1 - \mu \alpha_n F_2) \sum_{j=1}^{\ell} \theta_j Q_j (y_n - \frac{\delta}{\lambda} A_2^* (2A_2(y_n) - A_1(x_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);

(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (1 - \mu \alpha_n F_3) \sum_{k=1}^{s} \omega_k G_k (z_n - \frac{\delta}{\lambda} A_3^* (2A_3(z_n) - A_1(x_n) - A_2(y_n))) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in \mathcal{V}(\mu F - \gamma V, \Lambda) \).

**Proof.** Let

(i) \( T_1 = \sum_{i=1}^{m} \zeta_i M_i \);

(ii) \( T_2 = \sum_{j=1}^{\ell} \theta_j Q_j \);

(iii) \( T_3 = \sum_{k=1}^{s} \omega_k G_k \).

By Proposition 2.9,

(i) \( T_1 \) is a demiclosed \( \rho_1\)-strongly quasi-nonexpansive mapping for some \( \rho_1 > 0; \)
By Proposition 2.10,
(i) \( \text{Fix}(T_1) = \bigcap_{i=1}^{m} \text{Fix}(M_i) \);
(ii) \( \text{Fix}(T_2) = \bigcap_{j=1}^{\ell} \text{Fix}(Q_j) \);
(iii) \( \text{Fix}(T_3) = \bigcap_{k=1}^{n} \text{Fix}(G_k) \).

Let \( \Omega = \{(x,y,z) \in \prod_{1 \leq i \leq 3} H_i : x \in \text{Fix}(T_1), y \in \text{Fix}(T_2), z \in \text{Fix}(T_3)\}, A_1(x) = A_2(y) = A_3(z) \} \).

It is easy to see that \( \Omega = \Lambda \neq \emptyset \). Then Theorem 3.3 follows from Theorem 3.1.

Corollary 3.4. For each \( i \in \{1, 2, 3\} \), let \( V_i : H_1 \to H_1 \) be \( L_i \)-Lipschitz continuous, \( F_i : H_1 \to H_1 \) be \( \kappa_i \)-Lipschitz continuous and \( \eta_i \)-strongly monotone with \( \kappa_i > 0 \) and \( \eta_i > 0 \).

For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}, \) let \( \sigma_i > 0, \tau_j > 0 \) and \( \delta_k > 0 \), and let
(i) \( M_i : H_1 \to H_1 \) be a demiclosed \( \sigma_i \)-strongly quasi-nonexpansive mapping;
(ii) \( Q_j : H_1 \to H_1 \) be a demiclosed \( \tau_j \)-strongly quasi-nonexpansive mapping;
(iii) \( G_k : H_1 \to H_1 \) be a demiclosed \( \delta_k \)-strongly quasi-nonexpansive mapping.

Let \( (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s \). Suppose that
\[
\Lambda = \{(x,x,x) : x \in H_1, x \in \bigcap_{i=1}^{m} \text{Fix}(M_i) \bigcap_{j=1}^{\ell} \text{Fix}(Q_j) \bigcap_{k=1}^{n} \text{Fix}(G_k) \} \neq \emptyset.
\]

Let \( x_1 \in H_1, y_1 \in H_1, z_1 \in H_1 \), and let the sequences \( \{(x_n,y_n,z_n)\}_{n \in \mathbb{N}} \) be defined by
(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (1-\mu \alpha_n F_1) \sum_{i=1}^{m} \zeta_i M_i(x_n - \frac{\mu}{2} (2x_n - y_n - z_n)) \) for all \( n \in \mathbb{N} \);
(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (1-\mu \alpha_n F_2) \sum_{j=1}^{\ell} \theta_j Q_j(y_n - \frac{\mu}{2} (2y_n - x_n - z_n)) \) for all \( n \in \mathbb{N} \);
(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (1-\mu \alpha_n F_3) \sum_{k=1}^{s} \omega_k G_k(z_n - \frac{\mu}{2} (2z_n - x_n - y_n)) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n,y_n,z_n) \in V(\mu F - \gamma V, \Lambda) \).

Proof. Let \( H_1 = H_2 = H_3 = H_4, A_1 = I_1 = A_2 = A_3 \) in Theorem 3.3, then Corollary 3.4 follows from Theorem 3.3.

Theorem 3.5. For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}, \) let
(i) \( M_i : H_1 \to H_1 \) be a demiclosed \( \sigma_i \)-demicontractive mapping;
(ii) \( Q_j : H_2 \to H_2 \) be a demiclosed \( \tau_j \)-demicontractive mapping;
(iii) \( G_k : H_3 \to H_3 \) be a demiclosed \( \delta_k \)-demicontractive mapping.

Let
(i) \( M_i \lambda_i = (1- \lambda_i) I_1 + \lambda_i M_i \) for \( \lambda_i \in (0, 1- \sigma_i) \);
(ii) \( Q_j \beta_j = (1- \beta_j) I_2 + \beta_j Q_j \) for \( \beta_j \in (0, 1- \tau_j) \);
(iii) \( G_k \eta_k = (1- \eta_k) I_3 + \eta_k G_k \) for \( \eta_k \in (0, 1- \delta_k) \).

Let \( (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s \). Suppose that \( \Lambda = \{(x,y,z) \in \prod_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^{m} \text{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), z \in \bigcap_{k=1}^{n} \text{Fix}(G_k)\}, A_1(x) = A_2(y) = A_3(z) \} \neq \emptyset \).

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3, \) and let the sequences \( \{(x_n,y_n,z_n)\}_{n \in \mathbb{N}} \) be defined by
(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (1- \mu \alpha_n F_1) \sum_{i=1}^{m} \zeta_i M_i \lambda_i (x_n - \frac{\mu}{2} A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);
(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (1- \mu \alpha_n F_2) \sum_{j=1}^{\ell} \theta_j Q_j \beta_j (y_n - \frac{\mu}{2} A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);
Corollary 3.7. For each \( \eta \in (0, 1) \), let \( \Lambda \) be a demiclosed quasi-nonexpansive mapping.

\[ n \to \infty \]
\[ \lim \, (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda). \]

Proof. Since for each \( i \in \{1, 2, \ldots, m_j \} \), \( j \in \{1, 2, \ldots, \ell \} \), \( k \in \{1, 2, \ldots, s \} \),

(i) \( M_i : H_i \to H_i \) be a demiclosed \( \sigma_i \)-demicontractive mapping;
(ii) \( Q_j : H_2 \to H_2 \) be a demiclosed \( r_j \)-demicontractive mapping;
(iii) \( G_k : H_3 \to H_3 \) be a demiclosed \( \delta_k \)-demicontractive mapping.

It follows from Lemma 2.3 for each \( i \in \{1, 2, \ldots, m_j \} \), \( j \in \{1, 2, \ldots, \ell \} \), \( k \in \{1, 2, \ldots, s \} \), \( \lambda_i \in (0, 1 - \sigma_i) \), \( \beta_j \in (0, 1 - r_j) \), and \( \eta_k \in (0, 1 - \delta_k) \) that

(i) \( M_{i\lambda_i} \) is a demiclosed \( (1 - \sigma_i - \lambda_i) \)-strongly quasi-nonexpansive mapping;
(ii) \( Q_j\beta_j \) is a demiclosed \( (1 - r_j - \beta_j) \)-strongly quasi-nonexpansive mapping;
(iii) \( G_k\eta_k \) is a demiclosed \( (1 - \delta_k - \eta_k) \)-strongly quasi-nonexpansive mapping.

It is easy to see that

(i) \( \text{Fix}(M_{i\lambda_i}) = \text{Fix}(M_i) \);
(ii) \( \text{Fix}(Q_j\beta_j) = \text{Fix}(Q_j) \);
(iii) \( \text{Fix}(G_k\eta_k) = \text{Fix}(G_k) \).

Let \( \Omega = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^m \text{Fix}(M_i) \}, y \in \bigcap_{j=1}^\ell \text{Fix}(Q_j) \}, z \in \bigcap_{k=1}^s \text{Fix}(G_k) \}, A_1(x) = A_2(y) = A_3(z) \).

It is easy to see that \( \Omega = \Lambda \neq \emptyset \). Then Theorem 3.5 follows from Theorem 3.3. \( \square \)

Remark 3.6.

(i) Theorem 3.5 improves and generalizes [24, Theorem 3.2] and [22, Theorem 3.3]. In [22, Theorem 3.3], \( V = (f_1, f_2) \) is a contraction mapping and \( F = (I_1, I_2) \).
(ii) Theorem 3.5 extends [14, Theorem 4.1] from variational inequality over split equality fixed points of \( m \) demicontractive mappings to variational inequality over split equality of three families of demicontractive mappings.

Corollary 3.7. For each \( i \in \{1, 2, \ldots, m_j \} \), \( j \in \{1, 2, \ldots, \ell \} \), \( k \in \{1, 2, \ldots, s \} \), let

(i) \( M_i : H_1 \to H_1 \) be a demiclosed quasi-nonexpansive mapping;
(ii) \( Q_j : H_2 \to H_2 \) be a demiclosed quasi-nonexpansive mapping;
(iii) \( G_k : H_3 \to H_3 \) be a demiclosed quasi-nonexpansive mapping.

Let

(i) \( M_{i\lambda_i} = (1 - \lambda_i)I_1 + \lambda_i M_i \) for \( \lambda_i \in (0, 1) \);
(ii) \( Q_j\beta_j = (1 - \beta_j)I_2 + \beta_j Q_j \) for \( \beta_j \in (0, 1) \);
(iii) \( G_k\eta_k = (1 - \eta_k)I_3 + \eta_k G_k \) for \( \eta_k \in (0, 1) \).

Let \( (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s \). Suppose that

\[ \Lambda = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^m \text{Fix}(M_i) \}, y \in \bigcap_{j=1}^\ell \text{Fix}(Q_j) \}, z \in \bigcap_{k=1}^s \text{Fix}(G_k) \}, A_1(x) = A_2(y) = A_3(z) \neq \emptyset \).

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3 \), and let the sequences \( \{(x_n, y_n, z_n)\}_{n \in \mathbb{N}} \) be defined by

(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu \alpha_n F_1) \sum_{i=1}^m \zeta_i M_{i\lambda_i}(x_n - \frac{\xi}{3} A_1^*(2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);
(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_2 - \mu \alpha_n F_2) \sum_{j=1}^\ell \theta_j Q_j\beta_j(y_n - \frac{\xi}{3} A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda) \).

Proof. Since a quasi-nonexpansive mapping is a 0-demicontractive mapping, Corollary 3.7 follows from Theorem 3.5.

Remark 3.8.

(i) Corollary 3.7 improves and generalizes [13, Theorem 3.1]. [13, Theorem 3.1] established a strong convergence theorem for \( VI(L - \gamma f, \Lambda) \), where \( L \) is a strongly positive bounded self-adjoint linear operator, \( f \) is a contraction mapping, and \( \Lambda \) is a multiple sets split fixed point of quasi-nonexpansive mappings. Since a strongly positive bounded self-adjoint operator is a Lipschitz continuous and strongly monotone operator.

(ii) Corollary 3.7 generalizes [5, Corollary 5.1] which established a weak convergence of multiple sets split fixed point theorem of quasi-nonexpansive mappings.

(iii) Corollary 3.7 also extends [27, Theorems 3.2 and 3.4]. In [27, Theorems 3.2 and 3.4], \( V = (f_1, f_2) \) is a contraction mapping and \( F = (I_1, I_2) \).

Theorem 3.9. For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\} \), let

(i) \( M_i : H_1 \to H_1 \) be a \( \sigma_i \)-strictly pseudo-nonspradling mapping;

(ii) \( Q_j : H_2 \to H_2 \) be a \( r_j \)-strictly pseudo-nonspradling mapping;

(iii) \( G_k : H_3 \to H_3 \) be a demiglicic \( \delta_k \)-strictly pseudo-nonspradling mapping.

Let

(i) \( M_{i\lambda_i} = (1 - \lambda_i) I_1 + \lambda_i M_i \) for \( \lambda_i \in (\sigma_i, 1) \);

(ii) \( Q_{j\beta_j} = (1 - \beta_j) I_2 + \beta_j Q_j \) for \( \beta_j \in (r_j, 1) \);

(iii) \( G_{k\eta_k} = (1 - \eta_k) I_3 + \eta_k G_k \) for \( \eta_k \in (\delta_k, 1) \).

Let \( (\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s \). Suppose that

\[ \Lambda = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^m \text{Fix}(M_i), y \in \bigcap_{j=1}^\ell \text{Fix}(Q_j), z \in \bigcap_{k=1}^s \text{Fix}(G_k), A_1(x) = A_2(y) = A_3(z) \} \neq \emptyset. \]

Let \( x_n, y_n, z_n \in H_1, y_n \in H_2, z_n \in H_3, \) and let the sequences \( \{ (x_n, y_n, z_n) \}_{n \in \mathbb{N}} \) be defined by

(i) \( x_{n+1} = \alpha_n \gamma V(x_n) + (1 - \lambda_n M_i) \sum_{i=1}^m \theta_i M_{i\lambda_i} x_n - \frac{\lambda_i}{2} A_i^*(2A_i(x_n) - A_2(y_n) - A_3(z_n)) \) for all \( n \in \mathbb{N} \);

(ii) \( y_{n+1} = \alpha_n \gamma V(y_n) + (1 - \mu_n M_2) \sum_{j=1}^\ell \theta_j Q_{j\beta_j} y_n - \frac{\beta_j}{2} A_j^*(2A_j(y_n) - A_1(x_n) - A_3(z_n)) \) for all \( n \in \mathbb{N} \);

(iii) \( z_{n+1} = \alpha_n \gamma V(z_n) + (1 - \mu_n M_3) \sum_{k=1}^s \omega_k G_{k\eta_k} z_n - \frac{\omega_k}{2} A_k^*(2A_k(z_n) - A_1(x_n) - A_2(y_n)) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda) \).

Proof. By assumptions and Lemma 2.4, for each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\} \), and for \( \lambda_i \in (\sigma_i, 1) \), \( \beta_j \in (r_j, 1) \) and \( \eta_k \in (\delta_k, 1) \) we have that

(i) \( \text{Fix}(M_{i\lambda_i}) = \text{Fix}(M_i), \text{Fix}(Q_{j\beta_j}) = \text{Fix}(Q_j), \text{Fix}(G_{k\eta_k}) = \text{Fix}(G_k) \);

(ii) \( M_{i\lambda_i}, Q_{j\beta_j}, \) and \( G_{k\eta_k} \) are demiglicic;

(iii) \( \|M_{i\lambda_i} x - M_{i\lambda_i} y\| \leq \|x - y\| + \frac{\lambda_i}{2} \|x - M_{i\lambda_i} x - (y - M_{i\lambda_i} y)\| \);\( \|Q_{j\beta_j} x - Q_{j\beta_j} y\| \leq \|x - y\| + \frac{\beta_j}{2} \|x - Q_{j\beta_j} x - (y - Q_{j\beta_j} y)\| \);

(iv) \( \|G_{k\eta_k} x - G_{k\eta_k} y\| \leq \|x - y\| + \frac{\eta_k}{2} \|x - G_{k\eta_k} x - (y - G_{k\eta_k} y)\| \);\( \|G_{k\eta_k} x - G_{k\eta_k} y\| \leq \|x - y\| + \frac{\eta_k}{2} \|x - G_{k\eta_k} x - (y - G_{k\eta_k} y)\| \).
It is easy to see that for each \( i \in \{1,2,\ldots,m\}, j \in \{1,2,\ldots,\ell\}, k \in \{1,2,\ldots,s\} \), and for \( \lambda_i \in (\sigma_i,1), \beta_j \in (\tau_j,1) \) and \( \eta_k \in (\delta_k,1) \) that

(i) \( M_{i\lambda_i} \) is a demiclosed \( (\lambda_i - \sigma_i) \)-strongly quasi-nonexpansive mapping;
(ii) \( Q_{j\beta_j} \) is a demiclosed \( (\beta_j - \tau_j) \)-strongly quasi-nonexpansive mapping;
(iii) \( G_{k\eta_k} \) is a demiclosed \( (\eta_k - \delta_k) \)-strongly quasi-nonexpansive mapping.

Let

\[
\Omega = \left\{ (x,y,z) \in \bigotimes_{1 \leq i \leq 3} \bigcap_{i=1}^{m} \text{Fix}(M_{i\lambda_i}), y \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_{j\beta_j}), z \in \bigcap_{k=1}^{s} \text{Fix}(G_{k\eta_k}), A_1(x) = A_2(y) = A_3(z) \right\}.
\]

It is easy to see that \( \Omega = \Lambda \neq \emptyset \). Then Theorem 3.9 follows from Theorem 3.3. \( \square \)

Remark 3.10.

(i) Theorem 3.9 improves and generalizes [10, Theorem 3.1]. [10, Theorem 3.1] established a weak convergence theorem for split equality multiple sets fixed point of strictly pseudo-nonspreading mappings. The simultaneous iteration in Theorem 3.9 is different from the simultaneous iterations in [10, Theorem 3.1].

(ii) In [14, Theorem 3.7], the authors studied variational inequality problem over split equality fixed point for \( m \) strictly pseudo-nonspreading mappings, but Theorem 3.9 studies variational inequality problem over split equality fixed point for three finite families of strictly pseudo-nonspreading mappings.

Theorem 3.11. For each \( i \in \{1,2,\ldots,m\}, j \in \{1,2,\ldots,\ell\}, k \in \{1,2,\ldots,s\} \), let

(i) \( M_i : H_1 \to H_1 \) be a \( \sigma_i \)-strictly pseudo-nonscattering mapping;
(ii) \( Q_j : H_2 \to H_2 \) be a \( \tau_j \)-strictly pseudo-contractive mapping;
(iii) \( G_k : H_3 \to H_3 \) be a demiclosed \( \delta_k \)-demicontractive mapping.

Let

(i) \( M_{i\lambda_i} = (1 - \lambda_i)I_1 + \lambda_i M_i \) for \( \lambda_i \in (\sigma_i,1) \);
(ii) \( Q_{j\beta_j} = (1 - \beta_j)I_2 + \beta_j Q_j \) for \( \beta_j \in (0,1 - \tau_j) \);
(iii) \( G_{k\eta_k} = (1 - \eta_k)I_3 + \eta_k G_k \) for \( \eta_k \in (0,1 - \delta_k) \).

Let \( (\zeta_1,\zeta_2,\ldots,\zeta_m) \in \Delta_m, (\theta_1,\theta_2,\ldots,\theta_\ell) \in \Delta_\ell, (\omega_1,\omega_2,\ldots,\omega_s) \in \Delta_s \). Suppose that

\( \Lambda = \left\{ (x,y,z) \in \bigotimes_{1 \leq i \leq 3} \bigcap_{i=1}^{m} \text{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), z \in \bigcap_{k=1}^{s} \text{Fix}(G_k), A_1(x) = A_2(y) = A_3(z) \right\} \neq \emptyset \).

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3 \), and let the sequences \((x_n,y_n,z_n)\) for \( n \in \mathbb{N} \) be defined by

(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n F_1) \sum_{i=1}^{m} \zeta_i M_{i\lambda_i}(x_n - \frac{\theta_i}{\beta_i} A_i^*(2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);
(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (1 - \mu \alpha_n F_2) \sum_{j=1}^{\ell} \theta_j Q_{j\beta_j}(y_n - \frac{\omega_j}{\beta_j} A_2^*(2A_2(y_n) - A_1(x_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);
(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (1 - \mu \alpha_n F_3) \sum_{k=1}^{s} \omega_k G_{k\eta_k}(z_n - \frac{\omega_k}{\beta_k} A_3^*(2A_3(z_n) - A_1(x_n) - A_2(y_n))) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n,y_n,z_n) \) in \( V(\mu F - \gamma V, \Lambda) \).

Proof. We see in the proof of Theorem 3.9 that \( M_{i\lambda_i} \) is a demiclosed \( (\lambda_i - \sigma_i) \)-strongly quasi-nonexpansive mapping and \( \text{Fix}(M_{i\lambda_i}) = \text{Fix}(M_i) \) for each \( i \in \{1,2,\ldots,m\} \).

Since for each \( j \in \{1,2,\ldots,\ell\}, Q_j : H_2 \to H_2 \) is a \( \tau_j \)-strictly pseudo-contractive mapping. It is easy to see that \( Q_j \) is a \( \tau_j \)-demicontractive mapping for each \( j \in \{1,2,\ldots,\ell\} \). For each \( j \in \{1,2,\ldots,\ell\} \), by Lemma 2.13, \( Q_j \) is demiclosed.

It follows from Lemma 2.3 for each \( j \in \{1,2,\ldots,\ell\}, k \in \{1,2,\ldots,s\}, \beta_j \in (0,1 - \tau_j) \) and \( \eta_k \in (0,1 - \delta_k) \) that
Corollary 3.12. Let $V_i : H_1 \to H_1$ be $L_i$-Lipschitz continuous, $F_i : H_1 \to H_1$ be $k_i$-Lipschitz continuous and $\eta_i$-strongly monotone with $k_i > 0$ and $\eta_i > 0$. For each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}, let$

(i) $M_i : H_1 \to H_1$ be a $\sigma_i$-strictly pseudo-nonspreading mapping;
(ii) $Q_j : H_1 \to H_1$ be a $r_j$-strictly pseudo-contractive mapping;
(iii) $G_k : H_1 \to H_1$ be a demiclosed $\delta_k$-demicontractive mapping.

Let

(i) $M_i \lambda_i = (1 - \lambda_i) I_1 + \lambda_i M_i$ for $\lambda_i \in (\sigma_i, 1)$;
(ii) $Q_j \beta_j = (1 - \beta_j) I_2 + \beta_j Q_j$ for $\beta_j \in (0, 1 - r_j)$;
(iii) $G_k \eta_k = (1 - \eta_k) I_3 + \eta_k G_k$ for $\eta_k \in (0, 1 - \delta_k)$.

Let $(\zeta_i, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x, y, z) : x, y, z \in \bigcap_{i=1}^{m} \text{Fix}(M_i), y, z \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), z \in \bigcap_{k=1}^{s} \text{Fix}(G_k) \} \neq \emptyset.$$ 

Proof. Let $H_1 = H_2 = H_3 = H_4, A_1 = I_1 = A_2 = A_3$ in Theorem 3.11, then Corollary 3.12 follows from Theorem 3.11.

Theorem 3.13. Let $V_i : H_1 \to H_1$ be $L_i$-Lipschitz continuous, $F_i : H_1 \to H_1$ be $k_i$-Lipschitz continuous and $\eta_i$-strongly monotone with $k_i > 0$ and $\eta_i > 0$. For each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}, let$

(i) $M_i : H_1 \to H_1$ be a $\sigma_i$-strictly pseudo-nonspreading mapping;
(ii) $Q_j : H_2 \to H_2$ be a $r_j$-strictly pseudo-contractive mapping;
(iii) $G_k : H_2 \to H_3$ be a demiclosed $\delta_k$-demicontractive mapping.

Let

(i) $M_i \lambda_i = (1 - \lambda_i) I_1 + \lambda_i M_i$ for $\lambda_i \in (\sigma_i, 1)$;
(ii) $Q_j \beta_j = (1 - \beta_j) I_2 + \beta_j Q_j$ for $\beta_j \in (0, 1 - r_j)$;
(iii) $G_k \eta_k = (1 - \eta_k) I_3 + \eta_k G_k$ for $\eta_k \in (0, 1 - \delta_k)$.

Let $(\zeta_i, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x, y, z) : x, y, z \in \bigcap_{i=1}^{m} \text{Fix}(M_i), y, z \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), z \in \bigcap_{k=1}^{s} \text{Fix}(G_k) \} \neq \emptyset.$$ 

Let $x_1 \in H_1, y_1 \in H_1, z_1 \in H_1$, and let the sequences $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be defined by
Proof. Theorem 3.11 shows that for each 

(iii) \( \beta_j \) is a demiclosed \( (1 - \tau_j - \beta_j) \)-strongly quasi-nonexpansive mapping;
(ii) \( G_{k\eta_j} \) is a demiclosed \( (1 - \delta_k - \eta_j) \)-strongly quasi-nonexpansive mapping.

Let

(i) \( V_j \beta_j = (I_1 - \frac{1}{\|\beta_j\|^2} B_1^*(I_2 - Q_j \beta_j) B_1) \);
(ii) \( W_{k\eta_j} = (I_1 - \left[ \frac{1}{\|\beta_j\|^2} B_2^*(I_3 - G_{k\eta_j}) B_2 \right] \).

By Proposition 2.8, for \( \beta_j \in (0, 1 - \tau_j) \), and for \( \eta_j \in (0, 1 - \delta_k) \),
(i) \( \text{Fix}(V_j \beta_j) = B_1^{-1} \text{Fix}(Q_j) \) and \( \text{Fix}(W_{k\eta_j}) = B_2^{-1} \text{Fix}(G_k) \).
(ii) \( V_j \beta_j : H_1 \to H_1 \) is a demiclosed \( (1 - \tau_j - \beta_j) \)-strongly quasi-nonexpansive mapping, and \( W_{k\eta_j} : H_1 \to H_1 \) is a demiclosed \( (1 - \delta_k - \eta_j) \)-strongly quasi-nonexpansive mapping.

Theorem 3.11 shows that \( M_{i\lambda_i} \) is a demiclosed \( \lambda_i - \sigma_i \)-strongly quasi-nonexpansive mapping for \( \lambda_i \in (\sigma_i, 1) \), and \( \text{Fix}(M_{i\lambda_i}) = \text{Fix}(M_i) \).

Let

\[ \Omega = \{ (x, x, x) : x \in H_j, x \in \bigcap_{i=1}^m \text{Fix}(M_{i\lambda_i}), x \in \bigcap_{j=1}^\ell \text{Fix}(V_j \beta_j), x \in \bigcap_{k=1}^s \text{Fix}(W_{k\eta_j}) \} . \]

It is easy to see that \( \Omega = \Lambda \neq \emptyset \). Then Theorem 3.13 follows from Corollary 3.12. \( \square \)

Theorem 3.14. For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\} \), let

(i) \( P_i : H_1 \to H_1 \) be a \( \sigma_i \)-Lipschitz continuous demiclosed quasi-pseudocontractive mapping;
(ii) \( R_j : H_2 \to H_2 \) be a \( \rho_j \)-Lipschitz continuous demiclosed quasi-pseudocontractive mapping;
(iii) \( W_k : H_3 \to H_3 \) be a \( \delta_k \)-Lipschitz continuous demiclosed quasi-pseudocontractive mapping.

For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, \) and \( k \in \{1, 2, \ldots, s\} \), let

(i) \( M_i = (1 - \xi_i) I_1 + \xi_i P_i (1 - \gamma_i) I_1 + \gamma_i P_i I_1 \);
(ii) \( Q_j = (1 - \sigma_j I_2 + \sigma_j R_j (1 - \nu_j) I_2 + \nu_j R_j I_2 \);
(iii) \( G_k = (1 - \rho_k) I_3 + \rho_k W_k (1 - \delta_k) I_3 + \delta_k W_k \),

where \( 0 < \xi_i < \gamma_i < \frac{1}{1 + \sqrt{1 + \sigma_i^2}}, i \in \{1, 2, \ldots, m\}, 0 < \omega_j < \nu_j < \frac{1}{1 + \sqrt{1 + \rho_j^2}}, j \in \{1, 2, \ldots, \ell\}, \) and \( 0 < \rho_k < \xi_k < \frac{1}{1 + \sqrt{1 + \delta_k^2}}, k \in \{1, 2, \ldots, s\} \), and let

(i) \( M_{i\lambda_i} = (1 - \lambda_i) I_1 + \lambda_i M_i \) for \( \lambda_i \in (0, 1) \);
(ii) \( Q_j \beta_j = (1 - \beta_j) I_2 + \beta_j Q_j \) for \( \beta_j \in (0, 1) \);
(iii) \( G_{k\eta_j} = (1 - \eta_k) I_3 + \eta_k G_k \) for \( \eta_k \in (0, 1) \).

Let \( (\xi_1, \xi_2, \ldots, \xi_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s \). Suppose that

\[ \Lambda = \{ (x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_1 : x \in \bigcap_{i=1}^m \text{Fix}(P_i), y \in \bigcap_{j=1}^\ell \text{Fix}(R_j), z \in \bigcap_{k=1}^s \text{Fix}(W_k), A_1(x) = A_2(y) = A_3(z) \} \neq \emptyset . \]

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3 \), and let the sequences \( \{ (x_n, y_n, z_n) \}_{n \in \mathbb{N}} \) be defined by
\( x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n F_1) \sum_{i=1}^{m} \xi_i M_i \lambda_i (x_n - \frac{\rho_i}{3} \Lambda_i (2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);

(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (1 - \mu \alpha_n F_2) \sum_{j=1}^{\ell} \theta_j Q_j \beta_j (y_n - \frac{\rho_j}{3} \Lambda_j (2A_2(y_n) - A_1(x_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);

(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (1 - \mu \alpha_n F_3) \sum_{k=1}^{s} \omega_k G_{knk} (z_n - \frac{\rho_k}{3} \Lambda_k (2A_3(z_n) - A_1(x_n) - A_2(y_n))) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda) \).

**Proof.** By Lemma 2.11, for each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\} \),

(i) \( \text{Fix}(M_i) = \text{Fix}(P_i), \text{Fix}(R_j) = \text{Fix}(Q_j), \text{Fix}(W_k) = \text{Fix}(G_k) \);

(ii) \( M_i, Q_j, \) and \( G_k \) are demiclosed at 0;

(iii) \( M_i, Q_j, \) and \( G_k \) are quasi-nonexpansive mappings.

Let

\[ \Omega = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^{m} \text{Fix}(M_i), y \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), z \in \bigcap_{k=1}^{s} \text{Fix}(G_k), A_1(x) = A_2(y) = A_3(z) \} \neq \emptyset. \]

It is easy to see that \( \Omega = \Lambda \neq \emptyset \) and Theorem 3.14 follows from Corollary 3.7.

**Remark 3.15.** Since firmly quasinonexpansive mapping, pseudo-contractive mappings, k-strict pseudo-contractive mapping, k-strict pseudo-nonsplitting mapping, demi-contractive mappings, and directed operators are special cases of quasi-pseudo-contractive mappings, we see that Theorem 3.14 extends many results on fixed point problems, multiple sets split fixed point problems and split equality fixed point problems existing in the literature.

**Corollary 3.16.** For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\} \), let

(i) \( P_i : H_1 \to H_1 \) be a \( \sigma_i \)-Lipschitz continuous demiclosed quasi-pseudo-contractive mapping;

(ii) \( R_j : H_2 \to H_2 \) be a \( \rho_j \)-Lipschitz continuous demiclosed quasi-pseudo-contractive mapping;

(iii) \( W_k : H_3 \to H_3 \) be a \( \delta_k \)-Lipschitz continuous demiclosed quasi-pseudo-contractive mapping.

For each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, \) and \( k \in \{1, 2, \ldots, s\} \), let

(i) \( M_i = (1 - \xi_i)I_1 + \xi_i P_i (1 - \gamma_i)I_1 + \gamma_i P_i \); 

(ii) \( Q_j = (1 - \sigma_j)I_2 + \sigma_j R_j (1 - \nu_j)I_2 + \nu_j R_j \); 

(iii) \( G_k = (1 - \rho_k)I_3 + \rho_k W_k (1 - \delta_k)I_3 + \delta_k W_k \).

where \( 0 < \xi_i < \gamma_i < \frac{1}{1 + \sqrt{1 + \sigma_i^2}}, i \in \{1, 2, \ldots, m\} \), \( 0 < \omega_j < \nu_j < \frac{1}{1 + \sqrt{1 + \rho_j^2}}, j \in \{1, 2, \ldots, \ell\} \), and \( 0 < \rho_k < \xi_k < \frac{1}{1 + \sqrt{1 + \delta_k^2}}, k \in \{1, 2, \ldots, s\} \), and let

(i) \( M_i \lambda_i = (1 - \lambda_i)I_1 + \lambda_i M_i \) for \( \lambda_i \in (0, 1) \);

(ii) \( Q_j \beta_j = (1 - \beta_j)I_2 + \beta_j Q_j \) for \( \beta_j \in (0, 1) \);

(iii) \( G_{knk} = (1 - \eta_k)I_3 + \eta_k G_k \) for \( \eta_k \in (0, 1) \).

Let \( (\xi_1, \xi_2, \ldots, \xi_m), (\eta_1, \eta_2, \ldots, \eta_\ell), (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta _\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta s \).

Suppose that

\[ \Lambda = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^{m} \text{Fix}(P_i), y \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), z \in \bigcap_{k=1}^{s} \text{Fix}(W_k), A_1(x) = A_2(y) = A_3(z) \} \neq \emptyset. \]

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3, \) and let the sequences \( \{(x_n, y_n, z_n)\}_{n \in \mathbb{N}} \) be defined by

(i) \( x_{n+1} = \alpha_n x_n + (1 - \alpha_n) \sum_{i=1}^{m} \xi_i M_i \lambda_i (x_n - \frac{\rho_i}{3} \Lambda_i (2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N} \);
Theorem 3.18. For each $\eta \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, \ell\}$, $k \in \{1, 2, \ldots, s\}$, let

(i) $M_i : H_1 \to H_1$ be a hemi-continuous, locally bounded pseudocontractive mapping;
(ii) $Q_j : H_2 \to H_2$ be a hemi-continuous, locally bounded monotone mapping;
(iii) $G_k : H_3 \to H_3$ be a demiclosed $\delta_k$-demicontinuous mapping.

Let $\tau > 0$, for $x \in H_1$, and $u \in H_2$, and set

(i) $S_i(x) = \{z \in H_1 : (y - z, M_i z) - \frac{1}{\tau}(y - z, (1 + \tau)z - x) \leq 0, \forall y \in H_1\};$
(ii) $P_j(u) = \{z \in D_j : (y - z, Q_j(z)) + \frac{1}{\tau}(y - z, z - u) \geq 0, \forall y \in D_j\};$
(iii) $G_{k\eta} = (1 - \eta_k)I_3 + \eta_k G_k$ for $\eta_k \in [0, 1 - \delta_k].$

Let $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m$, $(\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell$, $(\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^m \text{Fix}(M_i), y \in \bigcap_{j=1}^\ell \text{VI}(Q_j, D_j), z \in \bigcap_{k=1}^s \text{Fix}(G_k), A_1(x) = A_2(y) = A_3(z) \neq \emptyset.\}$$

Let $x_1 \in H_1$, $y_1 \in H_2$, $z_1 \in H_3$, and let the sequences $(x_n, y_n, z_n)_{n \in \mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n y_1 + (1 - \alpha_n)\sum_{i=1}^m \zeta_i S_i(x_n - \frac{1}{2}A_1(x_n - A_2(y_n) - A_3(z_n)))$ for all $n \in \mathbb{N};$
(ii) $y_{n+1} = \alpha_n y_2 + (1 - \alpha_n)\sum_{i=1}^\ell \theta_i P_j(y_n - \frac{1}{2}A_2(y_n - A_1(x_n) - A_3(z_n)))$ for all $n \in \mathbb{N};$
(iii) $z_{n+1} = \alpha_n y_3 + (1 - \alpha_n)\sum_{k=1}^s \omega_k G_{k\eta_k}(z_n - \frac{1}{2}A_3(z_n - A_1(x_n) - A_2(y_n)))$ for all $n \in \mathbb{N}.$

Then $\lim_{n \to \infty} (x_n, y_n, z_n) \in \text{VI}(\mu F - \gamma V, \Lambda).$
(ii) $P_j$ is a demiclosed $\gamma_j$-strongly quasi-nonexpansive mapping for some $\gamma_j > 0$.

Let

$$\Omega = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^{m} \text{Fix}(S_i), y \in \bigcap_{j=1}^{\ell} \text{Fix}(P_j), z \in \bigcap_{k=1}^{s} \text{Fix}(G_k \eta_k), A_1(x) = A_2(y) = A_3(z)\}.$$ 

It is easy to see that $\Omega = A \neq \emptyset$. Then Theorem 3.18 follows from Theorem 3.3. 

We apply Theorem 3.3 and argue as Theorems 3.5 and 3.13, we can study the variational inequality problem over split fixed point of three finite families of demicontractive mappings.

**Theorem 3.19.** Let $V_1 : H_1 \to H_1$ be $L_1$-Lipschitz continuous, $F_i : H_1 \to H_1$ be $k_i$-Lipschitz continuous and $\eta_1$-strongly monotone with $k_i > 0$, and $\eta_1 > 0$. For each $i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}$, let

(i) $M_i : H_1 \to H_1$ be a demiclosed $\sigma_i$-demicontractive;
(ii) $Q_j : H_2 \to H_2$ be a demiclosed $\tau_j$-demicontractive mapping;
(iii) $G_k : H_3 \to H_3$ be a demiclosed $\delta_k$-demicontractive mapping.

Let

(i) $M_i \lambda_i = (1 - \lambda_i)I_1 + \lambda_i M_i$ for $\lambda_i \in (0, 1 - \sigma_i)$;
(ii) $Q_j \beta_j = (1 - \beta_j)I_2 + \beta_j Q_j$ for $\beta_j \in (0, 1 - \tau_j)$;
(iii) $G_k \eta_k = (1 - \eta_k)I_3 + \eta_k G_k$ for $\eta_k \in (0, 1 - \delta_k)$.

Let $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x, x, x) : x \in H_i, x \in \bigcap_{i=1}^{m} \text{Fix}(M_i), B_1(x) \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), B_2(x) \in \bigcap_{k=1}^{s} \text{Fix}(G_k)\} \neq \emptyset.$$ 

Let $x_1 \in H_1, y_1 \in H_1, z_1 \in H_1$, and let the sequences $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu \alpha_n F_1) \sum_{i=1}^{m} \zeta_i M_i \lambda_i (x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ for all $n \in \mathbb{N}$;
(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_1 - \mu \alpha_n F_2) \sum_{j=1}^{\ell} \theta_j (I_1 - \frac{1}{\|B_1\|^2} B_1 (I_2 - Q_1 \beta_j) B_1) (y_n - \frac{\xi}{3}(2y_n - x_n - z_n))$ for all $n \in \mathbb{N}$;
(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_1 - \mu \alpha_n F_3) \sum_{k=1}^{s} \omega_k (I_1 - \frac{1}{\|B_2\|^2} B_2^* (I_3 - G_k \eta_k) B_2) (z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ for all $n \in \mathbb{N}$.

Then $\lim_{n \to \infty} (x_n, y_n, z_n) \in V (\mu F - \gamma V, \Lambda)$.

**Remark 3.20.** Theorem 3.19 improves and generalizes [24, Theorem 3.2]. In [24, Theorem 3.2], the authors studied multiple sets split fixed point of demicontractive mappings and quasi-nonexpansive mappings.

4. Variational inequality over split equality solutions for finite families of nonlinear mappings

Let $B : H_1 \rightharpoonup H_1$ be a multivalued mapping. The effective domain of $B$ is denoted by $D(B)$, that is, $D(B) = \{x \in H_1 : Bx \neq \emptyset\}$. We say $B : H_1 \rightharpoonup H_1$ is monotone on $H_1$ if $(x - y, u - v) \geq 0$ for all $x, y \in D(B), u \in Bx, v \in By$. $B$ is maximal monotone on $H_1$ if $B$ is a monotone operator on $H_1$ and its graph is not properly contained in the graph of any other monotone operator on $H_1$. For a maximal monotone operator $B : H_1 \rightharpoonup H_1$ and $r > 0$, we may define a single-valued mapping $J_B^r : H_1 \to D(B)$ by $J_B^r = (I + rB)^{-1}$, and it is called the resolvent mapping of $B$ for $r$.

For each $i \in \{1, 2, \ldots, m\}$, let $C_i$ be a closed convex subset of $H_i$, let $M_{i1} : H_1 \rightharpoonup H_1$ be a maximum monotone multivalued mapping such that $D(M_{i1}) \subset C_i, L_{i1} : C_i \to H_1$ be a $\gamma_{i1}$-inverse strongly monotone
operator, \(h_{11} \in \Gamma_0(H_1), g_{11} \in \Gamma_0(H_1)\), and let \(g_{11}\) be Fréchet differentiable with \(\sigma_{11}\)-Lipschitz continuous Fréchet derivative \(\nabla g_{11}\).

For each \(j \in \{1, 2, \ldots, \ell\}\), let \(D_j\) be a closed convex subset of \(H_2\), \(M_{2j} : H_2 \to H_2\) be a maximum monotone operator such that \(D(M_{2j}) \subset D_j\), \(L_j : D_j \to H_2\) be a \(\gamma_{2j}\)-inverse strongly monotone operator, \(h_{2j} \in \Gamma_0(H_2), g_{2j} \in \Gamma_0(H_2)\), and let \(g_{2j}\) be Fréchet differentiable with \(\sigma_{2j}\)-Lipschitz continuous Fréchet derivative \(\nabla g_{2j}\). For each \(k \in \{1, 2, \ldots, s\}\), let \(E_k\) be a closed convex subset of \(H_3\), \(M_{3k} : H_3 \to H_3\) be a maximum monotone operator such that \(D(M_{3k}) \subset E_k\), \(L_{3k} : E_k \to H_3\) be a \(\gamma_{3k}\)-inverse strongly monotone operator, \(h_{3k} \in \Gamma_0(H_3), g_{3k} \in \Gamma_0(H_3)\), and let \(g_{3k}\) be Fréchet differentiable with \(\sigma_{3k}\)-Lipschitz continuous Fréchet derivative \(\nabla g_{3k}\). For each \(k \in \{1, 2, \ldots, s\}\), let \(G_k : H_3 \to H_3\) be a demiclosed \(\delta_k\)-demicontractive mapping. Throughout this section we use these notations and assumptions unless specified otherwise.

**Theorem 4.1.** Let \(\zeta_1, \zeta_2, \ldots, \zeta_m \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s\) and let \(\zeta > 0\). Suppose that

\[
\Lambda = \{(x, y, z) \in \prod_{1 \leq i \leq 3} H_i : x \in \bigcap_{l=1}^m (M_{1l} + L_{1l})^{-1} 0, y \in \bigcap_{k=1}^s \text{Fix}(G_k), z \in \bigcap_{j=1}^\ell (M_{2j} + L_{2j})^{-1} 0, A_1(x) = A_2(y) = A_3(z) = \emptyset \}
\]

Let \(x_1 \in H_1, y_1 \in H_2, z_1 \in H_3\), and let the sequences \(\{x_n, y_n, z_n\}_{n \in \mathbb{N}}\) be defined by

1. \(x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n F_1) \sum_{i=1}^m \zeta_i J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})(x_n - \frac{\zeta}{3} A_1^3(2A_1(x_n) - A_2(y_n) - A_3(z_n)))\) for all \(n \in \mathbb{N}\);
2. \(y_{n+1} = \alpha_n \gamma V_2(y_n) + (1 - \mu \alpha_n F_2) \sum_{j=1}^\ell \theta_j J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})(y_n - \frac{\zeta}{3} A_2^3(2A_2(y_n) - A_1(x_n) - A_3(z_n)))\) for all \(n \in \mathbb{N}\);
3. \(z_{n+1} = \alpha_n \gamma V_3(z_n) + (1 - \mu \alpha_n F_3) \sum_{k=1}^s \omega_k G_k(z_n - \frac{\zeta}{3} A_3^3(2A_3(z_n) - A_1(x_n) - A_2(y_n)))\) for all \(n \in \mathbb{N}\).

Then \(\lim_{n \to \infty} (x_n, y_n, z_n) \in \text{VI}(\mu F - \gamma V, \Lambda)\).

**Proof.** Argue as [25, Theorem 4.1], we see that \(J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})\) and \(J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})\) are averaged for each \(i \in \{1, 2, \ldots, m\}\) and each \(j \in \{1, 2, \ldots, \ell\}\).

It is easy to see that

1. \(\text{Fix}(J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})) = (M_{11} + L_{11})^{-1} 0\);
2. \(\text{Fix}(J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})) = (M_{2j} + L_{2j})^{-1} 0\).

By Proposition 2.2,

1. \(J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})\) is a \(\lambda_i\)-strongly quasi-nonexpansive mapping for some \(\lambda_i > 0\);
2. \(J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})\) is a \(\beta_j\)-strongly quasi-nonexpansive mapping for some \(\beta_j > 0\).

For each \(i \in \{1, 2, \ldots, m\}\), and each \(j \in \{1, 2, \ldots, \ell\}\), since \(J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})\) and \(J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})\) are averaged, \(J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})\) and \(J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})\) are nonexpansive. By Lemma 2.1, \(J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})\) and \(J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})\) are demiclosed.

It follows from Lemma 2.3 for each \(k \in \{1, 2, \ldots, s\}\), \(\eta_k \in (0, 1 - \delta_k)\) that \(G_k\) is a demiclosed \((1 - \delta_k - \eta_k)\)-strongly quasi-nonexpansive mapping. Let

\[
\Omega = \{x \in \bigcap_{l=1}^m \text{Fix}(J_{\zeta_i}^{M_{1i}}(I_1 - \zeta_1 L_{11})), z \in \bigcap_{k=1}^s \text{Fix}(G_k), y \in \bigcap_{j=1}^\ell \text{Fix}(J_{\zeta_j}^{M_{2j}}(I_2 - \zeta_2 L_{2j})), A_1(x) = A_2(y) = A_3(z)\}.
\]

It is easy to see that \(\Lambda = \Omega \neq \emptyset\) and Theorem 4.1 follows from Theorem 3.3.
Theorem 4.2. Let \((\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s\), let \(V_1 : H_1 \to H_1\) be \(L_1\)-Lipschitz continuous, \(F_1 : H_1 \to H_1\) be \(\kappa_1\)-Lipschitz continuous and \(\eta_1\)-strongly monotone with \(\kappa_1 > 0\), and \(\eta_1 > 0\), and let \(\kappa > 0\). Suppose that

\[
\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^m (M_{1i} + L_{1i})^{-1}0, B_1(x) \in \bigcap_{j=1}^\ell (M_{2i} + L_{2j})^{-1}0, B_2(x) \in \bigcap_{k=1}^s \text{Fix}(G_k) \neq \emptyset\}.
\]

Let \(x_1 \in H_1, y_1 \in H_1, z_1 \in H_1,\) and let the sequences \(\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}\) be defined by

(i) \(x_{n+1} = \alpha_n x + (1 - \alpha_n) y_1 + \sum_{i=1}^m \zeta_i \xi_{i1}(I_1 - \kappa L_{11}) (x_n - \frac{\xi}{\xi} (2x_n - y_n - z_n))\) for all \(n \in \mathbb{N}\);

(ii) \(y_{n+1} = \alpha_n y + (1 - \alpha_n) F_1(x_1) + \sum_{i=1}^\ell \theta_i (I_1 - \frac{1}{\|B_1\|^2} B_1^\dagger (I_2 - \frac{1}{\|I_2\|^2} B_2^\dagger (I_3 - G_k^{(n)}) B_2)) (y_n - \frac{\xi}{\xi} (2x_n - y_n - z_n))\) for all \(n \in \mathbb{N}\);

(iii) \(z_{n+1} = \alpha_n z + (1 - \alpha_n) F_2(x_1) + \sum_{i=1}^s \omega_i (I_1 - \frac{1}{\|B_1\|^2} B_1^\dagger (I_3 - G_k^{(n)}) B_2) (z_n - \frac{\xi}{\xi} (2x_n - y_n - z_n))\) for all \(n \in \mathbb{N}\).

Then \(\lim_{n \to \infty} (x_n, y_n, z_n) \in \mathcal{V}(\mu F - \gamma V, \Lambda)\).

Proof. Theorem 3.5 shows that for \(\eta_k \in (0, 1 - \kappa_k)\), \(G_k^{(n)}\) is a demiclosed \((1 - \delta_k - \eta_k)\)-strongly quasinonexpansive mapping. In Theorem 4.1, we show that

(i) \(I_1^{\kappa_{i1}} (I_1 - \kappa L_{11})\) is a \(\lambda_1\)-strongly quasinonexpansive mapping for some \(\lambda_1 > 0\);

(ii) \(I_2^{\kappa_{i2}} (I_2 - \kappa L_{2j})\) is a \(\beta_j\)-strongly quasinonexpansive mapping for some \(\beta_j > 0\).

Let

(i) \(U_1 = (I_1 - \frac{1}{\|B_1\|^2} B_1^\dagger (I_2 - \frac{1}{\|I_2\|^2} B_2^\dagger (I_3 - G_k^{(n)}) B_2)) B_1\);

(ii) \(W_{k^{(n)}} = (I_1 - \frac{1}{\|B_1\|^2} B_1^\dagger (I_3 - G_k^{(n)}) B_2)\).

By Proposition 2.8, for \(\eta_k \in (0, 1 - \delta_k)\),

(i) \(\text{Fix}(U_1) = B_1^{-1} \text{Fix}(I_2^{\kappa_{i2}} (I_2 - \kappa L_{2j}))\) and \(\text{Fix}(W_{k^{(n)})} = B_2^{-1} \text{Fix}(G_k)\);

(ii) \(U_1 : H_1 \to H_1\) is a demiclosed \(\beta_j\)-strongly quasinonexpansive mapping, and \(W_{k^{(n)}} : H_1 \to H_1\) is a demiclosed \((1 - \delta_k - \eta_k)\)-strongly quasinonexpansive mapping.

Let

\[
\Omega = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^m \text{Fix}(I_1^{\kappa_{i1}} (I_1 - \kappa L_{11})), x \in \bigcap_{j=1}^\ell \text{Fix}(U_j), x \in \bigcap_{k=1}^s \text{Fix}(W_{k^{(n)})}\}.
\]

It is easy to see that \(\Omega = \Lambda \neq \emptyset\). Then Theorem 4.2 follows from Theorem 3.3. \(\square\)

Remark 4.3. In [21] the authors introduced an iteration to study the following problem:

\[
\text{Find } x \in M^{-1}0 \text{ such that } Lx \in \text{Fix}(T),
\]

where \(M : H_1 \to H_1\) is a maximum monotone operator, and \(T : H_2 \to H_2\) is a nonexpansive operator. Theorems 4.1 and 4.2 improve and generalize [21, Theorems 4.2 and 4.3]. In [21, Theorems 4.2 and 4.3], the authors established weak convergence theorems of this problem.

Theorem 4.4. Let \((\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s\), let \(V_1 : H_1 \to H_1\) be \(L_1\)-Lipschitz continuous, \(F_1 : H_1 \to H_1\) be \(\kappa_1\)-Lipschitz continuous and \(\eta_1\)-strongly monotone with \(\kappa_1 > 0\), and \(\eta_1 > 0\), and let \(\kappa > 0\). Suppose that

\[
\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^m (M_{1i} + L_{1i})^{-1}0, B_1(x) \in \bigcap_{j=1}^\ell (M_{2i} + L_{2j})^{-1}0, B_2(x) \in \bigcap_{k=1}^s (M_{3k} + L_{3k})^{-1}) \neq \emptyset\}.
\]

Let \(x_1 \in H_1, y_1 \in H_1, z_1 \in H_1,\) and let the sequences \(\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}\) be defined by
Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda) \).

**Proof.** Theorem 4.1 shows that

(i) \( J_{\kappa}^{M_{11}}(I_1 - \kappa L_{11}) \) is a \( \lambda_1 \)-strongly quasi-nonexpansive mapping for some \( \lambda_1 > 0 \);
(ii) \( J_{\kappa}^{M_{2j}}(I_2 - \kappa L_{2j}) \) is a \( \beta_j \)-strongly quasi-nonexpansive mapping for some \( \beta_j > 0 \);
(iii) \( J_{\kappa}^{M_{3k}}(I_3 - \kappa L_{3k}) \) is a \( \delta_k \)-strongly quasi-nonexpansive mapping for some \( \delta_k > 0 \).

Let

(i) \( M_i = J_{\kappa}^{M_{11}}(I_1 - \kappa L_{11}); \)
(ii) \( Q_j = (I_1 - \frac{1}{\|b_{1j}\|^2} B_{1j}^* (I_2 - J_{\kappa}^{M_{2j}}(I_2 - \kappa L_{2j})) B_{1j}); \)
(iii) \( W_k = (I_1 - \frac{1}{\|b_{2j}\|^2} B_{2j}^* (I_3 - J_{\kappa}^{M_{3k}}(I_3 - \kappa L_{3k})) B_{2j}); \)

By Proposition 2.8,

(i) \( \text{Fix}(M_i) = H_1 \to H_1 \) is a demiclosed \( \lambda_i \)-strongly quasi-nonexpansive mapping, and \( \text{Fix}(M_i) = (\|M_{11} + L_{11}\|)^{-1} \); 
(ii) \( Q_j : H_1 \to H_1 \) is a demiclosed \( \beta_j \)-strongly quasi-nonexpansive mapping, and \( \text{Fix}(Q_j) = (B_{1j}^{-1}(M_{2j} + L_{2j})^{-1}) \); 
(iii) and \( W_k : H_1 \to H_1 \) is a demiclosed \( \delta_k \)-strongly quasi-nonexpansive mapping, and \( \text{Fix}(W_k) = (B_{2j}^{-1}(M_{3k} + L_{3k})^{-1}) \).

Let

\[ \Omega = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} \text{Fix}(M_i), x \in \bigcap_{j=1}^{\ell} \text{Fix}(Q_j), x \in \bigcap_{k=1}^{s} \text{Fix}(W_k)\}. \]

It is easy to see that \( \Omega = \Lambda \neq \emptyset \). Then Theorem 4.4 follows from Corollary 3.4.

\[ \square \]

**Theorem 4.5.** Let \((\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s, \) and let \( \kappa > 0 \). Suppose that

\[ \Lambda = \{(x, y, z) \in \bigotimes_{i=1}^{l} H_i : x \in H_1, x \in \bigcap_{i=1}^{m} (M_{1i} + L_{1i})^{-1} 0, z \in \bigcap_{k=1}^{s} (M_{3k} + L_{3k})^{-1} 0, y \in \bigcap_{j=1}^{\ell} (M_{2j} + L_{2j})^{-1} 0, \]

\[ A_1(x) = A_2(y) = A_3(z) \neq \emptyset. \]

Let \( x_1 \in H_1, y_1 \in H_2, z_1 \in H_3, \) and let the sequences \{\((x_n, y_n, z_n)\)\}_{n \in N} be defined by

(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu \alpha_n F_1) \sum_{i=1}^{m} \zeta_i J_{\kappa}^{M_{11}}(I_1 - \kappa L_{11})(x_n - \frac{\kappa}{3} A_1^* (2A_1(x_n) - A_2(y_n) - A_3(z_n))) \) for all \( n \in \mathbb{N}; \)
(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_2 - \mu \alpha_n F_2) \sum_{j=1}^{\ell} \theta_j J_{\kappa}^{M_{2j}}(I_2 - \kappa L_{2j})(y_n - \frac{\kappa}{3} A_2^* (2A_2(y_n) - A_1(x_n) - A_3(z_n))) \) for all \( n \in \mathbb{N}; \)
(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_3 - \mu \alpha_n F_3) \sum_{k=1}^{s} \omega_k J_{\kappa}^{M_{3k}}(I_3 - \kappa L_{3k})(z_n - \frac{\kappa}{3} A_3^* (2A_3(z_n) - A_1(x_n) - A_2(y_n))) \) for all \( n \in \mathbb{N}; \)

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda) \).
Proof. We apply Theorem 4.1 and argue as in Theorem 4.4, we can prove Theorem 4.5. □

Remark 4.6. Theorems 4.1 and 4.5 improve and generalize [23, Theorem 4.2].

Corollary 4.7. In Theorem 4.4, let $H_1 = H_2 = H_3 = H_4$, $I_1 = I_2 = I_3$, $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m$, $(\theta_1, \theta_2, \ldots, \theta_s) \in \Delta_s$, and $(\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$. Suppose that

$$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} (M_{i1} + L_{11})^{-1}0 \bigcap_{j=1}^{s} (M_{2j} + L_{2j})^{-1}0 \bigcap_{k=1}^{s} (M_{3k} + L_{3k})^{-1}0 \} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, and let the sequences $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n) \sum_{i=1}^{m} \zeta_i P_{C_i}(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ for all $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (1 - \mu \alpha_n) \sum_{j=1}^{s} \theta_j (I_1 - \frac{1}{\|B_1\|} B_1^* (I_1 - B_1) ) (y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ for all $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (1 - \mu \alpha_n) \sum_{k=1}^{s} \omega_k (I_1 - \frac{1}{\|B_2\|} B_2^* (I_3 - G_k) B_2) (z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ for all $n \in \mathbb{N}$.

Then $\lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Let $B_1 = I_1 = B_2 = I_2 = I_3$ in Theorem 4.4, then $\|B_1\| = \|B_2\| = 1$ and Corollary 4.7 follows from Theorem 4.4. □

Theorem 4.8. Let $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m$, $(\theta_1, \theta_2, \ldots, \theta_s) \in \Delta_s$, and $(\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$, let $V_i : H_1 \to H_2$ be $L_i$-Lipschitz continuous, $F_i : H_1 \to H_1$ be $k_i$-Lipschitz continuous and $\eta_i$-strongly monotone with $k_i > 0$, and $\eta_i > 0$. Suppose that

$$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} C_i, B_1(x) \in \bigcap_{j=1}^{s} D_j, B_2(x) \in \bigcap_{k=1}^{s} \text{Fix}(G_k) \} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, and let the sequences $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n) \sum_{i=1}^{m} \zeta_i P_{C_i}(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ for all $n \in \mathbb{N}$;

(ii) $y_{n+1} = \alpha_n \gamma V_2(y_n) + (1 - \mu \alpha_n) \sum_{j=1}^{s} \theta_j (I_1 - \frac{1}{\|B_1\|} B_1^* (I_1 - B_1) ) (y_n - \frac{\xi}{3}(y_n - x_n - z_n))$ for all $n \in \mathbb{N}$;

(iii) $z_{n+1} = \alpha_n \gamma V_3(z_n) + (1 - \mu \alpha_n) \sum_{k=1}^{s} \omega_k (I_1 - \frac{1}{\|B_2\|} B_2^* (I_3 - G_k) B_2) (z_n - \frac{\xi}{3}(2z_n - x_n - y_n))$ for all $n \in \mathbb{N}$.

Then $\lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)$.

Proof. Let $L_1 = 0, L_2 = 0, M_1 = \partial C_i, M_2 = \partial D_j$ in Theorem 4.2, then $J_{k_i}^{1} = P_{C_i}, J_{k_j}^{1} = P_{D_j}$, and theorem 4.8 follows from Theorem 4.2. □

Remark 4.9. Theorem 4.8 improves and generalizes [4, Theorem 3.1]. In [4, Theorem 3.1], the authors established a strongly convergence theorem for split feasibility problem and fixed point problem of k-strictly pseudo-contractive mapping.

Corollary 4.10. Let $(\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m$, $(\theta_1, \theta_2, \ldots, \theta_s) \in \Delta_s$, and $(\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s$, let $V_i : H_1 \to H_1$ be $L_i$-Lipschitz continuous, $F_i : H_1 \to H_1$ be $k_i$-Lipschitz continuous and $\eta_i$-strongly monotone with $k_i > 0$, and $\eta_i > 0$. Suppose that

$$\Lambda = \{(x, x, x) : x \in H_1, x \in \bigcap_{i=1}^{m} C_i, B_1(x) \in \bigcap_{j=1}^{s} D_j, B_2(x) \in \bigcap_{k=1}^{s} E_k \} \neq \emptyset.$$

Let $x_1 \in H_1$, $y_1 \in H_1$, $z_1 \in H_1$, and let the sequences $\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}$ be defined by

(i) $x_{n+1} = \alpha_n \gamma V_1(x_n) + (1 - \mu \alpha_n) \sum_{i=1}^{m} \zeta_i P_{C_i}(x_n - \frac{\xi}{3}(2x_n - y_n - z_n))$ for all $n \in \mathbb{N}$;
(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_1 - \mu \alpha_n F_2) \sum_{i=1}^\ell \theta_i (I_1 - \frac{1}{\|B_1\|} B_1^* (I_1 - P_{D_i}) B_1) (y_n - \frac{\xi}{\delta} (y_n - x_n - z_n)) \) for all \( n \in \mathbb{N} \);

(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_1 - \mu \alpha_n F_3) \sum_{k=1}^s \omega_k (I_1 - \frac{1}{\|B_2\|} B_2^* (I_1 - P_{E_k}) B_2) (z_n - \frac{\xi}{\delta} (2z_n - x_n - y_n)) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in V(\mu F - \gamma V, \Lambda) \).

**Proof.** For each \( k \in \{1, 2, \ldots, s\} \), let \( G_{3k} = P_{E_k} \). Since \( P_{E_k} \) is a firmly nonexpansive mapping, \( P_{E_k} \) is averaged and \( P_{E_k} \) is demiclosed. By Proposition 2.2, \( G_k \) is a \( \delta_k \)-strongly quasi-nonexpansive mapping for some \( \delta_k > 0 \). Hence \( G_k \) is a \( \delta_k \)-demicontractive mapping for some \( \delta_k > 0 \). Then Corollary 4.10 follows from Theorem 4.8.

**Theorem 4.11.** Let \((\xi_1, \xi_2, \ldots, \xi_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s, \) let \( V_1 : H_1 \to H_1 \) be \( L_1 \)-Lipschitz continuous, \( F_i : H_1 \to H_1 \) be \( \kappa_i \)-Lipschitz continuous and \( \eta_i \)-strongly monotone with \( \kappa_i > 0 \), and \( \eta_i > 0 \) and \( \lambda > 0 \). Suppose that

\[
\Lambda = \{ (x, x, x) : x \in H_1, x \in \bigcap_{i=1}^m \arg \min_{u \in H_1} (h_{1i} + g_{1i}(u)), B_1(x) \in \bigcap_{j=1}^\ell \arg \min_{u \in H_2} (h_{2j} + g_{2j})(u), B_2(x) \in \bigcap_{k=1}^s \text{Fix}(G_k) \} \neq \emptyset.
\]

Let \( x_1, y_1 \in H_1, z_1 \in H_1, \) and let the sequences \( \{ (x_n, y_n, z_n) \} \) be defined by

(i) \( x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu \alpha_n F_1) \sum_{i=1}^m \xi_i \text{prox}_{\eta_i \delta_1/\eta_1} (I_1 - \kappa \nabla g_{1i})(x_n - \xi_i (2x_n - y_n - z_n)) \) for all \( n \in \mathbb{N} \);

(ii) \( y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_1 - \mu \alpha_n F_2) \sum_{i=1}^\ell \theta_i (I_1 - \frac{1}{\|B_1\|} B_1^* (I_1 - P_{D_i}) B_1) (y_n - \frac{\xi}{\delta} (y_n - x_n - z_n)) \) for all \( n \in \mathbb{N} \);

(iii) \( z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_1 - \mu \alpha_n F_3) \sum_{k=1}^s \omega_k (I_1 - \frac{1}{\|B_2\|} B_2^* (I_1 - P_{E_k}) B_2) (z_n - \frac{\xi}{\delta} (2z_n - x_n - y_n)) \) for all \( n \in \mathbb{N} \).

Then \( \lim_{n \to \infty} (x_n, y_n, z_n) \in V(\mu F - \gamma V, \Lambda) \).

**Proof.** Apply Lemma 2.3 and argue as the proof II of [25, Theorem 4.2], we see that for each \( i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, \)

(i) \( \delta h_{1i}, \delta h_{ij} \) are maximum monotone operator;

(ii) \( \nabla g_{1i} \) is a \( \frac{1}{\eta_i} \)-inverse strongly monotone operator, \( \nabla g_{2j} \) is a \( \frac{1}{\delta_j} \)-inverse strongly monotone operator;

(iii) \( \text{prox}_{\kappa \eta_i \delta_1/\eta_1} (I_1 - \kappa \nabla g_{1i}) = \frac{1}{\kappa \delta_1/\eta_1} \text{prox}_{\kappa \delta_1/\eta_1} (I_1 - \kappa \nabla g_{1i}), \text{prox}_{\kappa \eta_j \delta_2/\eta_2} (I_1 - \kappa \nabla g_{2j}) = \frac{1}{\kappa \delta_2/\eta_2} \text{prox}_{\kappa \delta_2/\eta_2} (I_1 - \kappa \nabla g_{2j}) \);

(iv) \( \arg \min_{x \in H_1} (h_{1i} + g_{1i})(x) = (\delta h_{1i} + \nabla g_{1i})^{-1} 0, \arg \min_{x \in H_2} (h_{2j} + g_{2j})(x) = (\delta h_{2j} + \nabla g_{2j})^{-1} 0 \).

Then Theorem 4.11 follows from Theorem 4.2.

**Remark 4.12.** Since a strictly pseudo-contractive mapping is a demicontractive mapping. It is easy to see that Theorem 4.11 extends [4, Theorem 2.9].

**Theorem 4.13.** Let \((\xi_1, \xi_2, \ldots, \xi_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s, \) let \( V_1 : H_1 \to H_1 \) be \( L_1 \)-Lipschitz continuous, \( F_i : H_1 \to H_1 \) be \( \kappa_i \)-Lipschitz continuous and \( \eta_i \)-strongly monotone with \( \kappa_i > 0 \), and \( \eta_i > 0 \) and \( \lambda > 0 \). Suppose that

\[
\Lambda = \{ (x, x, x) : x \in H_1, x \in \bigcap_{i=1}^m \arg \min_{u \in H_1} (h_{1i} + g_{1i}(u)), B_1(x) \in \bigcap_{j=1}^\ell \arg \min_{y \in H_2} (h_{2j} + g_{2j})(y), B_2(x) \in \bigcap_{k=1}^s \arg \min_{z \in H_3} (h_{3k} + g_{3k})(z) \} \neq \emptyset.
\]

Let \( x_1, y_1 \in H_1, z_1 \in H_1, \) and let the sequences \( \{ (x_n, y_n, z_n) \} \) be defined by
Then \(\lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda)\).

Proof. For each \(k \in \{1, 2, \ldots, s\}\), let \(G_k = \text{prox}_{\kappa h_{3k}}(I_2 - \kappa \nabla g_{3k})\). We show in Theorems 4.1 and 4.11 that

(i) \(\text{prox}_{\kappa h_{3k}}(I_1 - \kappa \nabla g_{3k}) = \frac{1}{\kappa} h_{3k}(I_1 - \kappa \nabla g_{3k})\); 
(ii) \(\arg\min_{x \in H_2} (h_{3k} + g_{3k})(x) = (h_{3k} + \nabla g_{3k})^{-1}(0)\); 
(iii) \(\frac{1}{\kappa} h_{3k}(I_1 - \kappa \nabla g_{3k})\) is a \(\delta_k\)-strongly quasi-nonexpansive mapping for some \(\delta_k > 0\).

Then Theorem 4.13 follows from Theorem 4.4.

Corollary 4.14. In Theorem 4.13, let \(H_1 = H_2 = H_3 = H_4, I_1 = I_2 = I_3, \) and \((\zeta_1, \zeta_2, \ldots, \zeta_m) \in \mathcal{A}_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \mathcal{A}_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \mathcal{A}_s, \kappa > 0.\) Suppose that

\[\Lambda = \{ (x, y, z) : x \in H_1, y \in H_2, z \in H_3, \text{ and } \{ (x_n, y_n, z_n) \} \in \mathbb{N} \text{ be defined by} \]

(i) \(x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu \alpha_n f_1) \sum_{i=1}^{m} \xi_i (\text{prox}_{\kappa h_{3i}}(I_1 - \kappa \nabla g_{1i})(x_n - \frac{1}{\kappa}(2x_n - y_n - z_n))) \text{ for all } n \in \mathbb{N};\)
(ii) \(y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_1 - \mu \alpha_n f_2) \sum_{j=1}^{\ell} \theta_j (I_2 - \text{prox}_{\kappa h_{3j}}(I_2 - \kappa \nabla g_{2j})(y_n - \frac{1}{\kappa}(2y_n - x_n - z_n))) \text{ for all } n \in \mathbb{N};\)
(iii) \(z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_1 - \mu \alpha_n f_3) \sum_{k=1}^{s} \omega_k (I_3 - \text{prox}_{\kappa h_{3k}}(I_3 - \kappa \nabla g_{3k})(z_n - \frac{1}{\kappa}(2z_n - x_n - y_n))) \text{ for all } n \in \mathbb{N}.\)

Then \(\lim_{n \to \infty} (x_n, y_n, z_n) \in VI(\mu F - \gamma V, \Lambda).\)

Proof. Let \(B_1 = I_1 = B_2\) in Theorem 4.13, then Corollary 4.14 follows from Theorem 4.13.

Remark 4.15. Corollaries 3.4, 3.12, 4.7, 4.10, and 4.14 have real applications in the large scale of nonlinear problems and optimization problems. Indeed if the scale of nonlinear problems is large, we can group these problems into finite families of nonlinear problems, then we use simultaneous iteration to find the solutions of these problems.

Theorem 4.16. For each \(i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, \ell\}, k \in \{1, 2, \ldots, s\}, \) let \(\kappa > 0,\) let

(i) \(f_i : C_i \times C_i \to \mathbb{R}\) be a bifunction which satisfies conditions (A1)-(A4); 
(ii) \(Q_j : H_2 \to H_2\) be a hemicontinuous, locally bounded monotone mapping; 
(iii) \(H_{3k} \in \mathcal{I}(H_3), g_{3k} \in \mathcal{I}(H_3), g_{3k} \text{ Fréchet differentiable with } \sigma_{3k}\text{-Lipschitz continuous Fréchet derivative } \nabla g_{3k}.\)

For \(r > 0, x \in H_1,\) and \(u \in H_2,\) let

(i) \(M_i : H_1 \to C_i\) be defined by \(M_i(x) = \{ z \in C_i : f_i(z, u) + \frac{1}{r}(u - z, z - x) \geq 0, \, \forall \, u \in C_i\};\)
(ii) \(P_j : H_2 \to D_j\) be defined by \(P_j(u) = \{ z \in D_j : \langle y - z, Q_j(z) \rangle + \frac{1}{r}(y - z, z - u) \geq 0, \, \forall \, y \in D_j\}.\)
Let \((\zeta_1, \zeta_2, \ldots, \zeta_m) \in \Delta_m, (\theta_1, \theta_2, \ldots, \theta_\ell) \in \Delta_\ell, (\omega_1, \omega_2, \ldots, \omega_s) \in \Delta_s, \eta > 0\). Suppose that

\[
\Lambda = \{(x, y, z) \in \bigotimes_{1 \leq i \leq 3} H_i : x \in \bigcap_{i=1}^m \text{EP}(f_i), y \in \bigcap_{j=1}^\ell \text{VI}(Q_j, D_j), z \in \bigcap_{k=1}^s \arg \min_{w \in H_3} (h_{3k} + g_{3k})(w),
\]

\[A_1(x) = A_2(y) = A_3(z) \neq \emptyset.\]

Let \(x_1 \in H_1, y_1 \in H_2, z_1 \in H_3\), and let the sequences \(\{(x_n, y_n, z_n)\}_{n \in \mathbb{N}}\) be defined by

(i) \(x_{n+1} = \alpha_n \gamma V_1(x_n) + (I_1 - \mu \alpha_n f_1) \sum_{i=1}^m \zeta_i M_i(x_n - \frac{x}{\zeta} A_1^*(\lambda A_1(x_n) - A_2(y_n) - A_3(z_n)))\) for all \(n \in \mathbb{N}\);

(ii) \(y_{n+1} = \alpha_n \gamma V_2(y_n) + (I_2 - \mu \alpha_n f_2) \sum_{j=1}^\ell \theta_j P_j(y_n - \frac{x}{\zeta} A_2^*(\lambda A_2(y_n) - A_1(x_n) - A_3(z_n)))\) for all \(n \in \mathbb{N}\);

(iii) \(z_{n+1} = \alpha_n \gamma V_3(z_n) + (I_3 - \mu \alpha_n f_3) \sum_{k=1}^s \omega_k (I_3 - \text{prox}_{\kappa h_{3k}})(I_3 - \kappa \nabla g_{3k})(z_n - \frac{x}{\zeta} A_3^*(\lambda A_3(z_n) - A_1(x_n) - A_2(y_n)))\) for all \(n \in \mathbb{N}\).

Then \(\lim_{n \to \infty} (x_n, y_n, z_n) \in \text{VI}(\mu F - \gamma V, \Lambda)\).

**Proof.** It follows from Theorem 2.5 that for each \(i \in \{1, 2, \ldots, m\},\)

(i) \(M_i\) is single-valued;

(ii) \(M_i\) is firmly nonexpansive;

(iii) \(\{x \in H_1 : M_i x = x\} = \{x \in C_i : f_i(x, u) \geq 0, \forall u \in C_i\}\).

By Theorem 2.6,

(i) \(P_j\) is single-valued;

(ii) \(P_j\) is a firmly nonexpansive mapping;

(iii) \(\{x \in H : P_j x = x\} = \text{VI}(Q_j, D_j)\).

As in the proof of Theorem 3.18, we see that

(i) \(M_i\) is a \(\lambda_i\)-strongly quasi-nonexpansive mapping for some \(\lambda_i > 0\);

(ii) \(P_j\) is a demiclosed \(\beta_j\)-strongly quasi-nonexpansive mapping for some \(\beta_j > 0\).

We show in Theorems 4.13 that

(i) \(\text{prox}_{\kappa h_{3k}}(I_1 - \kappa \nabla g_{3k}) = J_k^3 h_{3k}(I_1 - \kappa \nabla g_{3k});\)

(ii) \(\arg \min_{x \in H_3} (h_{3k} + g_{3k})(x) = (\partial h_{3k} + \nabla g_{3k})^{-1} 0;\)

(iii) \(J_k^3 h_{3k}(I_1 - \kappa \nabla g_{3k})\) is a \(\delta_k\)-strongly quasi-nonexpansive mapping for some \(\delta_k > 0\).

Then Theorem 4.16 follows from Theorem 3.3. \(\square\)

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**References**


