Integral transforms and partial sums of certain meromorphically $p$-valent starlike functions

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Abstract

In this paper, we introduce two new subclasses of meromorphically $p$-valent starlike functions. Inclusion relation, integral transforms, and partial sums for each of these classes are discussed.

Keywords: Analytic function, meromorphic function, $p$-valent function, starlike function, subordination, inclusion relation, integral transforms, partial sum.

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1. Introduction

In this paper, we assume that

\[-1 \leq B < 0, \quad B < A \leq -B, \quad \lambda \geq 1 \quad \text{and} \quad k \in \mathbb{N} \setminus \{1\}.\]  \hspace{1cm} (1.1)

For functions $f$ and $g$ analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$, the function $f$ is said to be subordinate to $g$, written $f(z) \prec g(z)$ ($z \in U$), if there exists an analytic function $w$ in $U$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

A function $f$ which is analytic in a domain $D \subset \mathbb{C}$ is called $p$-valent in $D$ if for every complex number $w$, the equation $f(z) = w$ has at most $p$ roots in $D$ and there will be a complex number $w_0$ such that the equation $f(z) = w_0$ has exactly $p$ roots in $D$. Let $\Sigma_p$ denote the class of functions of the form

\[f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad (p \in \mathbb{N}),\]  \hspace{1cm} (1.2)

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which are analytic in the punctured open unit disk $U_0 = U \setminus \{0\}$. We denote by $S^*_p$ the well-known class of meromorphically $p$-valent starlike functions. It is defined as follows

$$S^*_p = \left\{ f \in \Sigma_p : \text{Re} \frac{zf'(z)}{f(z)} < 0, z \in U \right\}.$$ 

Let

$$f_j(z) = z^{-p} + \sum_{n=0}^{\infty} a_{n,j} z^n \in \Sigma_p \quad (j = 1, 2).$$

Then the Hadamard product (or convolution) of $f_1$ and $f_2$ is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n = (f_1 * f_2)(z).$$

**Lemma 1.1.** Let $f \in \Sigma_p$ defined by (1.2) satisfies

$$\sum_{n=p}^{\infty} \left\{ p(1 - A) + (1 - B)[\lambda n + p(\lambda - 1)\delta_{n,p,k}] \right\} \leq p(A - B). \tag{1.3}$$

Then

$$\frac{p(1 - \lambda) f_{p,k}(z) - \lambda zf'(z)}{pf(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in U), \tag{1.4}$$

where

$$f_{p,k}(z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^j f(\varepsilon_k^j z), \quad \varepsilon_k = \exp \left( \frac{2\pi i}{k} \right) \tag{1.5}$$

and

$$\delta_{n,p,k} = \begin{cases} 0, & \left( \frac{n+p}{k} \notin \mathbb{N} \right), \\ 1, & \left( \frac{n+p}{k} \in \mathbb{N} \right). \end{cases} \tag{1.6}$$

**Proof.** The function $f_{p,k}$ in (1.5) can be expressed as

$$f_{p,k}(z) = z^{-p} + \sum_{n=p}^{\infty} \delta_{n,p,k} a_n z^n \tag{1.7}$$

with

$$\delta_{n,p,k} = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^j \left( \frac{n+p}{k} \right) = \left\{ \begin{array}{ll} 0, & \left( \frac{n+p}{k} \notin \mathbb{N} \right), \\ 1, & \left( \frac{n+p}{k} \in \mathbb{N} \right). & \end{array} \right.$$ 

According to (1.1) and (1.6), we see that

$$pA - B[p(1 - \lambda)\delta_{n,p,k} - \lambda n] \leq -B[p - p(\lambda - 1)\delta_{n,p,k} - \lambda n] \leq 0 \quad (n \geq p). \tag{1.8}$$

Let the inequality (1.3) hold. Then from (1.7) and (1.8), we deduce that

$$\left| \frac{p(1 - \lambda) f_{p,k}(z) - \lambda zf'(z)}{pf(z)} - 1 \right| \leq \frac{-\sum_{n=p}^{\infty} [p(\lambda - 1)\delta_{n,p,k} + \lambda n + p] a_n z^{n+p}}{p(A - B) + \sum_{n=p}^{\infty} [pA - B[p(1 - \lambda)\delta_{n,p,k} - \lambda n]] a_n z^{n+p}} \leq 1.$$ 

Hence, by the maximum modulus theorem, we have (1.4). The proof is completed.  \[\square\]
We now introduce the following two subclasses of $\Sigma_p$.

**Definition 1.2.** A function $f \in \Sigma_p$ defined by (1.2) is said to be in the class $M_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

**Definition 1.3.** A function $f \in \Sigma_p$ defined by (1.2) is said to be in the class $R_{p,k}(\lambda, A, B)$ if and only if it satisfies the coefficient inequality

$$\sum_{n=p}^{\infty} n[p(1-A)+(1-B)[\lambda n + p(\lambda -1)\delta_n,p,k]] \leq p^2(A-B).$$

For $f \in \Sigma_p$, we have

$$2z^{-p} + \frac{zf'(z)}{p} = z^{-p} + \sum_{n=p}^{\infty} \frac{n}{p} \alpha_n z^n,$$

which implies that

$$f \in R_{p,k}(\lambda, A, B) \quad \text{if and only if} \quad 2z^{-p} + \frac{zf'(z)}{p} \in M_{p,k}(\lambda, A, B). \quad (1.9)$$

If we write

$$\alpha_n = \frac{p(1-A)+(1-B)[\lambda n + p(\lambda -1)\delta_n,p,k]}{p(A-B)} \quad \text{and} \quad \beta_n = \frac{n}{p} \alpha_n \quad (n \geq p), \quad (1.10)$$

then it is easy to verify that

$$\frac{\partial \beta_n}{\partial \lambda} = \frac{n}{p} \frac{\partial \alpha_n}{\partial \lambda} > 0, \quad \frac{\partial \beta_n}{\partial A} = \frac{n}{p} \frac{\partial \alpha_n}{\partial A} < 0, \quad \text{and} \quad \frac{\partial \beta_n}{\partial B} = \frac{n}{p} \frac{\partial \alpha_n}{\partial B} \geq 0.$$

Thus, we obtain the following inclusion relations. If

$$1 \leq \lambda_0 \leq \lambda, \quad -1 \leq B_0 < B < A \leq -B, \quad \text{and} \quad A \leq A_0 \leq -B_0,$$

then

$$R_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda_0, A_0, B_0) \subset M_{p,k}(1,1,-1) \subseteq S_p^* = \left\{ f \in \Sigma_p : \Re \frac{zf'(z)}{f(z)} < 0, z \in U \right\}.$$ 

Therefore, by Lemma 1.1, we see that each function in the classes $M_{p,k}(\lambda, A, B)$ and $R_{p,k}(\lambda, A, B)$ is meromorphically $p$-valent starlike function. Meromorphic (and analytic) functions which are starlike have been extensively investigated by several authors (see, e.g., [1–22] and the references therein). In this paper we study some properties such as inclusion relation, integral transforms, and partial sums for the above-defined classes $M_{p,k}(\lambda, A, B)$ and $R_{p,k}(\lambda, A, B)$.

2. Inclusion relation

In this section we shall generalize the above mentioned inclusion relation

$$R_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda, A, B). \quad (2.1)$$

**Theorem 2.1.** If $-1 \leq D \leq B$, then

$$R_{p,k}(\lambda, A, B) \subset M_{p,k}(\lambda, C(D), D),$$

where

$$C(D) = D + \frac{(1-D)(A-B)}{1-B}.$$ 

The number $C(D)$ cannot be decreased for each $D$. 

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Proof. Since $-1 \leq D < B < 0$ and $B < A \leq -B$, we see that

$$D < C(D) \leq D - \frac{2B(1-D)}{1-B} \leq -D.$$  

Let $f \in \mathbb{R}_{p,k}(\lambda, A, B)$. In order to prove that $f \in \mathbb{M}_{p,k}(\lambda, C(D), D)$, we need only to find the smallest $C$ ($D < C \leq -D$) such that

$$\frac{p(1-C) + (1-D)[\lambda n + p(\lambda - 1)\delta_{n,p,k}]}{p(C-D)} \leq \frac{n[p(1-A) + (1-B)[\lambda n + p(\lambda - 1)\delta_{n,p,k}]]}{p(A-B)}$$  

for all $n \geq p$, that is, that

$$\frac{(1-D)[\lambda n + p + p(\lambda - 1)\delta_{n,p,k}]}{p(C-D)} - 1 \leq \frac{n}{p} \left\{ \frac{(1-B)[\lambda n + p + p(\lambda - 1)\delta_{n,p,k}]}{p(A-B)} - 1 \right\}. \quad (2.3)$$

For $n \geq p$ and $\frac{n+p}{k} \notin \mathbb{N}$, (2.3) becomes

$$C \geq D + \frac{1-D}{\frac{n(1-B)}{p(A-B)} - \frac{n-\lambda n + p}{\lambda n + p}} := \varphi(n).$$

Noting that (1.1), a simple calculation shows that $\varphi(n)$ ($n \geq p$) is decreasing in $n$. Therefore,

$$\varphi(n) \leq \begin{cases} \varphi(p+1), & \left( \frac{2p+1}{k} \in \mathbb{N} \right) \\ \varphi(p), & \left( \frac{2p}{k} \notin \mathbb{N} \right) \end{cases}.$$  

For $n \geq p$ and $\frac{n+p}{k} \in \mathbb{N}$, (2.3) is equivalent to

$$C \geq D + \frac{1-D}{\frac{n(1-B)}{p(A-B)} - \frac{n-p}{(n+p)}} := \psi(n).$$

Also, $\psi(n)(n \geq p)$ is decreasing in $n$. Thus

$$\psi(n) \leq \begin{cases} \psi(p), & \left( \frac{2p+1}{k} \in \mathbb{N} \right) \\ \psi \left( k \left[ \frac{2p}{k} \right] + 1 \right) - p, & \left( \frac{2p}{k} \notin \mathbb{N} \right) \end{cases}, \quad (2.4)$$

where $[x]$ in (2.4) denotes the integer part of a given real number $x$. Consequently, by taking

$$C = \varphi(p) = \psi(p) = D + \frac{(1-D)(A-B)}{1-B} = C(D),$$  

it follows from (2.2) to (2.5) that $f \in \mathbb{M}_{p,k}(\lambda, C(D), D)$. Furthermore, for $\frac{2p}{k} \in \mathbb{N}$ and $D < C_0 < C(D)$, we see that

$$\frac{1-C_0 + (2\lambda - 1)(1-D)}{C_0 - D} \cdot \frac{A-B}{1-A + (2\lambda - 1)(1-B)} \geq \frac{1-C(D) + (2\lambda - 1)(1-D)}{C(D) - D} \cdot \frac{A-B}{1-A + (2\lambda - 1)(1-B)} = 1,$$

which implies that the function

$$f(z) = z^p + \frac{A-B}{1-A + (2\lambda - 1)(1-B)} z^p \in \mathbb{R}_{p,k}(\lambda, A, B).$$
is not in the class $M_{p,k}(\lambda, C_0, D)$. Also, for $\frac{2p}{k} \notin \mathbb{N}$ and $D < C_0 < C(D)$, we have
\[
\frac{1 - C_0 + \lambda(1 - D)}{C_0 - D} \cdot \frac{A - B}{1 - A + \lambda(1 - B)} > \frac{1 - C_0 + \lambda(1 - D)}{C_0 - D} \cdot \frac{A - B}{1 - A + \lambda(1 - B)} = 1,
\]
which implies that the function
\[
f(z) = z^{-p} + \frac{A - B}{1 - A + \lambda(1 - B)} z^p \in R_{p,k}(\lambda, A, B)
\] (2.6)
is not in the class $M_{p,k}(\lambda, C_0, D)$. The proof of the theorem is completed. \qed

**Remark 2.2.** Putting $D = B$ in Theorem 2.1, we have the inclusion relation (2.1).

### 3. Integral transforms

**Theorem 3.1.** Let $p < \mu < p(2\lambda + 1)$. Suppose that $f \in M_{p,k}(\lambda, A, B)$ and
\[
I_{\mu}(z) = \frac{\mu - p}{z^\mu} \int_0^z t^{\mu - 1} f(t) \, dt.
\] (3.1)
Then $I_{\mu} \in M_{p,k}(\lambda, C_1(D), D)$, where $-1 \leq D \leq B$ and
\[
C_1(D) = D + \frac{(\lambda + 1)(\mu - p)(A - B)(1 - D)}{[\lambda + 1](\mu + p)(1 - B) - 2p(A - B)}.
\]
The number $C_1(D)$ cannot be decreased for each $D$.

**Proof.** Since $-1 \leq D \leq B < 0, B < A \leq -B$ and $p < \mu < p(2\lambda + 1)$, we can see that
\[
D < C_1(D) \leq D + \frac{(\lambda + 1)(\mu - p)(A - B)(1 - D)}{[\lambda + 1](\mu + p)(1 - B) - 2p(A - B)} \leq D - \frac{2B(1 - D)}{1 - B} \leq -D.
\]
For
\[
f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \in M_{p,k}(\lambda, A, B),
\]
it follows from (3.1) that
\[
I_{\mu}(z) = z^{-p} + \sum_{n=p}^{\infty} \frac{\mu - p}{\mu + n} a_n z^n.
\] (3.2)

In order to prove that $I_{\mu} \in M_{p,k}(\lambda, C_1(D), D)$, we need only to find the smallest $C$ $(D < C \leq -D)$ such that
\[
\frac{p(1 - C) + (1 - D) \lambda n + p(\lambda - 1) \delta_{n,p,k}}{p(C - D)} \cdot \frac{\mu - p}{\mu + n} \leq \frac{p(1 - A) + (1 - B) \lambda n + p(\lambda - 1) \delta_{n,p,k}}{p(A - B)}
\] (3.3)
for all $n \geq p$.

For $n \geq p$ and $\frac{n + p}{k} \notin \mathbb{N}$, (3.3) becomes
\[
C \geq D + \frac{1 - D}{\frac{p(n + p)}{[\mu - p][A - B]} - \frac{p(n + p)}{[\mu - p][\lambda n + p]}} := \varphi_1(n).
\]
It is easy to show that $\varphi_1(n)$ $(n \geq p)$ is a decreasing function of $n$ and so
\[
\varphi_1(n) \leq \begin{cases} 
\varphi_1(p + 1), & \left(\frac{2p}{k} \in \mathbb{N}\right), \\
\varphi_1(p), & \left(\frac{2p}{k} \notin \mathbb{N}\right).
\end{cases}
\]
For \( n \geq p \) and \( \frac{n+p}{k} \in \mathbb{N} \), (3.3) reduces to
\[
C \geq D + \frac{1 - D}{\frac{(\mu+n)(1-B)}{\mu-p}(A-B)} \frac{p}{\lambda(p-m)} := \psi_1(n)
\]
and we have
\[
\psi_1(n) \leq \begin{cases} \psi_1(p), & \left( \frac{2p}{k} \in \mathbb{N} \right) \\ \psi_1 \left( k \left( \frac{2p}{k} + 1 \right) - p \right), & \left( \frac{2p}{k} \notin \mathbb{N} \right) \end{cases} \quad (3.4)
\]
A simple calculation shows that \( \psi_1(p) \leq \varphi_1(p) \). Therefore, by taking
\[
C = \psi_1(p) = C_1(D),
\]
it follows from (3.3) to (3.4) that \( I_{\mu} \in M_{p,k}(\lambda, C_1(D), D) \).

Furthermore, the number \( C_1(D) \) is best possible for the function defined by (2.6). The proof of the
theorem is completed. \( \square \)

**Theorem 3.2.** Let \( p < \mu < p(2\lambda + 1) \). Also let \( I_{\mu} \) and \( C_1(D) \) be the same as in Theorem 3.1. If \( f \in R_{p,k}(\lambda, A, B) \),
then \( I_{\mu} \in R_{p,k}(\lambda, C_1(D), D) \) and the number \( C_1(D) \) cannot be decreased for each \( D \).

**Proof.** By (3.2) we have
\[
I_{\mu}(z) = \left( z^{-p} + \sum_{n=p}^{\infty} \frac{\mu-p}{\mu+n} z^n \right) * f(z)
\]
and so
\[
2z^{-p} + \frac{z[I_{\mu}(z)]'}{p} = \left( z^{-p} + \sum_{n=p}^{\infty} \frac{\mu-p}{\mu+n} z^n \right) * \left( 2z^{-p} + \frac{zf'(z)}{p} \right). \quad (3.5)
\]
In view of (3.5) and (1.9), an application of Theorem 3.1 yields Theorem 3.2. The proof of the theorem is
completed. \( \square \)

### 4. Partial sums

In this section, we let \( f \in \Sigma_p \) be given by (1.2) and define the partial sums \( s_1(z) \) and \( s_m(z) \) by
\[
s_1(z) = z^{-p} \quad \text{and} \quad s_m(z) = z^{-p} + \sum_{n=p}^{p+m-2} a_n z^n \quad (m \in \mathbb{N} \setminus \{1\}).
\]
For simplicity we use the notation \( \alpha_n \) (\( n \geq p \)) defined by (1.10).

**Theorem 4.1.** Let \( p \geq 2 \) and \( 1 \leq \lambda \leq \frac{p}{p-1} \). Suppose that \( f \in M_{p,k}(\lambda, A, B) \). Then for \( m \in \mathbb{N} \), we have
\[
\text{Re} \left( \frac{f(z)}{s_m(z)} \right) > 1 - \frac{1}{\alpha_{p+m-1}} \quad (z \in \mathbb{U}) \quad (4.1)
\]
and
\[
\text{Re} \left( \frac{s_m(z)}{f(z)} \right) > \frac{\alpha_{p+m-1}}{1 + \alpha_{p+m-1}} \quad (z \in \mathbb{U}). \quad (4.2)
\]
The bounds in (4.1) and (4.2) are sharp for each \( m \).

**Proof.** In view of the assumptions of the theorem, we see that
\[
\alpha_n = \frac{p(1-A) + (1-B)\lambda n + p(\lambda-1)\delta_{n,p,k}}{p(A-B)} \geq \frac{2-A-B}{A-B} \geq 1 \quad (4.3)
\]
and
\[
\alpha_{n+1} = \alpha_n + \frac{(1-B)[\lambda + p(\lambda-1)(\delta_{n+1,p,k} - \delta_{n,p,k})]}{p(A-B)} \geq \alpha_n + \frac{(1-B)[\lambda - p(\lambda-1)]}{p(A-B)} \geq \alpha_n. \quad (4.4)
\]
Let \( f \in M_{p,k}(\lambda, \Lambda, B) \). Then it follows from (4.3) and (4.4) that
\[
\sum_{n=p}^{p+m-2} |a_n| + \alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \alpha_n |a_n| \leq 1 \quad (m \in \mathbb{N} \setminus \{1\}). \tag{4.5}
\]

If we put
\[
p_1(z) = 1 + \alpha_{p+m-1} \left( \frac{f(z)}{s_m(z)} - 1 \right)
\]
for \( z \in \mathbb{U} \) and \( m \in \mathbb{N} \setminus \{1\} \), then \( p_1(0) = 1 \) and we deduce from (4.5) that
\[
\frac{|p_1(z) - 1|}{|p_1(z) + 1|} \leq \frac{\alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} a_n z^{n+p}}{2 \left( 1 + \sum_{n=p}^{p+m-2} a_n z^{n+p} \right) + \alpha_{p+m-1} \sum_{n=p+m-1}^{\infty} a_n z^{n+p}} \leq 1.
\]
This implies that \( \text{Re} \; p_1(z) > 0 \) (\( z \in \mathbb{U} \)), and so (4.1) holds for \( m \in \mathbb{N} \setminus \{1\} \).

Similarly, by setting
\[
p_2(z) = (1 + \alpha_{p+m-1}) \frac{s_m(z)}{f(z)} - \alpha_{p+m-1},
\]
it follows from (4.5) that
\[
\frac{|p_2(z) - 1|}{|p_2(z) + 1|} \leq \frac{(1 + \alpha_{p+m-1}) \sum_{n=p+m-1}^{\infty} a_n z^{n+p}}{2 \left( 1 + \sum_{n=p}^{p+m-2} a_n z^{n+p} \right) + (1 - \alpha_{p+m-1}) \sum_{n=p+m-1}^{\infty} a_n z^{n+p}} \leq 1.
\]
Hence, we have (4.2) for \( m \in \mathbb{N} \setminus \{1\} \).

For \( m = 1 \), replacing (4.5) by
\[
\alpha_p \sum_{n=p}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \alpha_n |a_n| \leq 1
\]
and proceeding as the above, we see that (4.1) and (4.2) are also true.

Furthermore, taking the function
\[
f(z) = z^{-p} + \frac{z^{p+m-1}}{\alpha_{p+m-1}} \in M_{p,k}(\lambda, \Lambda, B),
\]
we have \( s_m(z) = z^{-p} \),
\[
\text{Re} \frac{f(z)}{s_m(z)} \to 1 - \frac{1}{\alpha_{p+m-1}} \quad \text{as} \quad z \to \exp \left( \frac{\pi i}{2p + m - 1} \right)
\]
and
\[
\text{Re} \frac{s_m(z)}{f(z)} \to \frac{\alpha_{p+m-1}}{1 + \alpha_{p+m-1}} \quad \text{as} \quad z \to 1.
\]
The proof of the theorem is completed. \( \square \)
Theorem 4.2. Let $p \geq 2$ and $1 \leq \lambda \leq \frac{p}{p-1}$. Suppose that $f \in \mathcal{R}_{p,k}(\lambda, A, B)$. Then for $m \in \mathbb{N}$, we have

$$\text{Re} \frac{f(z)}{s_m(z)} > 1 - \frac{p}{(p + m - 1)\alpha_{p+m-1}} \quad (z \in \mathbb{U})$$

(4.6)

and

$$\text{Re} \frac{s_m(z)}{f(z)} > \frac{(p + m - 1)\alpha_{p+m-1}}{p + (p + m - 1)\alpha_{p+m-1}} \quad (z \in \mathbb{U}).$$

(4.7)

The bounds in (4.6) and (4.7) are sharp for the function

$$f(z) = z^{-p} + \frac{pz^{p+m-1}}{(p + m - 1)\alpha_{p+m-1}} \in \mathcal{R}_{p,k}(\lambda, A, B).$$

(4.8)

Proof. According to the assumptions of the theorem, it follows from (4.3) and (4.4) that

$$\sum_{n=p}^{p+m-2} |a_n| + \frac{(p + m - 1)\alpha_{p+m-1}}{p} \sum_{n=p+m-1}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \frac{n}{p} \alpha_n |a_n| \leq 1 \quad (m \in \mathbb{N} \setminus \{1\})$$

(4.9)

and

$$\alpha_p \sum_{n=p}^{\infty} |a_n| \leq \sum_{n=p}^{\infty} \frac{n}{p} \alpha_n |a_n| \leq 1.$$  

(4.10)

If we put

$$p_1(z) = 1 + \frac{(p + m - 1)\alpha_{p+m-1}}{p} \left[ \frac{f(z)}{s_m(z)} - 1 \right]$$

and

$$p_2(z) = \left[ 1 + \frac{(p + m - 1)\alpha_{p+m-1}}{p} \right] \frac{s_m(z)}{f(z)} - \frac{(p + m - 1)\alpha_{p+m-1}}{p},$$

then (4.9) and (4.10) lead to $\text{Re} p_j(z) > 0$ ($z \in \mathbb{U}; m \in \mathbb{N}; j = 1, 2$). The proof of the theorem is completed.

\[ \Box \]

Theorem 4.3. Let $p \geq 2$ and $1 \leq \lambda \leq \frac{p}{p-1}$. Suppose that $f \in \mathcal{R}_{p,k}(\lambda, A, B)$. Then for $m \in \mathbb{N}$, we have

$$\text{Re} \frac{f'(z)}{s_m'(z)} > 1 - \frac{1}{\alpha_{p+m-1}} \quad (z \in \mathbb{U})$$

(4.11)

and

$$\text{Re} \frac{s_m'(z)}{f'(z)} > \frac{\alpha_{p+m-1}}{1 + \alpha_{p+m-1}} \quad (z \in \mathbb{U}).$$

(4.12)

The bounds in (4.11) and (4.12) are sharp.

Proof. By virtue of the assumptions of the theorem, it follows from (4.3) and (4.4) that

$$\frac{1}{p} \sum_{n=p}^{p+m-2} n|a_n| + \frac{\alpha_{p+m-1}}{p} \sum_{n=p+m-1}^{\infty} n|a_n| \leq \sum_{n=p}^{\infty} \frac{n}{p} \alpha_n |a_n| \leq 1 \quad (m \in \mathbb{N} \setminus \{1\})$$

(4.13)

and

$$\frac{\alpha_p}{p} \sum_{n=p}^{\infty} n|a_n| \leq \sum_{n=p}^{\infty} \frac{n}{p} \alpha_n |a_n| \leq 1.$$  

(4.14)
By considering the functions

\[\begin{align*}
p_1(z) &= 1 + \alpha_{p+m-1} \left( \frac{f'(z)}{s_m(z)} - 1 \right) \quad \text{and} \quad p_2(z) = \left( 1 + \alpha_{p+m-1} \right) \frac{s_m(z)}{f'(z)} - \alpha_{p+m-1},
\end{align*}\]

we deduce from (4.13) and (4.14) that \( \text{Re} \ p_1(z) > 0 \) \((z \in U; m \in \mathbb{N}; j = 1, 2)\). Thus (4.11) and (4.12) hold true.

Furthermore, the bounds in (4.11) and (4.12) are best possible for the function defined by (4.8). The proof of the theorem is completed. \(\square\)

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