Topological coincidence principles

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Abstract

In this paper a number of general coincidence principles are presented for set valued maps defined on subsets of completely regular topological spaces.

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1. Introduction

In this paper we present coincidence principles for multimaps. We present two approaches. The first approach is based on the new notion of Φ-essential and d-Φ-essential maps (see [1, 3–5]) and the second approach is based on the notion of extendability (see [2]). The arguments presented are based on a Urysohn type lemma and homotopy type arguments.

2. Continuation principles

Let $E$ be a completely regular topological space and $U$ an open subset of $E$.

We consider classes $A$ and $B$ of maps.

\textbf{Definition 2.1.} We say $F \in A(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \rightarrow 2^E$ and $F \in A(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$); here $2^E$ denotes the family of nonempty subsets of $E$.

In this section we fix a $\Phi \in B(\overline{U}, E)$.

\textbf{Definition 2.2.} We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here $\partial U$ denotes the boundary of $U$ in $E$.

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Definition 2.3. Let $E$ be a completely regular (respectively normal) topological space, and $U$ an open subset of $E$. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : U \to [0, 1]$ with $\eta(\partial U) = 0$. If $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, then $H_t(x) \cap \Phi(x) = \emptyset$ for some $t \in [0, 1]$ is compact (respectively closed); here $H_t(x) = H(x, t)$.

Definition 2.4. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J_{\partial U} = F_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$.

Theorem 2.5. Let $E$ be a completely regular (respectively normal) topological space, $U$ an open subset of $E$, and let $F \in A_{\partial U}(\overline{U}, E)$ be $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$. Suppose there exists a map $H : \overline{U} \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : U \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_0 = F$, and $\{x \in U : \Phi(x) \cap H(x, t) \neq \emptyset \}$ is compact (respectively closed). Then there exists $x \in U$ with $\Phi(x) \cap H_t(x) \neq \emptyset$; here $H_t(x) = H(x, t)$.

Proof. Let

$$\text{D} = \{x \in U : \Phi(x) \cap H(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}.$$  \hfill (2.1)

Notice $\text{D} \neq \emptyset$ since $F$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ (note from (2.1) that $F \cong F$ in $A_{\partial U}(\overline{U}, E)$). Also $\text{D}$ is compact (respectively closed) if $E$ is a completely regular (respectively normal) topological space. Note $\text{D} \cap \partial U = \emptyset$ (note $H_0 = F$ so for $t = 0$ we have $\Phi(x) \cap H_0(x) = \emptyset$ for $x \in \partial U$ since $F \in A_{\partial U}(\overline{U}, E)$). Thus there exists a continuous map $\mu : U \to [0, 1]$ with $\mu(\partial U) = 0$ and as a result $H_t(x) \cap \Phi(x) \neq \emptyset$.

It remains to show (2.2). Let $Q : U \times [0, 1] \to 2^E$ be given by $Q(x, t) = H(x, t\mu(x))$. Note $Q(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : U \to [0, 1]$ with $\eta(\partial U) = 0$ and (see (2.1) and Definition 2.3)

$$\{x \in U : \emptyset \neq \Phi(x) \cap Q(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

is compact (respectively closed). Note $Q_0 = F$ and $Q_1 = J$. Finally if there exists a $t \in [0, 1]$ and $x \in \partial U$ with $\Phi(x) \cap Q_t(x) \neq \emptyset$ then $\Phi(x) \cap H_{t\mu(x)}(x) \neq \emptyset$ so $x \in \text{D}$, and so $\mu(x) = 1$, i.e., $\Phi(x) \cap H_t(x) \neq \emptyset$, a contradiction. Thus (2.2) holds.

Remark 2.6. Suppose we change Definition 2.4 as follows. Let $F \in A_{\partial U}(\overline{U}, E)$. We say $F : \overline{U} \to 2^E$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$ if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J_{\partial U} = F_{\partial U}$ there exists $x \in U$ with $J(x) \cap \Phi(x) \neq \emptyset$. The argument above (note (2.2) is not needed) yields the following result. Let $E$ be a completely regular (respectively normal) topological space, $U$ an open subset of $E$ and let $F \in A_{\partial U}(\overline{U}, E)$ be $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$. Suppose there exists a map $H : U \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : U \to [0, 1]$ with $\eta(\partial U) = 0$, $\Phi(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_0 = F$ and $\{x \in U : \Phi(x) \cap H(x, t) \neq \emptyset \}$ is compact (respectively closed). Then there exists $x \in U$ with $\Phi(x) \cap H_t(x) \neq \emptyset$; here $H_t(x) = H(x, t)$.

Again we consider the map $F : \overline{U} \to 2^E$. In our (quite abstract) result we will assume that we have a homotopy extension type property (i.e., a $h : E \times [0, 1] \to 2^E$ with $H_I|_{\overline{U}} = F$).
Definition 2.7. We say $F \in A(E, E)$ if $F : E \to 2^E$ and $F \in A(E, E)$.

Definition 2.8. Let $E$ be a completely regular (respectively normal) topological space. If $F, G \in A(E, E)$, then we say $F \bowtie G$ in $A(E, E)$ if there exists a map $\Lambda : E \times [0, 1] \to 2^E$ with $\Lambda(\cdot, \eta(\cdot)) \in A(E, E)$ for any continuous function $\eta : E \to [0, 1]$, $\Lambda_1 = G$, $\Lambda_0 = F$ (here $\Lambda_t(x) = \Lambda(x, t)$) and \{ $x \in E : F(x) \cap \Lambda(x, t) \neq \emptyset$ for some $t \in [0, 1]$ \} is compact (respectively closed).

We now fix a $F \in B(E, E)$.

Theorem 2.9. Let $E$ be a completely regular (respectively normal) topological space and $U$ an open subset of $E$. Suppose there exists a map $H : E \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(E, E)$ for any continuous function $\eta : E \to [0, 1]$ and with $F(x) \cap H(x, 0) = \emptyset$ for $x \in E \setminus U$, and $F(x) \cap H_t(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, and \{ $x \in E : F(x) \cap H(x, t) \neq \emptyset$ for some $t \in [0, 1]$ \} is compact (respectively closed). In addition assume the following hold:

\[ \text{for any } J \in A(E, E) \text{ with } J \bowtie H_0 \text{ in } A(E, E) \text{ there exists } x \in E \text{ with } F(x) \cap J(x) \neq \emptyset, \quad (2.3) \]

\[ \{ x \in E \setminus U : F(x) \cap H_t(x) \neq \emptyset \text{ for some } t \in [0, 1] \} \text{ is closed}, \quad (2.4) \]

and

\[ \left\{ \begin{array}{l}
\text{if } \mu : E \to [0, 1] \text{ is any continuous map with } \mu(U) = 1, \text{ then } \\
\{ x \in E : F(x) \cap H(x, t\mu(x)) \neq \emptyset \text{ for some } t \in [0, 1] \} \text{ is closed.}
\end{array} \right. \quad (2.5) \]

Then there exists $x \in U$ with $F(x) \cap H_t(x) \neq \emptyset$; here $H_t(x) = H(x, t)$.

Proof. Let

\[ D = \{ x \in E \setminus U : F(x) \cap H_t(x) \neq \emptyset \text{ for some } t \in [0, 1] \}. \]

We consider two cases, as $D \neq \emptyset$ and $D = \emptyset$.

Case (i). $D = \emptyset$.

Then for every $t \in [0, 1]$ we have $F(x) \cap H_t(x) = \emptyset$ for $x \in E \setminus U$. Also from $H_1 \bowtie H_0$ in $A(E, E)$ and (2.3) we know there exists $y \in E$ with $F(y) \cap H_1(y) \neq \emptyset$. Since $F(x) \cap H_1(x) = \emptyset$ for $x \in E \setminus U$ we deduce that $y \in U$, and we are finished.

Case (ii). $D \neq \emptyset$.

Now (note $H_1 \bowtie H_0$ in $A(E, E)$ and (2.4)) $D$ is compact (respectively closed) and $D \cap U \neq \emptyset$ (since $F(x) \cap H_t(x) = \emptyset$ for $x \in \partial U$ and $t \in [0, 1]$). Then there exists a continuous map $\mu : E \to [0, 1]$ with $\mu(D) = 0$ and $\mu(U) = 1$. Define a map $R : E \to 2^E$ by

\[ R(x) = H(x, \mu(x)). \]

Now $R \in A(E, E)$. In fact $R \bowtie H_0$ in $A(E, E)$. To see this let $\Omega : E \times [0, 1] \to 2^E$ be given by

\[ \Omega(x, t) = H(x, t\mu(x)). \]

Note $\Omega(\cdot, \eta(\cdot)) \in A(E, E)$ for any continuous function $\eta : E \to [0, 1]$, and (note (2.5) and $H_1 \bowtie H_0$ in $A(E, E)$),

\[ \{ x \in E : F(x) \cap \Omega(x, t) = F(x) \cap H(x, t\mu(x)) \neq \emptyset \text{ for some } t \in [0, 1] \} \]

is compact (respectively closed). Also $\Omega_1 = R$ and $\Omega_0 = H_0$.

Now (2.3) guarantees that there exists $x \in E$ with $F(x) \cap R(x) = F(x) \cap H_{\mu(x)}(x) \neq \emptyset$. If $x \in E \setminus U$ then since $x \in D$ we have $\emptyset \neq F(x) \cap H(x, \mu(x)) = F(x) \cap H(x, 0)$, a contradiction. Thus $x \in U$ and so $\emptyset \neq F(x) \cap H(x, \mu(x)) = F(x) \cap H(x, 1)$.

Remark 2.10. In Definition 2.8 and in the statement of Theorem 2.9 we could replace, any continuous map $\eta : E \to [0, 1]$, with any continuous map $\eta : E \to [0, 1]$ with $\eta(U) = 1$. 


We now show that the ideas in this section can be applied to other natural situations. Let $E$ be a Hausdorff topological vector space (so automatically a completely regular space), $Y$ a topological vector space, and $U$ an open subset of $E$. Also let $L : \text{dom}(L) \subseteq E \to Y$ be a linear single valued map; here $\text{dom}(L)$ is a vector subspace of $E$. Finally $T : E \to Y$ will be a linear single valued map with $L + T : \text{dom}(L) \to Y$ a bijection; for convenience we say $T \in H_t(E,Y)$.

**Definition 2.11.** We say $F \in A(\overline{U}, Y, L, T)$ (respectively $F \in B(\overline{U}, Y, L, T)$) if $F : \overline{U} \to 2^Y$ and $(L + T)^{-1}(F + T) \in A(\overline{U}, E)$ (respectively $(L + T)^{-1}(F + T) \in B(\overline{U}, E)$).

We now fix a $\Phi \in B(\overline{U}, Y, L, T)$.

**Definition 2.12.** We say $F \in A_{\partial U}(\overline{U}, Y, L, T)$ if $F \in A(\overline{U}, Y, L, T)$ with $(L + T)^{-1}(F + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for $x \in \partial U$.

**Definition 2.13.** Let $F, G \in A_{\partial U}(\overline{U}, Y, L, T)$. We say $F \equiv G$ in $A_{\partial U}(\overline{U}, Y, L, T)$ if there exists a map $H : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_t = F, H_0 = G$ and

$$\{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

is compact; hence $H_t(x) = H(x, t)$.

**Definition 2.14.** Let $F \in A_{\partial U}(\overline{U}, Y, L, T)$.

We say $F$ is $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y, L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y, L, T)$ with $J|_{\partial U} = F|_{\partial U}$ and $J \equiv F$ in $A_{\partial U}(\overline{U}, Y, L, T)$ there exists $x \in U$ with $(L + T)^{-1}(J + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$.

**Theorem 2.15.** Let $E$ be a topological vector space (so automatically completely regular), $Y$ a topological vector space, $U$ an open subset of $E$, $L : \text{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in H_t(E,Y)$. Let $F \in A_{\partial U}(\overline{U}, Y, L, T)$ be $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y, L, T)$. Suppose there exists a map $H : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0, 1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1], H_0 = F$ (here $H_t(x) = H(x, t)$) and

$$\{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\}$$

is compact. In addition assume

$$\left\{ \begin{array}{l}
\mu \in A(\overline{U} \to [0, 1]) \text{ is any continuous map with } \mu(\partial U) = 0,

\{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t \mu(x) + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\}
\end{array} \right. \text{ is closed.}
$$

Then there exists $x \in U$ with $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$.

**Proof.** Let

$$D = \{x \in \overline{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0, 1]\}.$$

Note $D \neq \emptyset$ (note $F$ is $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y, L, T)$ and $D$ is compact, $D \cap \partial U = \emptyset$ so there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $J : \overline{U} \to 2^Y$ by $J(x) = H(x, \mu(x))$. Note $J \in A_{\partial U}(\overline{U}, Y, L, T)$ and $J|_{\partial U} = F|_{\partial U}$. Also note $J \equiv F$ in $A_{\partial U}(\overline{U}, Y, L, T)$ (to see this let $Q : \overline{U} \times [0, 1] \to 2^Y$ be given by $Q(x, t) = H(x, t \mu(x))$). Now since $F$ is $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y, L, T)$ there exists $x \in U$ with $(L + T)^{-1}(J + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$ (i.e., $(L + T)^{-1}(H_t \mu(x) + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$), and thus $x \in D$ so $\mu(x) = 1$ and we are finished.

**Remark 2.16.** Suppose we change Definition 2.14 as follows. Let $F \in A_{\partial U}(\overline{U}, Y, L, T)$. We say $F$ is $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y, L, T)$ if for every map $J \in A_{\partial U}(\overline{U}, Y, L, T)$ with $J|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $(L + T)^{-1}(J + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$. The argument above yields the following result. Let $E$ be a topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $L : \text{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in H_t(E,Y)$. Let $F \in A_{\partial U}(\overline{U}, Y, L, T)$ be $L$-$\Phi$-essential in $A_{\partial U}(\overline{U}, Y, L, T)$. Suppose there exists a map $H : \overline{U} \times [0, 1] \to 2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot)) + T(\cdot)) \in A(\overline{U}, E)$ for any continuous
function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in (0,1)$, $H_0 = F$ (here $H_t(x) = H(x, t)$) and \{x $\in \overline{U}$: $(L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset$ for some $t \in [0,1]$}\} is compact. Then there exists $x \in U$ with $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$.

**Remark 2.17.** If $E$ is a normal topological vector space then the assumption that $D$ (in the proof of Theorem 2.15) is compact, can be replaced by $D$ is closed, in the statement (and proof) of Theorem 2.15 and also the assumption that

$$\{x \in \overline{U}: (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact, can be replaced by

$$\{x \in \overline{U}: (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is closed, in Definition 2.13.

**Definition 2.18.** Let $F : E \to 2^Y$. We say $F \in A(E, Y; L, T)$ if $(L + T)^{-1}(F + T) \in A(E, E)$.

**Definition 2.19.** If $F, G \in A(E, Y; L, T)$ then we say $F \equiv G$ in $A(E, Y; L, T)$ if there exists a map $\Lambda : E \times [0,1] \to 2^Y$ with $(L + T)^{-1}(\Lambda(\cdot, \eta(\cdot)) + T) \in A(E, E)$ for any continuous function $\eta : E \to [0,1]$, $\Lambda_1 = F$, $\Lambda_0 = G$ (here $\Lambda_t(x) = \Lambda(x, t)$) and

$$\{x \in E : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(\Lambda_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact.

We now fix a $\Phi \in B(E, Y; L, T)$.

**Theorem 2.20.** Let $E$ be a completely regular topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $L : \text{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in \mathcal{H}_L(E, Y)$. Suppose there exists a map $H : E \times [0,1] \to 2^Y$ with $(L + T)^{-1}(H_t + T) \in A(E, E)$ for any continuous function $\eta : E \to [0,1]$, $(L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_0 + T)(x) = \emptyset$ for $x \in E \cup \mathcal{U}$, $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, and

$$\{x \in E : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact. In addition assume the following conditions holds:

$$\begin{cases} \text{for any } J \in A(E, Y; L, T) \text{ with } J \supseteq H_0 \text{ in } A(E, Y; L, T) \text{ there exists } x \in E \text{ with} \\ (L + T)^{-1}(J + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset, \end{cases} \quad (2.6)$$

and

$$\begin{cases} \{x \in E \cup \mathcal{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed,} \end{cases} \quad (2.7)$$

and

$$\begin{cases} \text{if } \mu : E \to [0,1] \text{ is any continuous map with } \mu(\overline{U}) = 1, \text{ then} \\ \{x \in E : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_{t\mu(x)} + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is closed.} \end{cases}$$

Then there exists $x \in \mathcal{U}$ with $(L + T)^{-1}(H_1 + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) \neq \emptyset$; here $H_1(x) = H(x, t)$.

**Proof.** Let

$$D = \{x \in E \cup \mathcal{U} : (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \neq \emptyset \text{ for some } t \in [0,1]\}.$$  

We consider two cases, as $D \neq \emptyset$ and $D = \emptyset$. 


Case (i). $D = \emptyset$.

Then for every $t \in [0, 1]$ we have $(L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) = \emptyset$ for $x \in E \setminus U$. Also from $H_t \equiv H_0$ in $A(E; Y; L, T)$ and (2.6) we see that there exists $y \in E$ with $(L + T)^{-1}(\Phi + T)(y) \cap (L + T)^{-1}(H_t + T)(y) \neq \emptyset$. Since $D = \emptyset$ we see that $y \in U$, and we are finished.

Case (ii). $D \neq \emptyset$.

Now (note $H_t \equiv H_0$ in $A(E; Y; L, T)$ and (2.7)) $D$ is compact, and $D \cap U \neq \emptyset$ (since $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for $x \in \partial U$ and $t \in [0, 1]$). Then there exists a continuous map $\mu : E \to [0, 1]$ with $\mu(D) = 0$ and $\mu(U) = 1$. Define a map $R : E \to 2^Y$ by $R(x) = H(x, \mu(x))$. Note $R \in A(E; Y; L, T)$. Also note $R \equiv H_0$ in $A(E; Y; L, T)$ (to see this let $\Omega : E \times [0, 1] \to 2^E$ be given by $\Omega(x, t) = H(x, t\mu(x))$). Also $\Omega_1 = R$ and $\Omega_0 = H_0$.

Now (2.6) guarantees that there exists $x \in E$ with $\emptyset \neq (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(H_{\mu(x)} + T)(x)$. If $x \in E \setminus U$ then since $x \in D$ we have $\emptyset \neq (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_{\mu(x)} + T)(x) \cap (L + T)^{-1}(H_t + T)(x)$, a contradiction. Thus $x \in U$ and so $\emptyset \neq (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_{\mu(x)} + T)(x) \cap (L + T)^{-1}(H_t + T)(x)$. \[\square\]

Remark 2.21. In Definition 2.19 and in the statement of Theorem 2.20 we could replace any continuous map $\eta : E \to [0, 1]$ with any continuous map $\eta : X \to [0, 1]$ with $\eta(U) = 1$.

Remark 2.22. There is an analogue of Remark 2.17 (for normal topological vector spaces) in the statement of Theorem 2.20 and in Definition 2.19.

3. Generalized continuation principles

Let $E$ be a completely regular topological space and $U$ an open subset of $E$. Again we consider classes $A$ and $B$ of maps.

In this section we fix a $\Phi \in B(U, E)$.

For any map $F \in A(U, E$) let $F^* = I \times F : U \to 2^{U \times E}$, with $I : U \to U$ given by $I(x) = x$, and let

$$d : \left\{ (F^*)^{-1}(B) \right\} \cup \{ \emptyset \} \to \Omega$$

be any map with values in the nonempty set $\Omega$; here $B = \left\{ (x, \Phi(x)) : x \in U \right\}$.

Definition 3.1. Let $E$ be a completely regular (respectively normal) topological space, and $U$ an open subset of $E$. Let $F, G \in A_{\partial U}(U, E)$. We say $F \equiv G$ in $A_{\partial U}(U, E)$ if there exists a map $H : U \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(U, E)$ for any continuous function $\eta : \partial U \to [0, 1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_1 = F$, $H_0 = G$ and $\{ x \in U : (x, \Phi(x)) \cap H^+(x, t) \neq \emptyset \}$ for some $t \in [0, 1]$ is compact (respectively closed); here $H^+(x, t) = (x, H(x, t))$ and $H_t(x) = H(x, t)$.

Definition 3.2. Let $F \in A_{\partial U}(U, E)$ with $F^* = I \times F$. We say $F^* : U \to 2^{U \times E}$ is $d$-$\Phi$-essential if for every map $J \in A_{\partial U}(U, E)$ with $J^* = I \times J$ and $J|\partial U = F|\partial U$ and $J \equiv F$ in $A_{\partial U}(U, E)$ we have that $d \left( (J^*)^{-1}(B) \right) = d \left( (F^*)^{-1}(B) \right) \neq d(\emptyset)$.

Remark 3.3. If $F^*$ is $d$-$\Phi$-essential then

$$\emptyset \neq (F^*)^{-1}(B) = \{ x \in U : F^*(x) \cap B \neq \emptyset \} = \{ x \in U : (x, \Phi(x)) \cap (x, \Phi(x)) \neq \emptyset \},$$

and this together with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ implies that there exists $x \in U$ with $(x, \Phi(x)) \cap F^*(x) \neq \emptyset$ (i.e., $\Phi(x) \cap F(x) \neq \emptyset$).

Theorem 3.4. Let $E$ be a completely regular (respectively normal) topological space, $U$ an open subset of $E$, $B = \{(x, \Phi(x)) : x \in U\}$, $d$ a map defined in (3.1) and let $F \in A_{\partial U}(U, E)$ and $F^*$ be $d$-$\Phi$-essential (here $F^* = I \times F$). Suppose there exists a map $H : U \times [0, 1] \to 2^E$ with $H(\cdot, \eta(\cdot)) \in A(U, E)$ for any continuous function $\eta : \partial U \to [0, 1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0, 1]$, $H_0 = F$ and
\{x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0,1]\} \text{ is compact (respectively closed); here } H^*(x, t) = (x, H(x, t)) \text{ and } H_t(x) = H(x, t). \text{ In addition assume }
\begin{array}{l}
\{ \text{if } \mu : \overline{U} \to [0,1] \text{ is any continuous map with } \mu(\partial U) = 0, \text{ then} \\
\{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, t\mu(x))) \neq \emptyset \text{ for some } t \in [0,1] \} \text{ is closed.}
\end{array}

Let \( H_t^* = I \times H_t \). Then
\[
\text{d} \left( (H_t^*)^{-1} (B) \right) = \text{d} \left( (F^*)^{-1} (B) \right) \neq \text{d}(\emptyset).
\]

**Proof.** Let
\[
D = \left\{ x \in \overline{U} : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0,1] \right\},
\]
where \( H^*(x, t) = (x, H(x, t)) \). Notice \( D \neq \emptyset \) since \( F^* \) is d-\( \Phi \)-essential. Also \( D \) is compact (respectively closed) if \( E \) is a completely regular (respectively normal) topological space and \( D \cap \partial U = \emptyset \). Thus there exists a continuous map \( \mu : \overline{U} \to [0,1] \) with \( \mu(\partial U) = 0 \) and \( \mu(D) = 1 \). Define \( R_\mu : \overline{U} \to 2^\overline{U} \) by \( R_\mu(x) = H(x, \mu(x)) \) and let \( R^*_\mu = I \times R_\mu \). Note \( R_\mu \in A_{\partial U}(\overline{U}, E) \) with \( R_\mu|\partial U = F|\partial U \) (note if \( x \in \partial U \) then \( R_\mu(x) = H_0(x) = F(x) \) and \( R_\mu(x) \cap \Phi(x) = F(x) \cap \Phi(x) = \emptyset \).

Next we note since \( \mu(D) = 1 \) that
\[
(R^*_\mu)^{-1} (B) = \left\{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset \right\} = \left\{ x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset \right\} = (H_t^*)^{-1} (B),
\]
so
\[
\text{d} \left( (R^*_\mu)^{-1} (B) \right) = \text{d} \left( (H_t^*)^{-1} (B) \right). 
\]

Also note \( R_\mu \cong F \in A_{\partial U}(\overline{U}, E) \) (to see this let \( Q : \overline{U} \times [0,1] \to 2^E \) be given by \( Q(x, t) = H(x, t\mu(x)) \)). As a result since \( F^* \) is d-\( \Phi \)-essential we have \( \text{d} \left( (R^*_\mu)^{-1} (B) \right) = \text{d} \left( (F^*)^{-1} (B) \right) \neq \text{d}(\emptyset) \). This together with (3.2) yields \( \text{d} \left( (H_t^*)^{-1} (B) \right) = \text{d} \left( (F^*)^{-1} (B) \right) \neq \text{d}(\emptyset) \).

**Remark 3.5.** Suppose we change Definition 3.2 as follows. Let \( F \in A_{\partial U}(\overline{U}, E) \) with \( F^* = I \times F \). We say \( F^* : \overline{U} \to 2^{\overline{U} \times E} \) is d-\( \Phi \)-essential if for every map \( J \in A_{\partial U}(\overline{U}, E) \) with \( J^* = I \times J \) and \( J|_{\partial U} = F|_{\partial U} \) we have that \( \text{d} \left( (F^*)^{-1} (B) \right) = \text{d} \left( (J^*)^{-1} (B) \right) \neq \text{d}(\emptyset) \). The argument above yields the following result. Let \( E \) be a completely regular (respectively normal) topological space, \( U \) an open subset of \( E \), \( B = \{(x, \Phi(x)) : x \in \overline{U} \} \), a map defined in (3.1) and let \( F \in A_{\partial U}(\overline{U}, E) \) and \( F^* \) be d-\( \Phi \)-essential (here \( F^* = I \times F \)). Suppose there exists a map \( H : \overline{U} \times [0,1] \to 2^E \) with \( H(., \eta(\cdot)) \in A(\overline{U}, E) \) for any continuous function \( \eta : \overline{U} \to [0,1] \) with \( \eta(\partial U) = 0 \), \( H_t(x) \cap \Phi(x) = \emptyset \) for any \( x \in \partial U \) and \( t \in [0,1] \), \( H_0 = F \) and \( \{ x \in U : (x, \Phi(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0,1] \} \) is compact (respectively closed); here \( H^*(x, t) = (x, H(x, t)) \) and \( H_t(x) = H(x, t) \). Then \( \text{d} \left( (H_t^*)^{-1} (B) \right) = \text{d} \left( (F^*)^{-1} (B) \right) \neq \text{d}(\emptyset) \).

**Remark 3.6.** Suppose the following conditions holds (which is common in the literature on topological degree):
\[
\begin{cases}
\text{if } F, G \in A_{\partial U}(\overline{U}, E) \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \cong G \\
in A_{\partial U}(\overline{U}, E), \text{ then } \text{d} \left( (F^*)^{-1} (B) \right) = \text{d} \left( (G^*)^{-1} (B) \right).
\end{cases}
\]

Then Definition 3.2 reduces to the following. Let \( F \in A_{\partial U}(\overline{U}, E) \) with \( F^* = I \times F \). We say \( F^* : \overline{U} \to 2^{\overline{U} \times E} \) is d-\( \Phi \)-essential if \( \text{d} \left( (F^*)^{-1} (B) \right) \neq \text{d}(\emptyset) \).

Next in this paper we use the notion of extendability to establish new continuation theorems.

**Definition 3.7.** We say \( F \in A(E, E) \) if \( F : E \to 2^E \) and \( F \in A(E, E) \).
We now fix a \( \Phi \in B(E, E) \).

For any map \( F \in A(E, E) \) let \( F^* = I \times F : E \to 2^E \times E \), with \( I : E \to E \) given by \( I(x) = x \), and let

\[
d : \{(F^*)^{-1}(B)\} \cup \{\emptyset\} \to \Omega \tag{3.4}
\]

be any map with values in the nonempty set \( \Omega \); here \( B = \{(x, \Phi(x)) : x \in E\} \). In our applications we will be interested in maps \( F : \overline{U} \to 2^E \) so \( F^* = I \times F : \overline{U} \to 2^{\overline{U}\times E} \) and in this case we consider

\[
d : \{(F^*)^{-1}(B_U)\} \cup \{\emptyset\} \to \Omega,
\]

where \( B_U = \{(x, \Phi(x)) : x \in U\} \).

**Definition 3.8.** If \( F, G \in A(E, E) \), then we say \( F \equiv G \) in \( A(E,E) \) if there exists a map \( \Lambda : E \times [0,1] \to 2^E \) with \( \Lambda(,\eta(\cdot)) \in A(E,E) \) for any continuous function \( \eta : E \to [0,1] \), \( \Lambda_1 = F \), \( \Lambda_0 = G \) (here \( \Lambda_1(x) = \Lambda(x,t) \)) and \( \{x \in E : (x, \Phi(x)) \cap \Lambda^*(x,t) \neq \emptyset \) for some \( t \in [0,1] \} \) is compact (respectively closed); here \( \Lambda^*(x,t) = (x,\Lambda(x,t)) \).

**Theorem 3.9.** Let \( E \) be a completely regular (respectively normal) topological space, \( U \) an open subset of \( E \) and \( d \) a map defined in (3.4). Suppose there exists a map \( H : E \times [0,1] \to 2^E \) with \( H(,\eta(\cdot)) \in A(E,E) \) for any continuous function \( \eta : E \to [0,1] \) and with \( \Phi(x) \cap H(x,0) = \emptyset \) for \( x \in E \setminus U \) and \( \Phi(x) \cap H_1(x) = \emptyset \) for any \( x \in \partial U \) and \( t \in [0,1] \), and

\[
\{x \in E : (x, \Phi(x)) \cap (x, H(x,t)) \neq \emptyset \text{ for some } t \in [0,1]\}
\]

is compact (respectively closed). In addition assume the following hold:

\[
\begin{cases}
\text{for any } J \in A(E,E) \text{ with } J^* = I \times J \text{ and } J \equiv H_0 \text{ in } A(E,E), \text{ we have that}\\
d \left( (J^*)^{-1}(B) \right) = d \left( (H_0^*)^{-1}(B) \right) \neq d(\emptyset),\\
\end{cases}
\]

\[
[x \in E \setminus U : (x, \Phi(x)) \cap H^*(x,t) \neq \emptyset \text{ for some } t \in [0,1] \}
\]

is closed,

and

\[
\begin{cases}
\text{if } \mu : E \to [0,1] \text{ is any continuous map with } \mu(U) = 1, \text{ then}\\
\{x \in E : \emptyset \neq (x, \Phi(x)) \cap (x, H(x, t\mu(x))) \text{ for some } t \in [0,1]\}
\end{cases}
\]

is closed;

here \( H_0^* = I \times H_0 \) and \( H^*(x,t) = (x, H(x,t)) \). Let \( H_1^* = I \times H_1 \). Then we have

\[
d \left( (H_1^*)^{-1}(B_U) \right) = d \left( (H_0^*)^{-1}(B_U) \right) \neq d(\emptyset).
\]

**Proof.** Let

\[
D = \{x \in E \setminus U : (x, \Phi(x)) \cap H^*(x,t) \neq \emptyset \text{ for some } t \in [0,1]\},
\]

where \( H^*(x,t) = (x, H(x,t)) \).

We consider two cases, as \( D \neq \emptyset \) and \( D = \emptyset \).

**Case (i).** \( D = \emptyset \).

Then for every \( t \in [0,1] \) we have \( \Phi(x) \cap H_t(x) = \emptyset \) for \( x \in E \setminus U \). Also from \( H_1 \equiv H_0 \) in \( A(E,E) \) and (3.5) we have

\[
d \left( (H_1^*)^{-1}(B) \right) = d \left( (H_0^*)^{-1}(B) \right) \neq d(\emptyset). \tag{3.7}
\]

**Note** \( (H_1^*)^{-1}(B) = \{x \in E : (x, \Phi(x)) \cap (x, H_1(x)) \neq \emptyset\} \). Consider \( y \in E \) and \( (y, \Phi(y)) \cap H_1^*(y) \neq \emptyset \). Then \( y \in E \) and \( \Phi(y) \cap H_1(y) \neq \emptyset \). Now since \( D = \emptyset \) we have \( y \in U \) and \( \Phi(y) \cap H_1(y) \neq \emptyset \) i.e., \( y \in U \) and \( (y, \Phi(y)) \cap H_1^*(y) \neq \emptyset \). Consequently \( (H_1^*)^{-1}(B) \subseteq (H_1^*)^{-1}(B_U) \) and on the other hand it is immediate that \( (H_1^*)^{-1}(B_U) \subseteq (H_1^*)^{-1}(B) \). Thus \( (H_1^*)^{-1}(B) = (H_1^*)^{-1}(B_U) \). It is also immediate that \( (H_1^*)^{-1}(B) = (H_1^*)^{-1}(B_U) \). Thus (3.7) implies \( d \left( (H_1^*)^{-1}(B_U) = d \left( (H_0^*)^{-1}(B_U) \right) \right) \neq d(\emptyset) \), and we are finished.
Case (ii). $D \neq \emptyset$.

Note $D$ is compact (respectively closed) and also note $D \cap \overline{U} \neq \emptyset$ (since $\Phi(x) \cap H_t(x) = \emptyset$ for $x \in \partial U$ and $t \in [0,1]$). Then there exists a continuous map $\mu : E \to [0,1]$ with $\mu(D) = 0$ and $\mu(\overline{U}) = 1$. Define a map $R : E \to 2^E$ by $R(x) = H(x, \mu(x))$. Note $R \in A(E,E)$. In fact $R \cong H_0$ in $A(E,E)$. To see this let $\Lambda : E \times [0,1] \to 2^E$ be given by $\Lambda(x,t) = H(x, t\mu(x))$. Note $\Lambda(\cdot, \eta(\cdot)) \in A(E,E)$ for any continuous function $\eta : E \to [0,1]$, and (note (3.6) and $H_t \cong H_0$ in $A(E,E)$),

$$\{x \in E : (x, \Phi(x)) \cap (x, H(x, t\mu(x))) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact (respectively closed). Also $A_t = R$ and $A_0 = H_0$.

Let $R^* = I \times R$. Now (3.5) guarantees that

$$d\left((R^*)^{-1}(B)\right) = d\left((H_0^*)^{-1}(B)\right) \neq d(\emptyset).$$ \hfill (3.8)

Note $(R^*)^{-1}(B) = \{x \in E : (x, \Phi(x)) \cap (x, R(x)) \neq \emptyset\}$. Consider $x \in E$ and $(x, \Phi(x)) \cap R^*(x) \neq \emptyset$. Then $x \in E$ and $\emptyset \neq \Phi(x) \cap R(x) = \Phi(x) \cap H_{\mu(x)}(x)$. If $x \in E \cup \emptyset$ then since $x \in D$ we have $\emptyset \neq \Phi(x) \cap H_{\mu(x)}(x) = \Phi(x) \cap H(x, 0)$, which is a contradiction. Thus $x \in \overline{U}$ and $\emptyset \neq \Phi(x) \cap R(x)$. Consequently $(R^*)^{-1}(B) \subseteq (R^*)^{-1}(B_U)$ and on the other hand it is immediate that $(R^*)^{-1}(B_U) \subseteq (R^*)^{-1}(B)$. Thus $(R^*)^{-1}(B) = (R^*)^{-1}(B_U)$. Also $(H_0^*)^{-1}(B) = (H_0^*)^{-1}(B_U)$. Thus (3.8) implies

$$d\left((R^*)^{-1}(B_U)\right) = d\left((H_0^*)^{-1}(B_U)\right) \neq d(\emptyset).$$ \hfill (3.9)

Finally notice (note $\mu(\overline{U}) = 1$) that

$$(R^*)^{-1}(B_U) = \{x \in U : (x, \Phi(x)) \cap (x, H(x, \mu(x))) \neq \emptyset\}$$

$$= \{x \in U : (x, \Phi(x)) \cap (x, H(x, 1)) \neq \emptyset\} = (H_1^*)^{-1}(B_U),$$

so from (3.9) we have $d\left((H_1^*)^{-1}(B_U)\right) = d\left((H_0^*)^{-1}(B_U)\right) \neq d(\emptyset)$. \hfill $\Box$

**Remark 3.10.** In Definition 3.8 and in the statement of Theorem 3.9 we could replace, any continuous map $\eta : E \to [0,1]$, with, any continuous map $\eta : E \to [0,1]$ with $\eta(\overline{U}) = 1$.

Let $E$ be a topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $L : \text{dom}(L) \subseteq E \to Y$ a linear single valued map, and $T \in H_L(E,Y)$.

We now fix a $\Phi \in B(\overline{U}, Y; L, T)$.

For any map $F \in A(\overline{U}, Y; L, T)$ let $F^* = I \times (L + T)^{-1}(F + T) : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^*)^{-1}(B) \right\} \cup \{\emptyset\} \to \Omega \hfill (3.10)$$

be any map with values in the nonempty set $\Omega$; here $B = \{ (x, (L + T)^{-1}(\Phi + T)(x)) : x \in \overline{U} \}$.

**Definition 3.11.** Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L + T)^{-1}(H(\cdot, \eta(\cdot))) + T(\cdot) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L + T)^{-1}(H_t + T)(x) \cap (L + T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\{x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x)) \cap H_t(x, t) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact; here $H_t(x) = H(x, t)$ and $H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda))$.

**Definition 3.12.** Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^* = I \times (L + T)^{-1}(F + T)$. We say $F^* : \overline{U} \to 2^{\overline{U} \times E}$ is $d$-L-$\Phi$-essential if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^* = I \times (L + T)^{-1}(J + T)$ and $J_{\partial U} = F_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ we have that $d\left((F^*)^{-1}(B)\right) = d\left((J^*)^{-1}(B)\right) \neq d(\emptyset)$.
Theorem 3.13. Let $E$ be a Hausdorff topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $B = \{(x, (L+T)^{-1}(\Phi + T)(x)) : x \in \overline{U}\}$, $L : \text{dom}(L) \subseteq E \to Y$ a linear single valued map, $T \in H_L(E,Y)$, a map defined in (3.10), and let $F \in A_{\partial U}(\overline{U},Y;L,T)$ and $F^*$ be $d$-$L$-$\Phi$-essential (here $F^* = I \times (L+T)^{-1}(F + T)$). Suppose here exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\eta,\eta)) + T(\cdot)) \in A(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_1 + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_0 = F$, and

$$\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap H^*(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$$

is compact; here $H_1(x) = H(x,t)$ and $H^*(x,\lambda) = (x, (L+T)^{-1}(H + T)(x,\lambda))$. In addition assume

$$\left\{ \begin{array}{ll}
\text{if } \mu : \overline{U} \to [0,1] \text{ is any continuous map with } \mu(\partial U) = 0, \\
\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap (x, (L+T)^{-1}(H_{t\mu(x)}(x) + T)(x)) \neq \emptyset \text{ for some } t \in [0,1]\} \end{array} \right.$$ is closed.

Let $H^*_1 = I \times (L+T)^{-1}(H_1 + T)$. Then

$$d \left( (H^*_1)^{-1} (B) \right) = d \left( (F^*)^{-1} (B) \right) \neq d(\emptyset).$$

Proof. Let

$$D = \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap H^*(x,t) \neq \emptyset \text{ for some } t \in [0,1]\},$$

where $H^*(x,\lambda) = (x, (L+T)^{-1}(H + T)(x,\lambda))$. Notice $D \neq \emptyset$, $D$ is compact, and $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define $R_{\mu} : \overline{U} \to 2^Y$ by $R_{\mu}(x) = H(x,\mu(x)) = H_{\mu(x)}(x)$ and let $R^*_{\mu} = I \times (L+T)^{-1}(R_{\mu} + T)$. Note $R_{\mu} \in A_{\partial U}(\overline{U},Y;L,T)$ with $R_{\mu}|_{\partial U} = F|_{\partial U}$ (note if $x \in \partial U$ then $R_{\mu}(x) = H_0(x) = F(x)$ and $R_{\mu}(x) \cap \Phi(x) = F(x) \cap \Phi(x) = \emptyset$).

Next we note, since $\mu(D) = 1$, that

$$(R^*_{\mu})^{-1} (B) = \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap (x, (L+T)^{-1}(H_{\mu(x)}(x) + T)(x)) \neq \emptyset\} = \{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap (x, (L+T)^{-1}(H_1 + T)(x)) \neq \emptyset\} = (H^*_1)^{-1} (B)$$

and so

$$d \left( (R^*_{\mu})^{-1} (B) \right) = d \left( (H^*_1)^{-1} (B) \right). \quad (3.11)$$

Also note $R_{\mu} \equiv F$ in $A_{\partial U}(\overline{U},Y;L,T)$ (to see this let $Q : \overline{U} \times [0,1] \to 2^Y$ be given by $Q(x,t) = H(x,t)\mu(x))$.

As a result since $F^*$ is $d$-$L$-$\Phi$-essential we have $d \left( (R^*_{\mu})^{-1} (B) \right) = d \left( (F^*)^{-1} (B) \right) \neq d(\emptyset)$. This together with (3.11) yields $d \left( (H^*_1)^{-1} (B) \right) = d \left( (F^*)^{-1} (B) \right) \neq d(\emptyset)$.

Remark 3.14. Suppose we change Definition 3.12 as follows. Let $F \in A_{\partial U}(\overline{U},Y;L,T)$ with $F^* = I \times (L+T)^{-1}(F + T)$. We say $F^* : \overline{U} \to 2^{\overline{U} \times E}$ is $d$-$L$-$\Phi$-essential if for every map $J \in A_{\partial U}(\overline{U},Y;L,T)$ with $J^* = I \times (L+T)^{-1}(J + T)$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d \left( (F^*)^{-1} (B) \right) = d \left( (J^*)^{-1} (B) \right) \neq d(\emptyset)$. The argument above yields the following result. Let $E$ be a Hausdorff topological vector space, $Y$ a topological vector space, $U$ an open subset of $E$, $B = \{(x, (L+T)^{-1}(\Phi + T)(x)) : x \in \overline{U}\}$, $L : \text{dom}(L) \subseteq E \to Y$ a linear single valued map, $T \in H_L(E,Y)$, a map defined in (3.10) and let $F \in A_{\partial U}(\overline{U},Y;L,T)$ and $F^*$ be $d$-$L$-$\Phi$-essential (here $F^* = I \times (L+T)^{-1}(F + T)$). Suppose here exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\eta,\eta)) + T(\cdot)) \in A(\overline{U},E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_1 + T)(x) \cap (L+T)^{-1}(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_0 = F$, $H_0 = G$ and

$$\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi + T)(x)) \cap H^*(x,t) \neq \emptyset \text{ for some } t \in [0,1]\}$$
is compact; here $H_1(x) = H(x,t)$ and $H^*(x,\lambda) = (x, (L+T)^{-1}(H + T)(x,\lambda))$. Let $H^*_1 = I \times (L+T)^{-1}(H_1 + T)$. Then $d \left( (H^*_1)^{-1} (B) \right) = d \left( (F^*)^{-1} (B) \right) \neq d(\emptyset)$. \qed
Definition 3.17. Suppose the following condition holds:

\[
\begin{cases}
\text{if } F, G \in A_{\partial U}(\overline{U}, Y; L, T) \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \cong G \\
in A_{\partial U}(\overline{U}, Y; L, T) \text{ then } d\left(\left((F^+)^{-1}(B)\right)\right) = d\left(\left((G^+)^{-1}(B)\right)\right).
\end{cases}
\]

Then Definition 3.12 reduces to the following. Let \( F \in A_{\partial U}(\overline{U}, Y; L, T) \) with \( F^* = I \times (L + T)^{-1}(F + T) \). We say \( F^* : \overline{U} \to 2^{\overline{U} \times E} \) is d-L-\( \Phi \)-essential if \( d\left(\left((F^+)^{-1}(B)\right)\right) \neq d(\emptyset) \).

Remark 3.16. If \( E \) is a normal topological vector space then the assumption that \( D \) (in the proof of Theorem 2.15) is compact, can be replaced by \( D \) is closed, in the statement (and proof) of Theorem 3.13 and also the assumption that

\[
\{x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x))) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}
\]

is compact, can be replaced by

\[
\{x \in \overline{U} : (x, (L + T)^{-1}(\Phi + T)(x))) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}
\]

is closed, in Definition 3.11; here \( H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda)) \).

We now fix a \( \Phi \in B(E, Y; L, T) \).

Definition 3.17. Let \( F : E \to 2^Y \). We say \( F \in A(E, Y; L, T) \) if \((L + T)^{-1}(F + T) \in A(E, E) \).

For any map \( F \in A(E, Y; L, T) \) let \( F^* = I \times (L + T)^{-1}(F + T) : E \to 2^{E \times E} \), with \( I : E \to E \) given by \( I(x) = x \), and let

\[
d : \left((F^+)^{-1}(B)\right) \cup \{\emptyset\} \to \Omega
\]

be any map with values in the nonempty set \( \Omega \); here \( B = \{(x, (L + T)^{-1}(\Phi + T)(x)) : x \in E\} \). In our applications we will be interested in maps \( F : \overline{U} \to 2^Y \) so \( F^* = I \times (L + T)^{-1}(F + T) : \overline{U} \to 2^{\overline{U} \times E} \) and in this case we consider

\[
d : \left((F^+)^{-1}(B_U)\right) \cup \{\emptyset\} \to \Omega,
\]

where \( B_U = \{(x, (L + T)^{-1}(\Phi + T)(x)) : x \in U\} \).

Definition 3.18. If \( F, G \in A(E, Y; L, T) \) then we say \( F \cong G \) in \( A(E, Y; L, T) \) if there exists a map \( \Lambda : E \times [0, 1] \to 2^Y \) with \((L + T)^{-1}(\Lambda(\cdot, \eta(\cdot)) + T) \in A(E, E) \) for any continuous function \( \eta : E \to [0, 1], \Lambda_1 = F \) and \( \Lambda_0 = G \) (here \( \Lambda_1(x) = (x, t) \)) and

\[
\{x \in E : (x, (L + T)^{-1}(\Phi + T)(x))) \cap \Lambda^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}
\]

is compact; here \( \Lambda^*(x, \lambda) = \Lambda(x, (L + T)^{-1}(\Lambda(x, t), \lambda)) \).

Theorem 3.19. Let \( E \) be a Hausdorff topological vector space, \( Y \) a topological vector space, \( U \) an open subset of \( E \), \( L : dom(L) \subseteq E \to Y \) a linear single valued map, \( T \in H_1(E, Y) \) and \( d \) a map defined in (3.12). Suppose there exists a map \( H : E \times [0, 1] \to 2^Y \) with \((L + T)^{-1}(H(\cdot, \eta(\cdot)) + T) \in A(E, E) \) for any continuous function \( \eta : E \to [0, 1], (L + T)^{-1}(\Phi + T)(x)) \cap (L + T)^{-1}(H_0 + T)(x) = \emptyset \) for any \( x \in \partial U \) and \( t \in [0, 1] \), and

\[
\{x \in E : (x, (L + T)^{-1}(\Phi + T)(x))) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1]\}
\]

is compact; here \( H^*(x, \lambda) = (x, (L + T)^{-1}(H + T)(x, \lambda)) \). In addition assume the following hold:

\[
\begin{cases}
\text{for any } J \in A(E, Y; L, T) \text{ with } J^* = I \times (L + T)^{-1}(J + T) \\
\text{and } \Phi \cong H_0 \text{ in } A(E, Y; L, T) \text{ we have that } d\left(\left((J^*)^{-1}(B)\right)\right) = d\left(\left((H_0^*)^{-1}(B)\right)\right) \neq d(\emptyset),
\end{cases}
\]
\[ \begin{cases} x \in E \setminus U : (x, (L + T)^{-1}(\Phi + T)(x)) \cap H^*(x, t) \neq \emptyset \text{ for some } t \in [0, 1] \\ \text{is closed} \end{cases} \]

and

\[ \begin{cases} \text{if } \mu : E \to [0, 1] \text{ is any continuous map with } \mu(U) = 1, \text{ then} \\ \{ x \in U : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (x, (L + T)^{-1}(H_{\mu(x)} + T)(x)) \neq \emptyset \text{ for some } t \in [0, 1] \} \text{ is closed.} \end{cases} \]

Here \( H_0^* = 1 \times (L + T)^{-1}(H_0 + T) \). Let \( H_1^* = 1 \times (L + T)^{-1}(H_1 + T) \). Then we have

\[ d \left( (H_1^*)^{-1}(B_1) \right) = \mathcal{d} \left( (H_0^*)^{-1}(B_1) \right) \neq d(\emptyset). \tag{3.14} \]

Note \( (H_1^*)^{-1}(B) = \{ x \in E : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (x, (L + T)^{-1}(H_1 + T)(x)) \neq \emptyset \} \). Consider \( y \in E \) and \( (y, (L + T)^{-1}(\Phi + T)(y)) \cap H_1^*(y) \neq \emptyset \). Then \( y \in E \) and \( \Phi(y) \cap (L + T)^{-1}(H_1 + T)(y) \neq \emptyset \). Now since \( D = \emptyset \) we have \( y \in U \) and \( \Phi(y) \cap (L + T)^{-1}(H_1 + T)(y) \neq \emptyset \) i.e., \( y \in U \) and \( (y, (L + T)^{-1}(\Phi + T)(y)) \cap H_1^*(y) \neq \emptyset \).

Consequently \( (H_1^*)^{-1}(B) \subseteq (H_0^*)^{-1}(B_1) \) and on the other hand it is immediate that \( (H_1^*)^{-1}(B_1) \subseteq (H_1^*)^{-1}(B) \). Thus \( (H_1^*)^{-1}(B) = (H_1^*)^{-1}(B_1) \). It is also immediate that \( (H_0^*)^{-1}(B) = (H_0^*)^{-1}(B_1) \).

Thus (3.14) implies \( \mathcal{d} \left( (H_1^*)^{-1}(B_1) \right) = \mathcal{d} \left( (H_0^*)^{-1}(B_1) \right) \neq d(\emptyset), \) and we are finished.

Case (ii). \( D \neq \emptyset \).

Note \( D \) is compact and also note \( D \cap U \neq \emptyset \). Then there exists a continuous map \( \mu : E \to [0, 1] \) with \( \mu(D) = 0 \) and \( \mu(U) = 1 \). Define a map \( R : E \to 2^Y \) by

\[ R(x) = H(x, \mu(x)). \]

Note \( R \in \mathcal{A}(E, Y; L, T) \). In fact \( R \equiv H_0 \) in \( \mathcal{A}(E, Y; L, T) \) (to see this let \( \Lambda : E \times [0, 1] \to 2^Y \) be given by \( \Lambda(x, t) = H(x, t \mu(x)) \)).

Let \( R^* = 1 \times (L + T)^{-1}(R + T) \). Now (3.13) guarantees that

\[ \mathcal{d} \left( (R^*)^{-1}(B) \right) = \mathcal{d} \left( (H_0^*)^{-1}(B) \right) \neq d(\emptyset). \tag{3.15} \]

Note \( (R^*)^{-1}(B) = \{ x \in E : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (x, (L + T)^{-1}(R + T)(x)) \neq \emptyset \} \). Consider \( x \in E \) and \( (x, (L + T)^{-1}(\Phi + T)(x)) \cap R^*(x) \neq \emptyset \). Then \( x \in E \) and \( \emptyset \neq (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(R + T)(x) = (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_{\mu(x)} + T)(x) \). If \( x \in E \setminus U \) then since \( x \in D \) we have \( \emptyset \neq (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(H_0 + T)(x) \) which is a contradiction. Thus \( x \in U \) and \( \emptyset \neq (L + T)^{-1}(\Phi + T)(x) \cap (L + T)^{-1}(R + T)(x) \). Consequently \( (R^*)^{-1}(B) \subseteq (R^*)^{-1}(B_1) \) and on the other hand it is immediate that \( (R^*)^{-1}(B) \subseteq (R^*)^{-1}(B_1) \). Thus \( (R^*)^{-1}(B) = (R^*)^{-1}(B_1) \). Also \( (H_0^*)^{-1}(B) = (H_0^*)^{-1}(B_1) \). Thus (3.15) implies

\[ \mathcal{d} \left( (R^*)^{-1}(B_1) \right) = \mathcal{d} \left( (H_0^*)^{-1}(B_1) \right) \neq d(\emptyset). \tag{3.16} \]
Finally notice

\[(\mathbb{R}^\ast)^{-1}(B_U) = \{x \in U : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (x, (L + T)^{-1}(H_{\mu(x)} + T)(x)) \neq \emptyset\}\]

\[= \{x \in U : (x, (L + T)^{-1}(\Phi + T)(x)) \cap (x, (L + T)^{-1}(H_1 + T)(x)) \neq \emptyset\} = (H_1^\ast)^{-1}(B_U),\]

so from (3.16) we have \(d\left((H_1^\ast)^{-1}(B_U)\right) = d\left((H_0^\ast)^{-1}(B_U)\right) \neq d(\emptyset).\)

\[\Box\]

**Remark 3.20.** In Definition 3.18 and in the statement of Theorem 3.19 we could replace, any continuous map \(\eta : E \to [0, 1]\), with, any continuous map \(\eta : E \to [0, 1]\) with \(\eta(U) = 1\).

**Remark 3.21.** There is an analogue of Remark 3.16 (for normal topological vector spaces) in the statement of Theorem 3.19 and in Definition 3.18.

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**References**


