Weakly \((s, r)\)-contractive multi-valued operators on \(b\)-metric space

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Abstract

In this paper we introduce the notion of weakly \((s, r)\)-contractive multi-valued operator on \(b\)-metric space and establish some fixed point theorems for this operator. In addition, an application to the differential equation is given to illustrate usability of obtained results.

Keywords: \(b\)-metric space, weakly \((s, r)\)-contractive multi-valued operator, fixed point theorem.

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1. Introduction

Banach fixed point theorem [1] says that every contractive mapping on a complete metric space has a unique fixed point. As it is well known, the Banach fixed point theorem is a very useful, simple and classical tool in modern analysis. There are a large number of generalizations for this interesting theorem, for example see [5, 9–11, 16, 20]. On the one hand, to get an analog result for multi-valued mappings, one has to equip the powerset of a set with some suitable metric. One such a metric is a Hausdorff metric. Markin [13] for the first time used the Hausdorff metric to study the fixed point theory of the multi-valued contractive mapping; Nadler [15] and Reich [18, 19] respectively introduced the fixed point theorem of the multi-valued contractive operator and generalized the compression conditions given by Nadler; Rus [23] introduced multi-valued weakly Picard operator; Popescu [17] introduced the definition of the \((s, r)\)-contractive multi-valued operator and showed that this operator is a weakly Picard operator. On the other hand, Czerwik [3] introduced the notions of the contractive mapping and the set-valued contractive mapping on \(b\)-metric space. Recently Kamran and Hussain [12] generalized the \((s, r)\)-contractive multi-valued operator and introduced the notion of the weakly \((s, r)\)-contractive multi-valued operator. They also obtained fixed points and strict fixed point theorems for the weakly \((s, r)\)-contractive multi-valued operators.

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operator. Thus it is worth for us to research fixed point theorems of the multi-valued operator in $b$-metric space.

Next, we present some elementary definitions and results which will be used throughout this paper. Details can be seen in [2–4, 6–8, 21, 24, 25].

**Definition 1.1** ([3]). Let $X$ be a nonempty set and $K \geq 1$ be a given constant. A function $d : X \times X \to \mathbb{R}^+$ is called a $b$-metric if the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq K[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair $(X, d)$ is called a $b$-metric space (with constant $K$).

It is easy to see that any metric space is a $b$-metric space with $K = 1$. The following example shows that a $b$-metric on $X$ need not be a metric on $X$.

**Example 1.2.** The set $\mathbb{R}$ of real numbers together with the function

$$d(x, y) := |x - y|^2$$

for all $x, y \in \mathbb{R}$ is a $b$-metric space with constant $K = 2$ but not a metric space.

**Definition 1.3** ([2]). Let $(X, d)$ be a $b$-metric space and $\{x_n\}$ be a sequence of $X$ such that

1. $\{x_n\}$ is convergent if there exists an $x$ in $X$ such that for any $\varepsilon > 0$, there exists an $n(\varepsilon) \in \mathbb{N}$, such that $n \geq n(\varepsilon)$, $d(x_n, x) < \varepsilon$.
2. $\{x_n\}$ is a Cauchy sequence if for any $\varepsilon > 0$, there exists an $n(\varepsilon) \in \mathbb{N}$, such that for all $m, n \geq n(\varepsilon)$, $d(x_n, x_m) < \varepsilon$.
3. $(X, d)$ is complete if and only if every Cauchy sequence in $X$ is convergent.

In contrast to the general metric, $b$-metric is not continuous. However we introduce the following lemma.

**Lemma 1.4** ([21]). Let $(X, d)$ be a $b$-metric space with the constant $K \geq 1$, and suppose that the sequences $\{x_n\}$ and $\{y_n\}$ converge to $x, y$, respectively. Then

$$\frac{1}{K^2} d(x, y) \leq \lim_{n \to \infty} d(x_n, y_n) \leq \lim_{n \to \infty} d(x_n, y) \leq K^2 d(x, y).$$

In particular, if $x = y$, then $\lim_{n \to \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$,

$$\frac{1}{K} d(x, z) \leq \lim_{n \to \infty} d(x_n, z) \leq \lim_{n \to \infty} d(x, z) \leq K d(x, z).$$

In order to study fixed point theorems of the multi-valued mapping, we introduce the concept of Hausdorff metric.

**Definition 1.5.** Let $(X, d)$ be a metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of $X$. For any $A, B \in CB(X)$, set

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$, then $(CB(X), H)$ is a metric space and $H(A, B)$ is called a Hausdorff metric.
Similarly, if \((X, d)\) is a b-metric space, then \((CB(X), H)\) is a b-metric space. \(H(A, B)\) is called a b-Hausdorff metric on \(CB(X)\). In the following, unless stated in particular, \(H(A, B)\) will always denote a b-Hausdorff metric.

**Remark 1.6.** Suppose that \((X, d)\) is a metric space, then \(H(A, B)=0\) iff \(A = B\).

**Definition 1.7** ([4]). Let \(X\) be a b-metric space and \(T : X \to CB(X)\) be a multi-valued operator. If there exists \(k \in [0, 1]\) such that \(H(Tx, Ty) \leq kd(x, y)\) for all \(x, y \in X\), then \(T\) is called a contractive multi-valued operator.

**Definition 1.8.** Let \((X, d)\) be a b-metric space and \(T : X \to CB(X)\) be a multi-valued operator. If there exist constants \(s, r\) with \(r \in [0, 1], s \geq r\) such that for all \(x, y \in X\),

\[
d(y, Tx) \leq Ksd(x, y) \Rightarrow H(Tx, Ty) \leq rM_T(x, y),
\]

where

\[
M_T(x, y) = \max \left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2K} \right\},
\]

then \(T\) is called a \((s, r)\)-contractive multi-valued operator.

The purpose of this paper is to generalize the results of Kamran [12] and introduce the notion of weakly \((s, r)\)-contractive multi-valued operator and establish some fixed point theorems for this operator on b-metric space.

### 2. Main results

In this section we introduce the notion of weakly \((s, r)\)-contractive multi-valued operator and present our results. We start this section with the following definition.

**Definition 2.1.** Let \((X, d)\) be a b-metric space and \(T : X \to CB(X)\) be a multi-valued operator. If there exist \(r \in [0, 1]\) and \(s \geq r, L \geq 0\) such that for any \(x, y \in X\),

\[
d(x, Ty) \leq Ksd(x, y) \Rightarrow H(Tx, Ty) \leq rM_T(x, y) + L \min\{d(x, y), d(y, Tx)\},
\]

where

\[
M_T(x, y) = \max \left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2K} \right\},
\]

then \(T\) is a weakly \((s, r)\)-contractive multi-valued operator on \(X\).

**Remark 2.2.** When \(L = 0\), the above definition reduces to Definition 1.8.

The following example shows that the notion of weakly \((s, r)\)-contractive operator properly generalizes the notion of \((s, r)\)-contractive operator.

**Example 2.3.** Let \(X = \{1, 2, 3\}\) endowed with the b-metric \(d(x, y) = |x - y|^2\). Then \((X, d)\) is a complete b-metric space with the constant \(K = 2\). Define \(T : X \to CB(X)\) by

\[
Tx = \begin{cases} 
\{1, 2\}, & x \in \{1, 2\}, \\
\{3\}, & x = 3.
\end{cases}
\]

Then \(H(T1, T1) = H(T2, T2) = H(T3, T3) = H(T1, T2) = H(T2, T1) = 0\). By choosing \(s = 0.4\),

\[
d(1, T3) = 4 > 3.2 = K \cdot s \cdot d(1, 3), \quad d(2, T3) = 1 > 0.8 = K \cdot s \cdot d(2, 3), \quad d(3, T2) = 1 < 0.8 = K \cdot s \cdot d(3, 2).
\]

Further, \(d(3, T1) = 1 < 3.2 = K \cdot s \cdot d(3, 1)\). Now if we choose \(L = 1\) and \(r = 0.2\), then

\[
H(T3, T1) = 1 < 4.8 = r \max \left\{d(3, 1), d(3, T3), d(1, T1), \frac{d(3, T1) + d(1, T3)}{2K} \right\} + L \min\{d(3, 1), d(1, T3)\}.
\]

This shows that \(T\) is weakly \((0.4, 0.2)\)-contractive map with \(L = 1\), but not \((0.4, 0.2)\)-contractive. Since
Lemma 2.4 ([14, 21]). Let \((X, d)\) be a complete \(b\)-metric space with the constant \(K \geq 1\) and \(\{x_n\}\) be a sequence in \(X\) such that \(d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1})\) for all \(n = 0, 1, 2, \ldots\), where \(0 \leq \alpha < 1\). If \(K\alpha < 1\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).

The following theorem generalizes the result of Kamran and Hussain [12] to the setting of \(b\)-metric space.

Theorem 2.5. Let \((X, d)\) be a complete \(b\)-metric space and \(T : X \to CB(X)\) be a weakly \((s, r)\)-contractive operator with \(r < \min\left(\frac{1}{K}, s\right)\). Then \(T\) has fixed points.

Proof. Take a real number \(r_1 > 1\) such that \(0 \leq r < r_1 < \min\left(\frac{1}{K}, s\right)\). Let \(x_1 \in X\) and \(x_2 \in TX_1\). Then \(d(x_2, TX_1) = 0 \leq Ksd(x_2, x_1)\) and using hypothesis,

\[
d(x_2, x_2) \leq H(Tx_1, Tx_2) \leq rM_T(x_1, x_2) + L\min\{d(x_1, x_2), d(x_2, Tx_1)\}
\]

\[
= r \max \left\{ d(x_1, x_2), d(x_2, Tx_2), d(x_1, Tx_1), \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{2K} \right\}
\]

\[
\leq r \max \left\{ d(x_1, x_2), d(x_2, Tx_2), \frac{d(x_1, x_2) + d(x_2, Tx_2)}{2} \right\}.
\]

(1) If \(d(x_1, x_2) \leq d(x_2, Tx_2)\), then \(d(x_2, Tx_2) \leq rd(x_2, Tx_2)\). Since \(r < 1\), we have \(d(x_2, Tx_2) = 0\), and \(x_2\) is a fixed point of \(T\).

(2) If \(d(x_1, x_2) > d(x_2, Tx_2)\), then \(d(x_2, Tx_2) \leq rd(x_1, x_2)\). Since \(r < 1\), it follows that there exists \(x_3 \in Tx_2\) such that \(d(x_2, x_3) < r_1d(x_1, x_2)\). Continuing in this manner a sequence \(\{x_n\}\) can be constructed in \(X\) such that \(x_{n+1} \in Tx_n\) and \(d(x_{n+1}, x_{n+2}) \leq r_1d(x_n, x_{n+1})\) for all \(n \in \mathbb{N}\).

Since \(Kr_1 < 1\), it implies \(\{x_n\}\) is a Cauchy sequence by using Lemma 2.4. Since \(X\) is a complete, there is \(z \in X\) such that \(\{x_n\}\) converges to \(z\). Now, we claim that there exists a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) such that

\[
d(z, Tx_{n_k}) \leq Ksd(z, x_{n_k}), \forall k \in \mathbb{N}.
\]

If not, there exists a positive integer \(N \in \mathbb{N}\) such that

\[
d(z, Tx_n) > Ksd(z, x_n), \forall n \geq N.
\]

This implies

\[
d(z, x_{n+1}) > Ksd(z, x_n), \forall n \geq N.
\]

By induction, we obtain

\[
d(z, x_{n+p}) > (Ks)^p d(z, x_n), \forall n \geq N, p \geq 1. \quad (2.1)
\]

Since

\[
d(x_n, x_{n+p}) \leq K(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})) \leq Kd(x_n, x_{n+1})(1 + Kr_1 + \ldots + K^{p-1}r_1^{p-1}) = \frac{K[1 - (Kr_1)^p]}{1 - Kr_1} d(x_n, x_{n+1}), \forall n \geq N, p \geq 1.
\]

Let \(p \to \infty\), using Lemma 1.4,

\[
\frac{1}{K} d(z, x_n) \leq \lim_{p \to \infty} d(x_n, x_{n+p}) \leq \frac{K}{1 - Kr_1} d(x_n, x_{n+1}), \forall n \geq N.
\]
Thus
\[ d(z, x_{n+p}) \leq \frac{K^2}{1 - Kr_1} d(x_{n+p}, x_{n+p+1}) \leq \frac{K^2r_1^p}{1 - Kr_1} d(x_n, x_{n+1}), \forall n \geq N, p \geq 1. \tag{2.2} \]

From (2.1) and (2.2), we obtain
\[ d(z, x_n) \leq \frac{K^2r_1^p}{(Ks)^p(1 - Kr_1)} d(x_n, x_{n+1}). \]

Set \( p \to \infty, d(z, x_n) = 0, \forall n \geq N, \) which contradicts to (1). Therefore there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that
\[ d(z, Tx_{n_k}) \leq Ksd(z, x_{n_k}), \forall k \in N. \]

Thus
\[ d(x_{n_k+1}, Tz) \leq H(Tx_{n_k}, Tz) \leq r \max \left\{ d(z, x_{n_k}), d(z, Tz), d(x_{n_k}, Tx), \frac{d(z, Tx_{n_k}) + d(z, Tz)}{2K} \right\} + L \min \{d(x_{n_k}, z), d(x_{n_k}, Tz)\}. \]

Letting \( k \to \infty, \)
\[ \lim_{k \to \infty} d(x_{n_k+1}, Tz) \leq r \max \left\{ d(z, Tz), \frac{d(z, Tz)}{2K} \right\} = rd(z, Tz). \]

By the triangle inequality,
\[ d(z, Tz) \leq K[d(z, x_{n_k+1}) + d(x_{n_k+1}, Tz)]. \]

Thus
\[ \lim_{k \to \infty} \frac{1}{K} d(z, Tz) \leq \lim_{k \to \infty} [d(z, x_{n_k+1}) + d(x_{n_k+1}, Tz)], \quad \frac{1}{K} d(z, Tz) \leq \lim_{k \to \infty} d(x_{n_k+1}, Tz) \leq rd(z, Tz). \]

As \( Kr < 1, d(z, Tz) = 0. \) Since \( Tz \in CB(X), z \in Tz, T \) has fixed point. \( \square \)

From the following example, one can see that under the condition of Theorem 2.5, the fixed point may not be unique.

**Example 2.6.** Let \( X = [1, \infty) \) and \( d(x, y) = |x - y|^2 \) for all \( x, y \in X. \) Then \( d \) is a complete \( b \)-metric but not a metric on \( X \) with the constant \( K = 2. \) Define \( T : X \to CB(X) \) by
\[ Tx = [2, 2 + \frac{x}{3}] \]
for all \( x \in X. \) Consider \( H(Tx, Ty) = \frac{1}{3}(x - y)^2 = \frac{1}{3}d(x, y), \) where we choose \( r = \frac{1}{3} \in (0, 1), s = \frac{1}{5} > r, \) \( L = 1 \geq 0. \) Then the conditions of Theorem 2.5 are satisfied. Moreover, 2 and 3 are the two fixed points of \( T. \)

It is necessary for us to consider the uniqueness of the fixed point of the weakly \((s, r)\)-contractive multi-valued operator.

**Corollary 2.7.** Let \((X, d)\) be a complete \( b \)-metric space and \( T : X \to X \) be a weakly \((s, r)\)-contractive single-valued operator with \( r < \min\left(\frac{1}{K}, s\right). \) Then \( T \) has a fixed point. Moreover, if \( Ks \geq 1 \) and \( r + L < 1, \) then \( T \) has a unique fixed point.

**Proof.** From Theorem 2.5, \( T \) has a fixed point. Let \( Ks \geq 1 \) and \( (r + L) < 1. \) Suppose that there exist two different fixed points \( x \) and \( y \) of \( T. \) Then
\[ d(y, Tx) = d(y, x) \leq Ksd(y, x). \]
Thus
\[
\begin{align*}
    d(Tx, Ty) &\leq r M_T(x, y) + L \min\{d(x, y), d(y, Tx)\}, \\
    d(x, y) &\leq r M_T(x, y) + L \min\{d(x, y), d(y, Tx)\} \\
    &= r \max \left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} + L \min\{d(x, y), d(y, Tx)\} \\
    &= rd(x, y) + Ld(x, y) = (r + L)d(x, y).
\end{align*}
\]
It is a contradiction, since \((r + L) < 1\).  

Next, we introduce the other theorem about the weakly \((s, r)\)-contractive multi-valued operator.

**Theorem 2.8.** Let \((X, d)\) be a complete \(\alpha\)-metric space and \(T : X \to \text{CB}(X)\) be a multi-valued operator. Assume that there exist constants \(r, s \in [0, 1)\) and \(r < s < \frac{1}{K}\) such that
\[
\frac{1}{K(1 + Kr)} d(x, Tx) \leq d(x, y) \leq \frac{K^2}{1 - Ks} d(Tx, x)
\]
implies
\[
H(Tx, Ty) \leq r M_T(x, y) + L \min\{d(x, y), d(y, Tx)\},
\]
where
\[
M_T(x, y) = \max \left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.
\]
Then \(T\) has a fixed point.

**Proof.** Take a real number \(r_1\) such that \(0 < r < r_1 < s < \frac{1}{K}\). Since \(\frac{1 - Kr_1}{1 - Ks} > 1\), it follows that for \(x_1 \in X\) there exists \(x_2 \in Tx_1\) such that
\[
d(x_1, x_2) \leq \frac{1 - Kr_1}{1 - Ks} d(x_1, Tx_1).
\]
Then
\[
\frac{1}{K(1 + Kr)} d(x_1, Tx_1) \leq d(x_1, x_2) \leq d(x_1, x_2) \leq \frac{1}{1 - Ks} d(x_1, Tx_1) \leq \frac{K^2}{1 - Ks} d(x_1, Tx_1),
\]
and by hypothesis
\[
d(x_1, Tx_2) \leq H(Tx_1, Tx_2) \leq r M_T(x_1, x_2) + L \min\{d(x_1, x_2), d(x_2, Tx_1)\} \\
    \leq r \max \left\{d(x_1, x_2), d(x_2, Tx_2), \frac{d(x_1, x_2) - d(x_2, Tx_2)}{2K} \right\} \\
    \leq r \max \left\{d(x_1, Tx_1), d(x_2, Tx_2), \frac{d(x_1 + x_2) + d(x_2, Tx_2)}{2} \right\}.
\]

(1) If \(d(x_1, x_2) \leq d(x_2, Tx_2)\), then \(d(x_2, Tx_2) \leq rd(x_2, Tx_2)\). Since \(r < 1\), we have \(d(x_2, Tx_2) = 0\). Then \(x_2\) is the fixed point of \(T\).

(2) If \(d(x_1, x_2) > d(x_2, Tx_2)\), then \(d(x_2, Tx_2) \leq rd(x_1, x_2)\). Since \(r < 1\), it follows that there exists \(x_3 \in Tx_2\) such that
\[
d(x_2, x_3) \leq r_1 d(x_1, x_2), \quad d(x_2, x_3) \leq \frac{1 - Kr_1}{1 - Ks} d(x_2, Tx_2).
\]
Therefore a sequence \(\{x_n\}\) can be constructed in \(X\) such that \(x_{n+1} \in Tx_n\) and
\[
d(x_{n+1}, x_{n+2}) \leq r_1 d(x_n, x_{n+1}), \quad \forall n \in \mathbb{N},
\]
\[
d(x_n, x_{n+1}) \leq \frac{1 - Kr_1}{1 - Ks} d(x_n, Tx_n), \quad \forall n \in \mathbb{N}. \quad (2.3)
\]
Since $K_{r_1} < 1$, it implies $\{x_n\}$ is a Cauchy sequence by using Lemma 2.4. Since $X$ is complete, there is $z \in X$ such that $x_n$ converges to $z$, that is

$$\lim_{n \to \infty} d(x_n, z) = 0.$$ 

Since

$$d(x_{n+p}, x_n) \leq Kd(x_n, x_{n+1}) + K^2d(x_{n+1}, x_{n+2}) + \cdots + K^p d(x_{n+p-1}, x_{n+p}),$$

$$d(x_{n+p}, x_n) \leq Kd(x_n, x_{n+1})(1 + Kr_1 + K^2r_1^2 + \cdots + K^{p-1}r_1^{p-1}) = \frac{K[1 - (Kr_1)^p]}{1 - Kr_1}d(x_n, x_{n+1}), \quad \forall n \geq N, p \geq 1.$$ 

Set $p \to \infty$,

$$\frac{1}{K}d(z, x_n) \leq \lim_{p \to \infty} d(x_{n+p}, x_n) \leq \frac{K}{1 - Kr_1}d(x_n, x_{n+1}).$$

Thus

$$d(z, x_n) \leq \frac{K^2}{1 - Kr_1}d(x_n, x_{n+1}), \quad \forall n \geq 1.$$ 

From (2.3),

$$d(z, x_n) \leq \frac{K^2}{1 - Ks}d(x_n, Tx_n), \quad \forall n \in N. \quad (2.4)$$

Now suppose that there exists $N > 0$ such that

$$d(z, x_n) \leq \frac{1}{K(1 + Kr)}d(x_n, Tx_n), \quad \forall n \geq N.$$ 

Therefore

$$d(x_n, x_{n+1}) \leq K(d(x_n, z) + d(z, x_{n+1})) \leq \frac{1}{1 + Kr}[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})]$$

$$\leq \frac{1}{1 + Kr}[d(x_n, Tx_n) + rd(x_n, x_{n+1})].$$

This implies

$$d(x_n, x_{n+1}) < d(x_n, Tx_n),$$

which is impossible. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(z, x_{n_k}) > \frac{1}{K(1 + Kr)}d(x_{n_k}, Tx_{n_k}), \quad \forall k \geq N. \quad (2.5)$$

From (2.4) and (2.5) and using the hypothesis,

$$d(x_{n_k+1}, Tz) \leq H(Tx_{n_k}, Tz) \leq rM_T(x_{n_k}, z) + L \min\{d(x_{n_k}, z), \ d(x_{n_k}, Tz)\}$$

$$= r \max\left\{d(x_{n_k}, z), \ d(x_{n_k}, Tx_{n_k}), \ d(z, Tz), \ \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2K}\right\}$$

$$+ L \min\{d(x_{n_k}, z), \ d(x_{n_k}, Tz)\}.$$ 

Therefore

$$\frac{1}{K}d(z, Tz) \leq \lim_{k \to \infty} d(x_{n_k+1}, Tz) \leq r \max\left\{d(z, Tz), \ \frac{d(z, Tz)}{2K}\right\} = rd(z, Tz).$$

As $Kr < 1$, we get $d(z, Tz) = 0$. Since $Tz \in CB(X), z \in Tz, T$ has the fixed point. \qed
Corollary 2.9. Let \((X, d)\) be a complete \(b\)-metric space and \(T : X \to X\) be a weakly \((s, r)\)-contractive single-valued operator. Assume there exists \(r \in [0, 1)\) and \(r < \frac{1}{K}\) such that \(\forall x, y \in X\)

\[
\frac{1}{K(1 + Kr)} d(x, Tx) \leq d(x, y) \leq \frac{K^2}{1 - Kr} d(x, Tx)
\]

\(\Rightarrow H(Tx, Ty) \leq rM_T(x, y) + L \min\{d(x, y), d(y, Tx)\}, \ \forall x, y \in X,\)

where

\[
M_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2K}\right\}.
\]

Then there exists \(z \in X\) such that \(Tz = z\).

Proof. For every \(x_1 \in X\) the sequence \(\{x_n\}\) is defined by \(x_{n+1} = Tx_n\). One can easily prove that \(d(x_{n+1}, x_{n+2}) \leq rd(x_n, x_{n+1})\) and \(\{x_n\}\) is a Cauchy sequence. Then there is a point \(z \in X\) such that \(\lim_{n \to \infty} x_n = z\). From above theorem we have \(d(x_n, z) \leq \frac{K^2}{1 - Kr} d(x_n, x_{n+1})\) for all \(n \geq 1\) and there exists a subsequence \(\{x_{n_k}\}\) such that

\[
d(z, x_{n_k}) \geq \frac{1}{K(1 - Kr)} d(x_{n_k}, x_{n_k+1}), \ \forall k \geq N.
\]

Therefore

\[
d(x_{n_k+1}, Tz) \leq H(Tx_{n_k}, Tz) \leq r \max\left\{d(x_{n_k}, z), d(x_{n_k}, Tx_{n_k}), d(z, Tz), \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2K}\right\}
\]

\[+ L \min\{d(x_{n_k}, z), d(x_{n_k}, Tz)\}.
\]

Letting \(k \to \infty\), using the triangle inequality,

\[
\frac{1}{K} d(z, Tz) \leq \lim_{k \to \infty} d(x_{n_k+1}, Tz) \leq r \max\left\{d(z, Tz), \frac{d(z, Tz)}{2K}\right\}.
\]

Then we get \(d(z, Tz) = 0\) as \(Kr < 1\). Since \(Tz \in CB(X)\), \(z \in Tz\), \(T\) has a fixed point. \(\square\)

3. Application

For fixed point theorems, there are a number of applications in differential equations and integral equations.

Let \(X\) be a set of the continuous functions on the closed interval \([a, b]\) and we define the \(b\)-metric by

\[
d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|^2, \ \forall x, y \in X.
\]

Then \((X, d)\) is a complete \(b\)-metric space with the constant \(K = 2\).

Consider the differential equation

\[
\begin{aligned}
\frac{dx}{dy} &= f(x, y), \\
y(x_0) &= y_0.
\end{aligned}
\]

The equation (3.1) is equivalent to the following integral equation,

\[
y(x) = y_0 + \int_{x_0}^{x} f(x, y(t)) \, dt.
\]

We choose a constant \(0 < \delta < 1\), and define a map \(T\) on the continuous functional space \(C[x_0 - \delta, x_0 + \delta]\) by

\[
Ty(x) = y_0 + \int_{x_0}^{x} f(x, y(t)) \, dt.
\]
Then the integral equation (3.2) has a solution which is equivalent to that the map $T$ has a fixed point. Now we suppose that 

1. there exist constants $r \in [0, 1]$, $s > 0$ and $r < \min\{1, s\}$, such that for all $y_1, y_2 \in X$,

$$|y_2 - [y_0 + \int_{x_0}^{x} f(x, y(t)) \, dt]|^2 \leq 2s|y_1 - y_2|^2 \Rightarrow |f(z, y_1) - f(z, y_2)|^2 \leq r|y_1 - y_2|^2.$$ 

We have

$$d(Ty_1, Ty_2) = \max_{|x-x_0|<\delta} \int_{x_0}^{x} \left| f(t, y_1(t)) - f(t, y_2(t)) \right|^2 \, dt$$

$$\leq \max_{|x-x_0|<\delta} \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_2(t))|^2 \, dt$$

$$\leq r\delta \max_{|t-t_0|<\delta} |y_1(t) - y_2(t)|^2$$

$$= r\delta d(y_1(t), y_2(t))$$

$$\leq rM_{T}(y_1(t), y_2(t)) + \min d(y_1(t), y_2(t), d(y_2(t), Ty_1(t)), \frac{d(y_1, Ty_2) + d(y_2, Ty_1)}{4}).$$

Then $T$ satisfies the conditions of Theorem 2.5 and $T$ has a fixed point. So there exists a continuous function $y_0(t)$ such that

$$y_0(t) = \int_{x_0}^{x} f(x, y_0(t)) \, dt, \forall x \in [x_0 - \delta, x_0 + \delta].$$

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