Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 375–382
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

# Bifurcation and periodically semicycles for fractional difference equation of fifth order

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Communicated by R. Saadati


## Abstract

Our paper takes into account a new bifurcation case of the cycle length and a fifth-order difference equation dynamics of

$$y_{m+1} = \frac{y_m y_{m-2}^{\alpha} y_{m-4}^{\beta} + y_m + y_{m-2}^{\alpha} + y_{m-4}^{\beta} + \gamma}{y_m y_{m-2}^{\alpha} + y_{m-2}^{\alpha} y_{m-4}^{\beta} + y_m y_{m-4}^{\beta} + \gamma + 1}, \quad m = 0, 1, 2, 3, \ldots,$$

where $\gamma \in [0, \infty)$ , $\alpha, \beta \in \mathbb{Z}^{+}$, and $y_{-4}, y_{-3}, y_{-1}, y_{-2}, y_0 \in (0, \ \infty)$ is took into consideration. The disturbance of initials lead to a distinction of cycle length principle of the non-trivial solutions of the equation. The principle of the track solutions structure for this equation is given. The consecutive periods of negative and positive semicycles of non-trivial solutions of this equation take place periodically with only prime period fifteen and in a period with the principles represented by either $\{3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-\}$ or $\{3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+\}$. From this rubric we will establish that the positive fixed point has global asymptotic stability.


**Keywords:** Semicycles, solutions, difference equations, oscillatory solution, global stability.

**2010 MSC:** 39A10, 39A28.



## 1. Introduction

Recently, the difference equations qualitative properties have been the matter studied. One of the purposes of this technique is the need for a number of techniques that can be used in the study of equations that arise in probability theory, psychology, biology, population, genetics, economics, etc..

The studies of fractional difference equations of order higher than one is perfectly challenging and remunerative because some patterns for developing the main theory of the behavior of non-linear difference equations come from results for fractional difference equations.

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doi: 10.22436/jnsa.011.03.06

*Received: 2017-10-15 Revised: 2017-11-07 Accepted: 2018-01-10*

Ladas [11] submitted the studying of the fractional difference equation

$$y_{m+1} = \frac{y_m + y_{m-1}y_{m-2} + \gamma}{y_m y_{m-1} + y_{m-2} + \gamma}.$$

From that time, rational difference equations whose have a unique positive fixed point $\varpi = 1$ have received considerable solicitude. For more results, we refer to [1–9, 12–16].

From the previous known works, we can grasp that it is troublesome to perceive perfectly the track framework of solutions of fractional difference equations in spite of they have simple semblances. If the change of a initial value or a parameter around a value leads to the essential change of the track structure principle of its solution, then it is called that a bifurcation of this equation takes place.

Furthermore, the critical value is said to be a bifurcation value. This concept looks like the definition of the bifurcation for any differential equation.

It is important to indicated that the fundamental change of the track structure principle of a difference equation contains many cases, such as, a solution from the boundedness to the unboundedness, and from the stability to the un-stability, from one period to other period, or the cycle length from one period to another period, etc.. Then, it is significance to examine the bifurcation theory of difference equation with reference to its own right.

In our current paper we take into account the fifth-order difference equation

$$y_{m+1} = \frac{y_m y_{m-2}^{\alpha} y_{m-4}^{\beta} + y_m + y_{m-2}^{\alpha} + y_{m-4}^{\beta} + \gamma}{y_m y_{m-2}^{\alpha} + y_{m-2}^{\alpha} y_{m-4}^{\beta} + y_m y_{m-4}^{\beta} + \gamma + 1}, \quad m = 0, 1, 2, \ldots, \tag{1.1}$$

where $\gamma \in [0, \infty)$, $\alpha, \beta \in \mathbb{Z}^+$and the initials $y_0, y_{-3}, y_{-2}, y_{-1}, y_{-4} \in (0, \infty)$.

By analyzing the principle of the length of semicycle to take place respectively, we characterize the principle of the track framework of its solutions and then we conclude the global asymptotic stabilization of fixed point of (1.1). It is considerably hard to employ methods in the previous known good manners, such as [10] to give the principle of track framework of solutions of (1.1).

It is achieved without great effort to find the fixed point $\varpi$ of (1.1) from

$$\varpi = \frac{\varpi^{\alpha+\beta+1} + \varpi + \varpi^{\alpha} + \varpi^{\beta} + \gamma}{\varpi^{1+\alpha} + \varpi^{1+\beta} + \varpi^{\alpha+\beta} + \gamma + 1}$$

and from it, we can find that (1.1) has a unique fixed point $\varpi = 1$.

Here, we also provide relevant definitions, finding some results that will benefit us in implementing of the behavior of solutions of (1.1). Consider $J$ is an interval in $\mathbb{R}$ and $g : J \times J \times J \to J$ be a differentiable continuously mapping. Then, for initials $y_{-1}, y_{-4}, y_{-3}, y_{-2}, y_0 \in J$, the difference equation

$$y_{m+1} = g(y_m, y_{m-2}, y_{m-4}), \quad m = 0, 1, 2, \ldots, \tag{1.2}$$

has a unique solution $\{y_m\}_{m=-4}^{\infty}$.

A point $\varpi$ is called an fixed point of (1.2) if $\varpi = g(\varpi, \varpi, \varpi)$. That is, $y_m = \varpi$ for $m \geqslant 0$, is a solution of (1.2).

**Definition 1.1.** Let $\varpi$ be a fixed of (1.2).

(a) A fixed point $\varpi$ is said to be stable if, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that if $y_{-3}, y_{-2}, y_{-4}, y_{-1}, y_0 \in J$ and $|-\varpi + y_{-4}| + |-\varpi + y_{-3}| + |y_{-2} - \varpi| + |y_{-1} - \varpi| + |y_0 - \varpi| < \delta$, then $|y_m - \varpi| < \epsilon$ for all $m \geqslant -4$.

(b) A fixed point $\varpi$ is called locally asymptotically stable if $\varpi$ is stable and if $\exists\ \delta > 0$ such that if $y_{-1}, y_{-3}, y_{-2}, y_{-4}, y_0 \in J$ and $|-\varpi + y_{-3}| + |y_{-4} - \varpi| + |y_{-2} - \varpi| + |y_{-1} - \varpi| + |y_0 - \varpi| < \delta$, then $\lim_{m\to\infty} y_m = \varpi$.

(c) A fixed point $\varpi$ is said to be a global attractor if

$$\lim_{m\to\infty} y_m = \varpi, \quad \forall y_{-2},\ y_{-3}, y_{-1}, y_{-4},\ y_0 \in J.$$

(d) The fixed point $\varpi$ is said to be globally asymptotically stable if $\varpi$ is global attractor and stable.
(e) A fixed point $\varpi$ is said to be unstable if isn't stable.
(f) A fixed point $\varpi$ is called a repeller if $\exists\ \delta > 0$ such that for $y_{-3}, y_{-4}, y_{-2}, y_{-1}, y_0 \in J$ and $|y_{-1} - \varpi| + |-\varpi + y_{-3}| + |y_{-2} - \varpi| + |y_{-4} - \varpi| + |-\varpi + y_0| < \delta$, $\exists\ M \geqslant -4$ such that $|y_M - \varpi| \geqslant \delta$.

Let

$$b_1 = \frac{\partial g(\varpi,\ \varpi,\ \varpi)}{\partial s}, \qquad b_2 = \frac{\partial g(\varpi,\ \varpi,\ \varpi)}{\partial t}, \qquad \text{and} \qquad b_3 = \frac{\partial g(\varpi,\ \varpi,\ \varpi)}{\partial u},$$

where $g(s,\ t, u)$ is the mapping in (1.2) and $\varpi$ is a fixed point of the equation. Hence

$$y_{m+1} = b_1 y_m + b_2 y_{m-2} + b_3 y_{m-4}, \qquad n = 0, 1, 2, 3, \ldots$$

is said to be linearized equation associated with (1.2) about the fixed point $\varpi$.

**Definition 1.2.** A positive semicycle of a solution $\{y_m\}_{m=-4}^{\infty}$ of (1.1) consists of a string of terms $\{y_h, y_{h+1}, \cdots, y_k\}$, all greater than or equal to the fixed point $\varpi$, with $h \geqslant -4$ and $k \leqslant \infty$ such that

$$\text{either} \quad h = -4 \quad \text{or} \quad h > -4 \quad \text{and} \quad y_{h-1} < \varpi$$

and

$$\text{either} \quad k = \infty \quad \text{or} \quad k < \infty \quad \text{and} \quad y_{k+1} < \varpi.$$

A negative semicycle of a solution $\{y_m\}_{m=-4}^{\infty}$ of (1.1) consists of a string of terms $\{y_h, y_{h+1}, \cdots, y_k\}$, all less than $\varpi$, with $h \geqslant -4$ and $k \leqslant \infty$ provided that

$$\text{either} \quad h = -4 \quad \text{or} \quad h > -4 \quad \text{and} \quad y_{h-1} \geqslant \varpi$$

and

$$\text{either} \quad k = \infty \quad \text{or} \quad k < \infty \quad \text{and} \quad y_{k+1} \geqslant \varpi.$$

Length for a semicycle represents number of all terms included in it.

The solution $\{y_m\}_{m=-4}^{\infty}$ of (1.1) is called eventually trivial if $y_m$ is eventually equal to $\varpi = 1$. Otherwise, the solution is called nontrivial. A solution $\{y_m\}_{m=-4}^{\infty}$ of (1.1) is called eventually negative (positive) if $y_m$ is eventually less (great) than $\varpi = 1$;

## 2. Main conclusions and their proofs

In current section we shall establish our main results. We concentrate on the non-trivial solutions, non-oscillation, oscillation, and global asymptotically stability for (1.1).

### 2.1. *Non-trivial solution*

**Theorem 2.1.** *A positive solution* $\{y_m\}_{m=-4}^{\infty}$ *of* (1.1) *is eventually trivial if and only if*

$$(-1 + y_{-3})(-1 + y_{-1})(y_{-2} - 1)(-1 + y_{-4})(y_0 - 1) = 0. \tag{2.1}$$

*Proof.* Let (2.1) holds. Thus it pursues from (1.1):

1) if $y_{-4} = 1$, hence $y_m = 1$ where $m \geqslant 1$;
2) $y_{-3} = 1$, hence $y_m = 1$ where $m \geqslant 2$;
3) $y_{-2} = 1$, hence $y_m = 1$ where $m \geqslant 1$;
4) $y_{-1} = 1$, hence $y_m = 1$ where $m \geqslant 2$;
5) $y_0 = 1$, hence $y_m = 1$ for $m \geqslant 1$.

Conversely, assume that

$$(-1+y_{-1})(y_{-4}-1)(-1+y_{-2})(y_{-3}-1)(y_0-1)\neq 0.$$

Then we can show that

$$y_m \neq 1 \quad \text{for any} \quad m \geqslant 1.$$

Assume to the contrary that, for some $1 \leqslant M$,

$$1 = y_M \quad \text{and that} \quad 1 \neq y_m \quad \text{for} \quad -4 \leqslant m \leqslant M-1.$$

We have,

$$1 = y_M = \frac{y_{M-1}y^{\alpha}_{M-3}y^{\beta}_{M-5} + y_{M-1} + y^{\alpha}_{M-3} + y^{\beta}_{M-5} + \gamma}{y_{M-1}y^{\alpha}_{M-3} + y^{\alpha}_{M-3}y^{\beta}_{M-5} + y_{M-1}y^{\beta}_{M-5} + \gamma + 1},$$

which implies $(y_{M-1}-1)(y_{M-3}-1)(y_{M-5}-1)=0$, which is a contradiction. □

*Remark* 2.2. Theorem 2.1 factually elucidates that a solution $\{y_m\}_{m=-4}^{\infty}$ of (1.1) is eventually non-trivial if and only if $(-1+y_{-4})(y_{-3}-1)(y_{-1}-1)(-1+y_{-2})(y_0-1)\neq 0$. Thus, if a solution $\{y_m\}_{m=-4}^{\infty}$ is non-trivial, then $y_m \neq 1$ for $-4 \leqslant m$.

### 2.2. *Non-Oscillation and Oscillation*

Firstly, we require the following important lemma.

**Lemma 2.3.** *For every non-trivial solution $\{y_m\}_{m=-4}^{\infty}$ of (1.1):*

(a) $(-1+y_{m+1})(-1+y_m)(y_{m-2}-1)(-1+y_{m-4})>0$, $m \geqslant 0$;
(b) $(y_{m+1}-y_m)(y_m-1)<0$ *for* $m \geqslant 0$;
(c) $(-y_{m-2}+y_{m+1})(-1+y_{m-2})<0$ *for* $m \geqslant 0$;
(d) $(y_{m+1}-y_{m-4})(-1+y_{m-4})<0$ *for* $m \geqslant 0$.

*Proof.* By (1.1), we find that

$$-1+y_{m+1} = \frac{(-1+y_m)(-1+y_{m-2})(-1+y_{m-4})f(y_{m-2},y_m,y_{m-4})}{y_my^{\alpha}_{m-2} + y^{\alpha}_{m-2}y^{\beta}_{m-4} + y_my^{\beta}_{m-4} + \gamma + 1}, \quad m = 0,1,2,\dots,$$

where $f(y_m,y_{m-2},y_{m-4})$ gives positive values for all $y_m,y_{m-2},y_{m-4}\in(0,\ \infty)$. This gives inequality (a). Moreover,

$$y_{m+1}-y_m = \frac{-(-1+y_m)g(y_{m-2},y_m,y_{m-4})}{y_my^{\alpha}_{m-2} + y^{\alpha}_{m-2}y^{\beta}_{m-4} + y_my^{\beta}_{m-4} + \gamma + 1}, \quad m = 0,1,2,3,\dots,$$

where $g(y_{m-4},y_{m-2},y_m)$ gives positive values for all $y_m,y_{m-4},y_{m-2}\in(0,\ \infty)$, which gives the inequality (b). Similarly we can get inequalities (c) and (d). So the proof terminates. □

**Theorem 2.4.** *There exist nonoscillatory solutions for (1.1), which have to be eventually negative. Eventually positive nonoscillatory solutions do not exist for (1.1).*

*Proof.* Take into account a solution of (1.1) with $y_{-1}<1, y_{-3}<1, y_{-4}<1,\ y_{-2}<1$ and $y_0<1$. By Lemma 2.3 (a) we have $y_m<1$ for $m\geqslant -4$. This solution is merely a nonoscillatory solution and over and above eventually negative.

Let there are eventually positive nonoscillatory solutions of (1.1). So, we have a positive integer M provided that $y_m>1$ for $m\geqslant M$. Therefore, for $m\geqslant M+2$, $(-1+y_{m+1})(y_m-1)(-1+y_{m-2})(-1+y_{m-4})<0$. Thus, there is no eventually positive nonoscillatory solutions of (1.1), as desired. □

Now we anatomize the principle of track framework of precisely solutions of (1.1).

**Theorem 2.5.** *Let $\{y_m\}_{m=-4}^{\infty}$ represents a precisely oscillatory solution of (1.1). The principle of periods of negative and positive semicycles of the solution respectively take place as either*

$$\cdots, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, \cdots$$

*or*

$$\cdots, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, \cdots l.$$

*Proof.* According to the precisely oscillatory nature of a solution, we have, for integer $\Upsilon \geqslant 0$, that subsequent cases take place as:

Case 1: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 < y_{\Upsilon-2}, 1 > y_{\Upsilon-1}, 1 < y_{\Upsilon}$;

Case 2: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 < y_{\Upsilon-2}, 1 > y_{\Upsilon-1}, y_{\Upsilon} < 1$;

Case 3: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 < y_{\Upsilon-2}, 1 < y_{\Upsilon-1}, y_{\Upsilon} < 1$;

Case 4: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 < y_{\Upsilon-2}, 1 < y_{\Upsilon-1}, y_{\Upsilon} > 1$;

Case 5: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 > y_{\Upsilon-2}, 1 > y_{\Upsilon-1}, y_{\Upsilon} > 1$;

Case 6: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 > y_{\Upsilon-2}, 1 > y_{\Upsilon-1}, y_{\Upsilon} < 1$;

Case 7: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 > y_{\Upsilon-2}, 1 < y_{\Upsilon-1}, y_{\Upsilon} < 1$;

Case 8: $y_{\Upsilon-4} > 1, y_{\Upsilon-3} < 1, 1 > y_{\Upsilon-2}, 1 < y_{\Upsilon-1}, y_{\Upsilon} > 1$.

If Case 1 takes place, we have, from Lemma 2.3 (a), that $1 < y_{\Upsilon+1}$, $y_{\Upsilon+2} > 1$, $1 < y_{\Upsilon+3}$, $1 > y_{\Upsilon+4}$, $1 < y_{\Upsilon+5}$, $1 > y_{\Upsilon+6}$, $1 < y_{\Upsilon+7}$, $1 > y_{\Upsilon+8}$, $1 > y_{\Upsilon+9}$, $1 < y_{\Upsilon+10}$, $1 < y_{\Upsilon+11}$, $1 > y_{\Upsilon+12}$, $1 < y_{\Upsilon+13}$, $1 > y_{\Upsilon+14}$, $1 < y_{\Upsilon+15}, \ldots$, and then the principle of the periods of negative and positive semicycles of the solution of (1.1) respectively take place as $\ldots, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^- \ldots$.

If Case 2 occurs, then we have $y_{\Upsilon+1} < 1$, $1 > y_{\Upsilon+2}$, $1 < y_{\Upsilon+3}$, $1 < y_{\Upsilon+4}, y_{\Upsilon+5} > 1$, $1 > y_{\Upsilon+6}$, $1 < y_{\Upsilon+7}$, $1 < y_{\Upsilon+8}, y_{\Upsilon+9} < 1$, $1 > y_{\Upsilon+10}$, $1 < y_{\Upsilon+11}$, $1 > y_{\Upsilon+12}$, $1 < y_{\Upsilon+13}$, $1 > y_{\Upsilon+14}$, $1 > y_{\Upsilon+15}, \ldots$. Thus the principle of terms numbers of negative and positive semicycles of solution of (1.1) respectively is $\ldots, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, \ldots$.

By the same way and by using Lemma 2.3 (a) we can find that the Cases 3, 6, 7 have the principle $\ldots, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, \ldots$ for negative and positive semicycles of solution as in Case 2. But the Cases 4, 5, 8 have the principle $\ldots, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, \ldots$ for the periods of negative and positive semicycles of solution as in Case 1. □

*Remark* 2.6. We know that specification Cases in Theorem 2.5 are happened by the disruption of the starting point round the fixed point. So, Theorem 2.5 in effect points out that disturbance of the initials may instruct to the distinction of the track framework principle of solution of (1.1).

### 2.3. *Global asymptotic stability*

In the beginning, we take into account local asymptotically stability of positive fixed $\varpi$ of (1.1).

**Theorem 2.7.** *Non-negative fixed point of (1.1) has locally asymptotically stability.*

*Proof.* For the positive fixed point $\varpi = 1$, we get linearized equation of (1.1)

$$y_{m+1} = y_{m-4} \times 0 + y_{m-2} \times 0 + 0 \times y_m, \quad m = 0, 1, \ldots.$$

By dint of [10], $\varpi$ is locally asymptotically stable. □

**Theorem 2.8.** *The Non-negative fixed point of (1.1) has global asymptotic stability.*

*Proof.* It is sufficient to show that each solution $\{y_m\}_{m=-4}^{\infty}$ of (1.1) converges to $\varpi$ as $m \to \infty$, i.e.,

$$\lim_{m\to\infty} y_m = \varpi = 1. \tag{2.2}$$

We can split solutions as 1) trivial solutions and 2) non-trivial solutions.

If solution is trivial, then (2.2) holds because $y_m = 1$ eventually. Hence (2.2) holds.

First, let solution be non-trivial. Thus we can split the solution to a) nonoscillatory solution and b) oscillatory solution.

If a) happens, thus we have an eventually negative solution. Thus, there works out an integer $M$ provided that $y_m < 1$ for $m \geqslant M$. From Lemma 2.3 (b) we have that solution is bounded and monotonic. Then, $\lim_{n\to\infty} y_m = U$ exists and finite. If we take limit of (1.1), we get

$$U = \frac{U^{1+\alpha+\beta} + U + U^{\alpha} + U^{\beta} + a}{U^{1+\alpha} + U^{1+\beta} + U^{\alpha+\beta} + a + 1}.$$

Hence $U = 1$. This shows that (2.2) is adequate for nonoscillatory solutions.

Hence, it is enough to show that (2.2) precises for the solution is oscillatory. This means that Case b) takes place.

Let $\{y_m\}_{m=-4}^{\infty}$ is precisely oscillatory about fixed point $\varpi$ of (1.1). First, we find that the principle of the periods of negative and positive semicycles of this solution respectively take place as either $\ldots, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, \ldots$ or $\ldots, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, \ldots$.

Now we do the case in which the principle of the periods of negative and positive semicycles take place respectively as $\ldots, 3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+, 3^-, 1^+, 2^-, 2^+, 1^-, \ldots$.

Now, we denote by $\{y_{\Upsilon}, y_{\Upsilon+1}, y_{\Upsilon+2}\}^-$, $\{y_{\Upsilon+3}\}^+$, $\{y_{\Upsilon+4}, y_{\Upsilon+5}\}^-$, $\{y_{\Upsilon+6}, y_{\Upsilon+7}\}^+$, $\{y_{\Upsilon+8}\}^-$, $\{y_{\Upsilon+9}\}^+$, $\{y_{\Upsilon+10}\}^-$, $\{y_{\Upsilon+11}, y_{\Upsilon+12}, y_{\Upsilon+13}, y_{\Upsilon+14}\}^+$. So, the principle of the negative and positive semicycles that take place, respectively can be put as $\{y_{\Upsilon+15m}, y_{\Upsilon+15m+1}, y_{\Upsilon+15m+2}\}^-$, $\{y_{\Upsilon+15m+3}\}^+$, $\{y_{\Upsilon+15m+4}, y_{\Upsilon+15m+5}\}^-$, $\{y_{\Upsilon+15m+6}, y_{\Upsilon+15m+7}\}^+$, $\{y_{\Upsilon+15m+8}\}^-$, $\{y_{\Upsilon+15m+9}\}^+$,$\{y_{\Upsilon+15m+10}\}^-$, $\{y_{\Upsilon+15m+11}, y_{\Upsilon+15m+12}, y_{\Upsilon+15m+13}, y_{\Upsilon+15m+14}\}^+$, $m = 0, 1, \ldots$.

We can see that:

(I1). $y_{\Upsilon+15m} < y_{\Upsilon+15m+1} < y_{\Upsilon+15m+4} < y_{\Upsilon+15m+5} < y_{\Upsilon+15m+10} < y_{\Upsilon+15m+15}$;

(I2). $y_{\Upsilon+15m+18} < y_{\Upsilon+15m+13} < y_{\Upsilon+15m+12} < y_{\Upsilon+15m+11} < y_{\Upsilon+15m+6} < y_{\Upsilon+15m+3}$ ;

(I3). $y_{\Upsilon+15m+2} < \frac{1}{y_{\Upsilon+15m+7}} < y_{\Upsilon+15m+8} < \frac{1}{y_{\Upsilon+15m+9}} < \frac{1}{y_{\Upsilon+15m+14}} < y_{\Upsilon+15m+15} < y_{\Upsilon+15m+16} < y_{\Upsilon+15m+17}$.

The inequalities (I1) and (I2) follow directly from Lemma 2.3 (b), Lemma 2.3 (c), and Lemma 2.3 (d), and inequality (I3) comes from Lemma 2.3 and (1.1). We can see from inequalities (I1), (I2), and (I3) that both of $\{y_{\Upsilon+15m}\}_{m=0}^{\infty}$ and $\{y_{\Upsilon+15m+2}\}_{m=0}^{\infty}$ will increase with upper bound 1, while $\{y_{\Upsilon+15m+3}\}_{m=0}^{\infty}$ will decrease with lower bound 1. Hence, their limits exist and are finite. Suppose that $\lim_{m\to\infty} y_{\Upsilon+15m} = U$, $\lim_{m\to\infty} y_{\Upsilon+15m+2} = V$, and $\lim_{m\to\infty} y_{\Upsilon+15m+3} = W$. So

$$\lim_{m\to\infty} y_{\Upsilon+15m} = \lim_{m\to\infty} y_{\Upsilon+15m+1} = \lim_{m\to\infty} y_{\Upsilon+15m+4} = \lim_{m\to\infty} y_{\Upsilon+15m+5} = \lim_{m\to\infty} y_{\Upsilon+15m+10} = U.$$

Furthermore, in light of (I2) we can get

$$\lim_{m\to\infty} y_{\Upsilon+15m+3} = \lim_{m\to\infty} y_{\Upsilon+15m+6} = \lim_{m\to\infty} y_{\Upsilon+15m+11} = \lim_{m\to\infty} y_{\Upsilon+15m+12} = \lim_{m\to\infty} y_{\Upsilon+15m+13} = W.$$

On the other hand, in light of (I3), we have

$$\lim_{m\to\infty} y_{\Upsilon+15m+2} = \lim_{m\to\infty} y_{\Upsilon+15m+8} = V,$$
$$\lim_{m\to\infty} y_{\Upsilon+15m+7} = \lim_{m\to\infty} y_{\Upsilon+15m+9} = \lim_{m\to\infty} y_{\Upsilon+15m+14} = \frac{1}{V}.$$

Now by using (1.1), we get

$$y_{\Upsilon+15m+5} = \frac{y_{\Upsilon+15m+4} y^{\alpha}_{\Upsilon+15m+2} y^{\beta}_{\Upsilon+15m} + y_{\Upsilon+15m+4} + y^{\alpha}_{\Upsilon+15m+2} + y^{\beta}_{\Upsilon+15m} + \gamma}{y_{\Upsilon+15m+4} y^{\alpha}_{\Upsilon+15m+2} + y_{\Upsilon+15m+4} y^{\beta}_{\Upsilon+15m} + y^{\alpha}_{\Upsilon+15m+2} y^{\beta}_{\Upsilon+15m} + \gamma + 1}, \tag{2.3}$$

$$y_{\Upsilon+15m+6} = \frac{y_{\Upsilon+15m+5}y^{\alpha}_{\Upsilon+15m+3}y^{\beta}_{\Upsilon+15m+1} + y_{\Upsilon+15m+5} + y^{\alpha}_{\Upsilon+15m+3} + y^{\beta}_{\Upsilon+15m+1} + \gamma}{y_{\Upsilon+15m+5}y^{\alpha}_{\Upsilon+15m+3} + y_{\Upsilon+15m+5}y^{\beta}_{\Upsilon+15m+1} + y^{\alpha}_{\Upsilon+15m+3}y^{\beta}_{\Upsilon+15m+1} + \gamma + 1}, \tag{2.4}$$

$$y_{\Upsilon+15m+7} = \frac{y_{\Upsilon+15m+6}y^{\alpha}_{\Upsilon+15m+4}y^{\beta}_{\Upsilon+15m+2} + y_{\Upsilon+15m+6} + y^{\alpha}_{\Upsilon+15m+4} + y^{\beta}_{\Upsilon+15m+2} + \gamma}{y_{\Upsilon+15m+6}y^{\alpha}_{\Upsilon+15m+4} + y_{\Upsilon+15m+6}y^{\beta}_{\Upsilon+15m+2} + y^{\alpha}_{\Upsilon+15m+4}y^{\beta}_{\Upsilon+15m+2} + \gamma + 1}. \tag{2.5}$$

Now let us take the limit on both sides of equalities (2.3)-(2.5). So, we have the following system

$$U = \frac{V^{\alpha}U^{\beta+1} + U + V^{\alpha} + U^{\beta} + \gamma}{UV^{\alpha} + U^{\beta+1} + V^{\alpha}U^{\beta} + 1 + \gamma}, \tag{2.6}$$

$$W = \frac{W^{\alpha}U^{\beta+1} + U + W^{\alpha} + U^{\beta} + \gamma}{UW^{\alpha} + U^{\beta+1} + W^{\alpha}U^{\beta} + 1 + \gamma}, \tag{2.7}$$

$$\frac{1}{V} = \frac{WU^{\alpha}V^{\beta} + W + U^{\alpha} + V^{\beta} + \gamma}{WU^{\alpha} + WV^{\beta} + U^{\alpha}V^{\beta} + 1 + \gamma}. \tag{2.8}$$

If we solve system (2.6)-(2.8) we get $U = V = W = 1$.

Up to this, we have shown $\lim_{m\to\infty} y_{\Upsilon+15m+k} = 1, k = 0, 1, 2, 3, \ldots, 14$. Thus $\lim_{m\to\infty} y_m = 1$.

Now, we examine the case in which the principle of lengths of negative and positive semicycles which take place respectively as $\ldots, 3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-, 3^+, 1^-, 2^+, 2^-, 1^+, \ldots$. Then the principle of the positive and negative semicycles to take place respectively can be periodically formulated as $\{y_{\Upsilon+15m}, y_{\Upsilon+15m+1}, y_{\Upsilon+15m+2}\}^+$, $\{y_{\Upsilon+15m+3}\}^-$, $\{y_{\Upsilon+15m+4}, y_{\Upsilon+15m+5}\}^+$, $\{y_{\Upsilon+15m+6}, y_{\Upsilon+15m+7}\}^-$, $\{y_{\Upsilon+15m+8}\}^+$, $\{y_{\Upsilon+15m+9}\}^-$, $\{y_{\Upsilon+15m+10}\}^+$, $\{y_{\Upsilon+15m+11}, y_{\Upsilon+15m+12}, y_{\Upsilon+15m+13}, y_{\Upsilon+15m+14}\}^-$, $m=0,1,\ldots$.

We have

(I4). $y_{\Upsilon+15m} > y_{\Upsilon+15m+1} > y_{\Upsilon+15m+4} > y_{\Upsilon+15m+5} > y_{\Upsilon+15m+10} > y_{\Upsilon+15m+15}$;

(I5). $y_{\Upsilon+15m+18} > y_{\Upsilon+15m+13} > y_{\Upsilon+15m+12} > y_{\Upsilon+15m+11} > y_{\Upsilon+15m+6} > y_{\Upsilon+15m+3}$;

(I6). $y_{\Upsilon+15m+2} > \frac{1}{y_{\Upsilon+15m+7}} > y_{\Upsilon+15m+8} > \frac{1}{y_{\Upsilon+15m+9}} > \frac{1}{y_{\Upsilon+15m+14}} > y_{\Upsilon+15m+15} > y_{\Upsilon+15m+16} > y_{\Upsilon+15m+17}$.

Indeed, the above inequalities (I4) and (I5) follow directly from Lemma 2.3 (b), Lemma 2.3 (c), and Lemma 2.3 (d) and inequality (I6) comes from Lemma 2.3 and (1.1). We can see from inequalities (I4), (I5), and (I6) that both of $\{y_{\Upsilon+15m}\}_{m=0}^{\infty}$ and $\{y_{\Upsilon+15m+2}\}_{m=0}^{\infty}$ will decrease with lower bound 1, while $\{y_{\Upsilon+15m+3}\}_{m=0}^{\infty}$ will increase with upper bound 1. Thus, their limits are finite and exist.

Let $\lim_{m\to\infty} y_{\Upsilon+15m} = U^*$, $\lim_{m\to\infty} y_{\Upsilon+15m+2} = V^*$, and $\lim_{m\to\infty} y_{\Upsilon+15m+3} = W^*$. So

$$\lim_{m\to\infty} y_{\Upsilon+15m} = \lim_{m\to\infty} y_{\Upsilon+15m+1} = \lim_{m\to\infty} y_{\Upsilon+15m+4} = \lim_{m\to\infty} y_{\Upsilon+15m+5} = \lim_{m\to\infty} y_{\Upsilon+15m+10} = U^*.$$

Furthermore, in light of (I5) we can get

$$\lim_{m\to\infty} y_{\Upsilon+15m+3} = \lim_{m\to\infty} y_{\Upsilon+15m+6} = \lim_{m\to\infty} y_{\Upsilon+15m+11} = \lim_{m\to\infty} y_{\Upsilon+15m+12} = \lim_{m\to\infty} y_{\Upsilon+15m+13} = W^*.$$

On the other hand, in light of (I6), we have

$$\lim_{m\to\infty} y_{\Upsilon+15m+2} = \lim_{m\to\infty} y_{\Upsilon+15m+8} = V^*,$$
$$\lim_{m\to\infty} y_{\Upsilon+15m+7} = \lim_{m\to\infty} y_{\Upsilon+15m+9} = \lim_{m\to\infty} y_{\Upsilon+15m+14} = \frac{1}{V^*}.$$

As in previous case and by using equations (2.3), (2.4), and (2.5), we get $U^* = W^* = V^* = 1$. So we have $\lim_{m\to\infty} y_{\Upsilon+15m+k} = 1, k = 0, 1, 2, 3, \ldots, 14$. Consequently $\lim_{m\to\infty} y_m = 1$.

So, we have completed the proof for the Theorem. □

### 2.4. *Principle of track structure*

We can summarize the general principle of the track framework of (1.1) solutions.

**Theorem 2.9.** *The principle of the track framework of a solution of* (1.1) *is divided into:*

1. *solution will be eventually trivial;*
2. *solution will be eventually non-trivial:*
   (a) *solution will be eventually negative nonoscillatory,*
   (b) *solution will be precisely oscillatory.*

*Moreover, the lengths for negative and positive semicycles take place periodically with only prime period fifteen and in a period exemplified by* $\{3^+, 1^-, 2^+, 2^-, 1^+, 1^-, 1^+, 4^-\}$ *or* $\{3^-, 1^+, 2^-, 2^+, 1^-, 1^+, 1^-, 4^+\}$. *By using this principle, positive fixed point* (1.1) *has global asymptotic stability.*

## Acknowledgment

The author would like to express his heartfelt thanks to referees for their suggestions and prosperous comments.

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