On the existence of generalized weak solutions to discontinuous fuzzy differential equations

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Abstract

In this paper, by means of replacing the Lebesgue integrability of support functions with its Henstock integrability, the definitions of the Henstock-Pettis integral of \(n\)-dimensional fuzzy-number-valued functions are defined. In addition, the controlled convergence theorems for such integrals are considered. As the applications of these integrals, we provide some existence theorems of generalized weak solutions to initial value problems for the discontinuous fuzzy differential equations under the strong GH-differentiability. ©2017 All rights reserved.

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1. Introduction

It is well-known that the Henstock integral includes the Riemann, improper Riemann, Lebesgue and Newton integrals [21, 25]. The Henstock integral is more powerful and simpler than the Lebesgue integral. It is also equal to the Denjoy and Perron integrals [26]. In the theory of integrals, there are some integrals based on the Banach space-valued functions such as Pettis and Bochner integrals [13, 14, 26, 29]. The integrals of fuzzy-number-valued functions, as a natural generalization of set-valued functions, have been discussed by Puri and Ralescu [27], Kaleva [22], and other authors [36, 37, 40]. Recently, Wu and Gong [15, 18, 19] have combined the fuzzy set theory and nonabsolute integration theory, and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva [22] integration. In order to complete the theory of fuzzy calculus and to meet the solving need of transferring a fuzzy differential equation into a fuzzy integral equation, we [15, 17] has defined the strong fuzzy Henstock integrals and

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discussed some of their properties and the controlled convergence theorem. However, for a fuzzy-valued function in the \( n \)-dimensional fuzzy number space \( E^n \), the integral and its characteristic theorems have not been defined or discussed. In this paper, by means of replacing the Lebesgue integrability of support functions with its Henstock integrability, we discuss the Henstock-Pettis integral of \( n \)-dimensional fuzzy-number-valued functions and its controlled convergence theorems.

Differential equations are used for modeling various physics. Unfortunately, many problems are too dynamical and complicated and an accurate differential equation model for such problems requires complex and time consuming algorithms hardly implementable in practice. Thus, a usage of fuzzy mathematics seems to be appropriate. The Cauchy problems for fuzzy differential equations have been studied by several authors [2, 4–8, 16, 22, 27, 30] on the metric space \((E^n, D)\) of normal fuzzy convex set with the distance \( D \) given by the maximum of the Hausdorff distance between the corresponding level sets. In 2002, Xue and Fu [39] established solutions to fuzzy differential equations with right-hand side functions satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets. However, there are discontinuous fuzzy systems in which the right-hand side functions \( f \) satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets.

The rest of the paper is organized as follows. In Section 2, we give some basic concepts and preliminary results and the definition of the Henstock-Pettis integral for fuzzy-number-valued functions. In Section 3, we prove a controlled convergence theorem for the fuzzy Henstock-Pettis integrals. As the applications of these integrals, we deal with the Cauchy problem of discontinuous fuzzy systems. And in Section 4, we present some concluding remarks.

2. Preliminaries

Let \( T \) be the closed interval on the real line \( R \), i.e., \( T = [a, b] \ (a, b \in R) \). \(|T|\) denotes the length of \( T \). If there exist \( T_1 \subseteq T \), \( \xi_i \in T_i \ (i = 1, 2, \ldots, k) \), such that \( \bigcup_{i=1}^{k} T_i = T \) (where \( T_1, T_2, \ldots, T_k \) are nonoverlapping subintervals of \( T \)), then a collection \( \{ \langle \xi_1, T_1 \rangle, \langle \xi_2, T_2 \rangle, \ldots, \langle \xi_k, T_k \rangle \} \) is called a devision of \( T \) and write

\[ \Pi = \{ \langle \xi_1, T_1 \rangle, \langle \xi_2, T_2 \rangle, \ldots, \langle \xi_k, T_k \rangle \} . \]

For brevity, we write \( \Pi = \{ \xi, [u, v] \} \), where \([u, v]\) denotes a typical interval in \( \Pi \) and \( \xi \) is the associated point of \([u, v]\).

Definition 2.1 ([26]). Let \( \delta(x) > 0 \) be a function on \( T \). A division \( \Pi \) of \( T \) is said to be \( \delta \)-fine, if \( \xi_i \in T_i \subset \left( \xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i) \right) \ (i = 1, 2, \ldots, k) \).

Definition 2.2 ([26]). A function \( F : T \to R^n \) is said to be Henstock integrable on \( T \) if for \( A \in R^n \) and every \( \varepsilon > 0 \), there is a function \( \delta(x) > 0 \), such that for any \( \delta \)-fine devision \( \Pi = \{ \xi, [u, v] \} \) we have

\[ \| \sum_{\Pi} F(\xi)(v - u) - A \| < \varepsilon . \]

As usual, we write \( (H) \int_T F(t) \, dt = A \). Here \( \| \cdot \| \) stands for the norm on the \( R^n \).

Throughout this paper, we use \( P_k(R^n) \) to denote the family of all nonempty compact convex subsets of \( R^n \). For \( A, B \in P_k(R^n), k \in R \), the addition and scalar multiplication are defined by the equations as follows respectively:

\[ A + B = \{ x + y \mid x \in A, y \in B \}, \quad aA = \{ ax \mid x \in A \}. \]
In addition, for $A, B \in P_k(R^n)$, the Hausdorff metric between them defined by

$$d(A, B) = \max \{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|b - a\| \}.$$ 

**Definition 2.3.** For $A \in P_k(R^n)$, $x \in S^{n-1}$, the support function of $A$ is defined by

$$\sigma(x, A) = \sup_{y \in A} \langle y, x \rangle,$$

where $S^{n-1}$ denotes the unit sphere of $R^n$, $\langle \cdot, \cdot \rangle$ is the inner product in $R^n$.

It is clear that for $A, B \in P_k(R^n)$, $x \in S^{n-1}$, we have

1. $\sigma(x, kA) = k\sigma(x, A)$ ($k \geq 0$);

2. $\sigma(x, A + B) = \sigma(x, A) + \sigma(x, B)$.

**Theorem 2.4 ([12]).** Let $A, B \in P_k(R^n)$. Then $d(A, B) = \sup_{x \in S^{n-1}} |\sigma(x, A) - \sigma(x, B)|$.

**Definition 2.5 ([12, 38]).** Let $E^n = \{ u \mid u : R^n \rightarrow [0, 1] \}$. For any $u \in E^n$, $u$ is said to be an $n$-dimensional fuzzy number if the following conditions are satisfied:

1. $u$ is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$, such that $u(x_0) = 1$;

2. $u$ is a convex fuzzy set, i.e., $u(tx + (1-t)y) \geq \min \{ u(x), u(y) \}$ for any $x, y \in R^n, t \in [0, 1]$;

3. $u$ is upper semi-continuous;

4. $\text{supp} u = \overline{\{ x \in R^n \mid u(x) > 0 \}}$ is compact, here $\overline{A}$ denotes the closure of $A$.

We define $D : E^n \times E^n \rightarrow [0, \infty)$ by the equation

$$D(u, v) = \sup_{t \in [0, 1]} d([u]^r, [v]^t), \quad u, v \in E^n,$$

then the metric space $(E^n, D)$ has a linear structure, it can be imbedded isomorphically as a convex cone with vertex 0 into the Banach space of functions $u^* : I \times S^{n-1} \rightarrow R$, where $S^{n-1}$ is the unit sphere in $R^n$, with an imbedding function $u^* = j(u)$ defined by

$$u^*(r, x) = \sup_{x \in [u]^x} < \alpha, x >$$

for all $r, x \in I \times S^{n-1}$.

**Theorem 2.6 ([38]).** There exists a real Banach space $X$ such that $E^n$ can be imbedding as a convex cone $C$ with vertex 0 into $X$. Furthermore the following conditions hold true:

1. the imbedding $j$ is isometric;

2. addition in $X$ induces addition in $E^n$;

3. multiplication by nonnegative real number in $X$ induces the corresponding operation in $E^n$;

4. $C - C$ is dense in $X$;

5. $C$ is closed.
A fuzzy-number-valued function \( \tilde{f} : [a, b] \to \mathbb{E}^n \) is said to satisfy the condition (H) on \([a, b]\), if for any \(x_1 < x_2 \in [a, b]\) there exists \(u \in \mathbb{E}^n\) such that \(f(x_2) = f(x_1) + u\). We call \(u\) the H-difference of \(f(x_2)\) and \(f(x_1)\), denoted \(\tilde{f}(x_2) - \tilde{f}(x_1)\) ([22]).

For brevity, we always assume that the condition (H) is satisfied when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

**Definition 2.7** ([3]). Let \(\tilde{f} : (a, b) \to \mathbb{E}^n\) and \(x_0 \in (a, b)\). We say that \(\tilde{f}\) is differentiable at \(x_0\), if there exists an element \(\tilde{f}'(x_0) \in \mathbb{E}^n\), such that

1. for all \(h > 0\) sufficiently small, there exist \(\tilde{f}(x_0 + h) - \tilde{f}(x_0), \tilde{f}(x_0 - h)\) and the limits (in the metric \(D\))
   \[
   \lim_{h \to 0} \frac{\tilde{f}(x_0 + h) - \tilde{f}(x_0)}{h} = \lim_{h \to 0} \frac{\tilde{f}(x_0 - h) - \tilde{f}(x_0)}{h} = \tilde{f}'(x_0),
   \]
   or;

2. for all \(h > 0\) sufficiently small, there exist \(\tilde{f}(x_0) - \tilde{f}(x_0 + h), \tilde{f}(x_0 - h) - \tilde{f}(x_0)\) and the limits
   \[
   \lim_{h \to 0} \frac{\tilde{f}(x_0) - \tilde{f}(x_0 + h)}{-h} = \lim_{h \to 0} \frac{\tilde{f}(x_0 - h) - \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0),
   \]
   \((h\text{ and }-h\text{ at denominators mean }\frac{1}{h}\text{ and }-\frac{1}{h'},\text{ respectively}).

**Theorem 2.8** ([12, 38]). If \(u, v \in \mathbb{E}^n, x, y \in S^{n-1}, r \in [0, 1]\), then

1. \(\sigma(x + y, [u]^r) \leq \sigma(x, [u]^r) + \sigma(y, [u]^r)\);
2. \(\sigma(k x, [u]^r) = k \sigma(x, [u]^r)\) whenever \(k \geq 0\);
3. \(\sigma(x, [u]^r)\) is bounded uniformly on \(S^{n-1} \times I\), and \(\sigma(x, [u]^r) \leq \sup_{a \in [u]^r} ||a||\);
4. \(\sigma(x, [u]^r)\) is nonincreasing left continuous for any \(x \in S^{n-1}\) with respect to \(r\), especially it is right continuous at \(r = 0\);
5. \(\sigma(x, [u]^r)\) is Lipschitz continuous uniformly for any \(r \in [0, 1]\) with respect to \(x\), and
   \[
   |\sigma(x, [u]^r) - \sigma(y, [u]^r)| \leq (\sup_{a \in [u]^r} ||a||)||x - y||;
   \]
6. \(d([u]^r, [v]^r) = |\sigma(x, [u]^r) - \sigma(x, [v]^r)|\).

In the following, we give the definition of Henstock-Pettis integral of fuzzy-number-valued functions and its representation theorems.

**Definition 2.9** ([36, 37]). A fuzzy-number-valued function \(\tilde{F} : T \to \mathbb{E}^n\) is said to be fuzzy Henstock integrable on \(T\) if there exists a fuzzy number \(\tilde{A} \in \mathbb{E}^n\) such that for every \(\varepsilon > 0\) there is a \(\varepsilon\)-fine division \(\Pi = \{\xi_i, [x_{i-1}, x_i]\}\) of \(T\), we have
\[
D(\tilde{A}, \sum_i \tilde{F}(\xi_i)(x_i - x_{i-1})) < \varepsilon.
\]
We write \((FH) \int_T \tilde{F}(x)dx = \tilde{A}\).

**Definition 2.10.** A fuzzy-number-valued function \(F : T \to \mathbb{E}^n\) is said to be Henstock-Pettis integrable on \(T\) if \([F(t)]^r\) is Henstock-Pettis integrable on \(T\) for every \(r \in [0, 1]\), and there exists a fuzzy number \(\tilde{A} \in \mathbb{E}^n\) such that for any \(x \in S^{n-1}\) we have
\[
(\sigma(x, [A]^r) = (H) \int_T \sigma(x, [F(t)]^r)dt).
\]
We write \(\tilde{A} = (FHP) \int_T \tilde{F}(t)dt\).
Remark 2.11. In particular, if $\tilde{F}$ is degenerated into $F : T \to \mathbb{R}^n$ and $\tilde{A}$ is degenerated into $A \in \mathbb{R}^n$, then
\[
\sigma(x, [A]^r) = \langle x, A \rangle.
\]

Remark 2.12. When $n = 1$, if the fuzzy-number-valued function $\tilde{F} : T \to E^1$ is Kaleva integrable on $T$ (refer to [36]), then $\tilde{F}$ is also Pettis integrable.

Similar to the methods of [36, 37], we easily obtain the following results.

**Theorem 2.13.** Suppose $\tilde{F}, \tilde{G} : T \to E^n$ are fuzzy-number-valued functions on $T$.

1. If $\tilde{F}, \tilde{G}$ are fuzzy Henstock-Pettis integrable on $T$, then $\alpha \tilde{F} + \beta \tilde{G}$ ($\alpha, \beta \in \mathbb{R}$) is also fuzzy Henstock-Pettis integrable on $T$, and
\[
(FHP) \int_T (\alpha \tilde{F}(t) + \beta \tilde{G}(t)) dt = \alpha (FHP) \int_T \tilde{F}(t) dt + \beta (FHP) \int_T \tilde{G}(t) dt.
\]

2. If $\tilde{F}$ is fuzzy Henstock-Pettis integrable on $T$, then $\tilde{F}$ is fuzzy Henstock-Pettis integrable on every subset of $T$, and for nonoverlapping $T_1, T_2, \ldots, T_m$ we have
\[
(FHP) \int_T \tilde{F}(t) dt = \sum_{i=1}^{m} (FHP) \int_{T_i} \tilde{F}(t) dt,
\]
where $T = \bigcup_{i=1}^{m} T_i$.

3. If $\tilde{F}$ is fuzzy Henstock-Pettis integrable on $T$ and $\tilde{F} = \tilde{G}$ almost everywhere on $T$, then $\tilde{G}$ is also fuzzy Henstock-Pettis integrable on $T$ and
\[
(FHP) \int_T \tilde{F}(t) dt = (FHP) \int_T \tilde{G}(t) dt.
\]

**Theorem 2.14.** A fuzzy-number-valued function $\tilde{F} : T \to E^n$ is fuzzy Henstock-Pettis integrable on $T$ if and only if for every $r \in [0, 1]$, real-valued function $\sigma(x, [F(t)]^r)$ is Henstock integrable uniformly on $T$ for any $x \in S^{n-1}$, and
\[
\sigma(x, (H) \int_T [F(t)]^r dt) = (H) \int_T \sigma(x, [F(t)]^r) dt.
\]

**Theorem 2.15.** If fuzzy-number-valued function $\tilde{G} : T \to E^n$ is Pettis integrable on $T$ and the null function is a selection of $[G(t)]^r$, $r \in [0, 1]$, then for every $A, B \subset T$ such that $A \subseteq B$ we have $w_A \leq w_B$, where $w_A = (FP) \int_A \tilde{G}(t) dt$, $w_B = (FP) \int_B \tilde{G}(t) dt$.

**Proof.** Since $A, B \subset T$, $A \subseteq B$ and $[G(t)]^r : T \to P_k(\mathbb{R}^n)$ is Pettis integrable for any $r \in [0, 1]$, we have $[w_A]^r \subseteq [w_B]^r$, i.e., $w_A \leq w_B$, where
\[
[w_A]^r = (P) \int_A [G(t)]^r dt, \quad [w_B]^r = (P) \int_B [G(t)]^r dt.
\]

**Example 2.16.** We present an example of function which is fuzzy Henstock-Pettis integrable and neither strong fuzzy Henstock integrable nor Keleva integrable.

Let $f : [0, 1] \to E^n$ and let $\tilde{f}(t) = \chi_{[F(t)]} + \tilde{A}(s) \cdot \tilde{F}'(t)$, where
\[
\tilde{F}(t) = \begin{cases} t^2 \sin t^{-2}, & t \neq 0, \\ 0, & t = 0, \end{cases}
\]
and 

\[ \tilde{\lambda}(s) = \begin{cases} 
  s, & 0 \leq s \leq 1, \\
  2 - s, & 1 < s \leq 2, \\
  0, & \text{others}. 
\end{cases} \]

Put \( \tilde{f}_1(t) = X_{[F_1(t)]} \) and \( \tilde{f}_2 = \tilde{\lambda}(s) \cdot F'(t) \). We can prove that a function \( f = f_1 + f_2 \) is integrable in the sense of fuzzy Henstock-Pettis. In fact, \( \sigma(x, [f(t)]^\ast) \) is Henstock integrable on \([0, 1]\). In addition, the function \( \tilde{f} \) is not Keleva integrable because \( j \circ f_2 \) is not Lebesgue integrable. Moreover, \( \tilde{f}_1 \) is not strong measurable, so this function is not strong fuzzy Henstock integrable.

3. The existence of generalized weak solutions to discontinuous fuzzy differential equations

Convergence theorems for a given integration theory are important for estimating the power of the theory. For real and Banach-valued Henstock integrable functions, there are a few convergence theorems, (see [26, 29] for instance). In order to generalize certain results on continuous dependence of solutions of ordinary differential equations with respect to parameters, Kurzweil introduced, in 1957, what he called generalised ordinary differential equations for Euclidean and Banach space-valued functions (see [25]).

The theory of generalized ordinary differential equations is extensively described in [28]. In [10] and [11], the authors extended the controlled convergence theorems and proved the existence theorems for the Cauchy problem for Banach space-valued functions under Henstock-Pettis integrability assumptions, respectively. In this section, we present a controlled convergence theorem for fuzzy Henstock-Pettis integral. At last, we also give an existence theorem for a Cauchy problem using the fuzzy Henstock-Pettis integral and its properties. The requirements on the right hands function \( \tilde{f} \) are not too restrictive.

**Definition 3.1.** Let \( X \subseteq [a, b] \). An \( n \)-dimensional fuzzy-number-valued function \( \tilde{F} \) defined on \( X \) is said to be AC\((X)\) if for every \( \varepsilon > 0 \) such that for every finite sequence of non-overlapping intervals \( \{[a_i, b_i]\} \), with \( \Sigma_{i=1}^{n} |b_i - a_i| < \eta \), we have

\[ \sum D(\tilde{F}(b_i), \tilde{F}(a_i)) < \varepsilon, \]

where the endpoints \( a_i, b_i \in X \) for all \( i \).

**Definition 3.2.** An \( n \)-dimensional fuzzy-number-valued function \( \tilde{F} \) defined on \( X \subseteq [a, b] \) is said to be uniformly AC\(^\ast\)(\( X \)) if for every \( \varepsilon > 0 \) there exists \( \eta > 0 \) such that for every finite sequence of non-overlapping intervals \( \{[a_i, b_i]\} \), satisfying \( \Sigma_{i=1}^{n} |b_i - a_i| < \eta \) where \( a_i, b_i \in X \) for all \( i \), we have

\[ \sum \omega([F, [a_i, b_i]]) < \varepsilon, \]

where \( \omega \) denotes the oscillation of \( \tilde{F} \) over \([a_i, b_i]\), i.e.,

\[ \omega(\tilde{F}, [a_i, b_i]) = \sup(\omega(\tilde{F}(y), \tilde{F}(x)) : x, y \in [a_i, b_i]). \]

**Definition 3.3.** An \( n \)-dimensional fuzzy-number-valued function \( \tilde{F} \) is said to be AC\(^\ast\) on \( X \), if \( X \) is the union of a sequence of closed sets \( \{X_i\} \) such that on each \( X_i \), \( \tilde{F} \) is AC\(^\ast\)(\( X_i \)).

An \( n \)-dimensional fuzzy-number-valued function \( \tilde{F} \) is said to be uniformly AC\(^\ast\) on \( X \), if \( X \) is the union of a sequence of closed sets \( \{X_i\} \) such that on each \( X_i \), \( \tilde{F} \) is uniformly AC\(^\ast\)(\( X_i \)).

**Theorem 3.4** (Controlled Convergence Theorem). Let \( \tilde{f}_n, \tilde{f} : T \to E^n \) be (FHP)-integrable functions on \( T \). Assume that

1. \( \sigma(x, [F_n(t)]^\ast) \longrightarrow \sigma(x, [f(t)]^\ast) \), a.e. on \( T \);

2. the family \( G = \{\sigma(x, [F_n(t)]^\ast), n = 1, 2, \cdots \} \) is uniformly AC\(^\ast\) and equicontinuous on \( T \).
Then $\tilde{f}$ is (FHP)-integrable and
\[
\lim_{n \to \infty} (\text{FHP}) \int_0^t f_n(s) \, ds = (\text{FHP}) \int_0^t \tilde{f}(s) \, ds.
\]

Proof.
(i) Since $f_n$ is (FHP)-integrable, the function $\sigma(x, [f_n(t)]^r)$ is (H)-integrable. So,
\[
\begin{align*}
\text{(a)} & \quad \sigma(x, [f_n(t)]^r) \longrightarrow \sigma(x, [f(t)]^r), \text{ a.e. on } T; \\
\text{(b)} & \quad G \text{ is uniformly } ACG^{*}; \\
\text{(c)} & \quad \text{the sequence } \sigma(x, [F_n(t)]^r) \text{ is uniformly convergent on } T.
\end{align*}
\]

By using the convergence theorem for Henstock integral [26], we have
\[
(\text{H}) \int_0^t \sigma(x, [f_n(s)]^r) \, ds \longrightarrow (\text{H}) \int_0^t \sigma(x, [f(s)]^r) \, ds.
\]

(ii) Fix an arbitrary $t \in T$, and let $g_n \subset f_n$. For every $\varepsilon > 0$ there exists $\sigma^*$ such that
\[
\sigma^*(x, [g_n(s)]^r) \subset \sigma(x, [f_n(s)]^r),
\]
with following conditions:
\[
\lim_{n \to \infty} (\text{H}) \int_0^t \sigma^*(x, [g_n(s)]^r) \, ds = (\text{H}) \int_0^t \sigma(x, [f(s)]^r) \, ds. \tag{3.1}
\]

Consider the set \{\sigma^*(x, [f_n(t)]^r) | n = 1, 2, \ldots\}, there exists a subsequence $\sigma^*(x, [g_n(s)]^r) \subset \sigma(x, [f_n(s)]^r)$ such that the limit $\lim_{k \to \infty} \sigma^*_k(x, [f(t)]^r)$ exists almost everywhere and
\[
\lim_{k \to \infty} \sigma^*_k(x, [f(t)]^r) = \sigma^*_0(x, [f(t)]^r). \tag{3.2}
\]

Since $\sigma(x, [f_m(s)]^r)$ is uniformly (H)-integrable, that is, for all $\varepsilon > 0$, there exists a $\delta$-fine partition on $T$ such that
\[
|\sum_{j=1}^{k} \sigma(x, [f_m(t_j)]^r \cdot (x_i - x_{j-1})) - (\text{H}) \int_a^b \sigma(x, [f_m(s)]^r) \, ds| < \varepsilon.
\]

By (3.1) and (3.2), we have
\[
|\sum_{j=1}^{k} \sigma^*_k(x, [f_m(t_j)]^r \cdot (x_i - x_{j-1})) - (\text{H}) \int_a^b \sigma^*_k(x, [f(s)]^r) \, ds| \\
\leq |\sum_{j=1}^{k} \sigma^*_k(x, [f_m(t_j)]^r \cdot (x_i - x_{j-1})) - \sum_{j=1}^{k} \sigma^*_k(x, [g_m(t_j)]^r \cdot (x_i - x_{j-1}))| \\
+ |(\text{H}) \int_a^b \sigma^*_k(x, [f(s)]^r) \, ds - (\text{H}) \int_a^b \sigma^*_k(x, [g(s)]^r) \, ds| \\
+ |(\text{H}) \int_a^b \sigma^*_k(x, [f(s)]^r) \, ds - (\text{H}) \int_a^b \sigma^*_k(x, [f(s)]^r) \, ds|.
\]

By condition (1), there exists $m_0 \in \mathbb{N}$ such that $m \geq m_0$ and we have
\[
|\sum_{j=1}^{k} \sigma^*_k(x, [f_m(t_j)]^r \cdot (x_i - x_{j-1})) - \sum_{j=1}^{k} \sigma^*_k(x, [g_m(t_j)]^r \cdot (x_i - x_{j-1}))| < \frac{\varepsilon}{3},
\]
According the uniform (H)-integrability of $\sigma^*(x, [f_m(s)]^T)$, we have

$$\left| \sum_{j=1}^{k} \sigma^*_k(x, [g_m(t_j)]^T \cdot (x_i - x_{i-1})) - (H) \int_a^b \sigma^*_k(x, [f(s)]^T)ds \right| < \frac{\epsilon}{3}.$$  

By (i),

$$|(H) \int_a^b \sigma^*_k(x, [f(s)]^T)ds - (H) \int_a^b \sigma^*_k(x, [f(s)]^T)ds| < \frac{\epsilon}{3}.$$  

So the set $\{\sigma^*(x, [f_n(t)]^T) \mid n = 1, 2, \ldots \}$ is uniformly integrable.

Now we are able to use the Vitali convergence theorem for real-valued Henstock integrable functions [26] and see that

$$\lim_{k \to \infty} (H) \int_0^t \sigma^*_k(x, [g_n(s)]^T)ds = (H) \int_0^t \sigma^*_0(x, [f(s)]^T)ds. \quad (3.4)$$  

From (3.2), we get that

$$\lim_{k \to \infty} (H) \int_0^t \sigma^*_k(x, [g_n(s)]^T)ds = (H) \int_0^t \sigma^*_0(x, [f(s)]^T)ds.$$  

Thus, for all $n > k$, we have

$$(H) \int_0^t \sigma^*_k(x, [g_n(s)]^T)ds > \epsilon,$$

and

$$(H) \int_0^t \sigma^*_k(x, [f(s)]^T)ds \geq \epsilon$$

for all $k = 1, 2, \ldots$. Passing to the limit with $k \to \infty$,

$$(H) \int_0^t \sigma^*_0(x, [f(s)]^T)ds \geq \epsilon.$$  

Since $\sigma^*_0(x, [g_n(s)]^T)$ is uniformly convergent to $\sigma^*_0(x, [g(s)]^T)$ for $n$, we have

$$\lim_{\alpha} (H) \int_0^t \sigma^*_\alpha(x, [g_n(s)]^T)ds = \lim_{\alpha} \sigma(x, [(FHP) \int_0^t g_n(s)ds]^T).$$  

Denote $z^*_n(t) = (FHP) \int_0^t g_n(s)ds$. In this case, we see that

$$\lim_{\alpha} \int_0^t \sigma^*_\alpha(x, [g_n(s)]^T)ds = \lim_{\alpha} \sigma(x, [z^*_n(t)]^T)$$

$$= \sigma^*_0(x, [(FHP) \int_0^t g_n(s)ds]^T)$$

$$= (H) \int_0^t \sigma^*_0(x, [g_n(s)]^T)ds.$$  

From (3.3), we have $\int_0^t \sigma^*_0(x, [f(s)]^T)ds = 0$ which contradicts (3.4). Since

$$(H) \int_0^t \sigma(x, [f_n(s)]^T)ds \longrightarrow (H) \int_0^t \sigma(x, [f(s)]^T)ds,$$

and according to (i), a sequence $\{(FHP) \int_0^t f_n(s)ds\}$ is Cauchy, so

$$\lim_{n \to \infty} (FHP) \int_0^t f_n(s)ds = (FHP) \int_0^t f(s)ds. \quad \Box$$
Since the convergence theorems are really applicable (fuzzy differential equations, for instance), we shall present a few particular case of Theorem 3.4.

**Corollary 3.5.** If condition (2) is replaced by condition:

(3) for each i, k ∈ N there exists real-valued Henstock integrable function g : T → R, such that
\[
|\sigma(x, [f_i(t)]) - \sigma(x, [f_k(t)])| \leq g.
\]

Then the conclusion of Theorem 3.4 also holds.

**Corollary 3.6.** If condition (2) is replaced by condition:

(4) for each n ∈ N there exist real-valued Henstock integrable functions g, h : T → R, such that
\[ g \leq \sigma(x, [f_n(t)]) \leq h; \]
then the conclusion of Theorem 3.4 also holds.

Next, we will deal with the Cauchy problem:
\[
\begin{cases}
  x'(t) = \tilde{f}(t, x(t)), & t \in [0, \alpha] = I_\alpha, \\
  x(0) = x_0,
\end{cases}
\tag{3.5}
\]
where \( \tilde{f} \) is fuzzy Henstock-Pettis integrable function. In fact, our existence theorem is based on an idea of Kurzweil from [25].

**Definition 3.6.** Let \( \tilde{F} : T \to \mathbb{R}^n \) be fuzzy-number-valued function and let \( A \subset T \). The function \( f : A \to \mathbb{R}^n \) is the weak derivative of \( F \) on \( A \), if the Banach-valued function \( j \circ \tilde{F} \) is differentiable almost everywhere on \( A \) and \( (j \circ \tilde{F})' = j \circ \tilde{f}, \) a.e. .

**Definition 3.7.** A fuzzy-number-valued function \( \tilde{f} : I_\alpha \to \mathbb{R}^n \) is weak continuous on \( I_\alpha \).

**Theorem 3.9.** Let \( \tilde{f} : [a, b] \to \mathbb{R}^n \) be (FHP)-integrable on \([a, b]\) and let \( \tilde{F}(x) = \int_a^x \tilde{f}(s)ds \). Then

(1) \( \sigma(x, [f(s)]) \) is Henstock integrable on \([a, b]\) and (H) \( \int_a^x \sigma(x, [f(s)])ds = \sigma(x, [|F(x)|]) \);

(2) the function \( F \) is weak continuous on \([a, b]\) and \( \tilde{f} \) is a weak derivative of \( F \) on \([a, b]\).

**Proof.**

(1) See Definition 2.10.

(2) Since the function \( \sigma(x, [f(s)]) \) is a real-valued and (H)-integrable, and
\[
(\text{H}) \int_a^x \sigma(x, [f(s)])ds = \sigma(x, [|F(x)|]),
\]
then \( G(x) = \int_a^x \tilde{f}(s)ds \) is continuous, that is \( \tilde{F} \) is weak continuous. By conclusion (a), there exists a set \( \Lambda \subset [a, b] \), such that \( G'(x) = \sigma(x, [f(s)]) \), but \( G'(x) = \sigma(x, [|F(x)|])' \).

**Definition 3.8.** A function \( x : I_\alpha \to \mathbb{R}^n \) is said to be a weak solution of Cauchy problem (3.5) if it satisfies the following conditions:

(1) \( x(\cdot) \) is ACG*;

(2) \( x(0) = x_0; \)
(3) there exists a set \( A \), with Lebesgue measure zero, such that for each \( t \not\in A \)

\[
j \circ (x'(t)) = j \circ (\tilde{f}(t, x(t))),
\]

where “\( r \)” denotes the weak derivative.

**Theorem 3.11.** If the function \( \tilde{f} : I_\alpha \to \mathbb{E}^n \) is (FHP)-integrable, then

\[
\int_I \tilde{f}(t) dt \in |I| \cdot \text{conv} \tilde{f}(1),
\]

where \( I \subset I_\alpha \) and \(|I|\) is the length of \( I \).

**Proof.** Since \( \sigma(x, [f(t)]^r) \) is \( (H) \)-integrable, by the mean value theorem for \( H \) integral we have

\[
(H) \int_I \sigma(x, [f(t)]^r) dt \in |I| \cdot \text{conv} \sigma(x, [f(I)]^r) = \sigma(x, |I| \cdot \text{conv} [f(I)]^r).
\]

However, by the definition of fuzzy Henstock-Pettis integral, there exists

\[
(H) \int_I \sigma(x, [f(t)]^r) dt = \sigma(x, [\int_I \tilde{f}(t) dt]^r).
\]

Therefore, \( \sigma(x, [\int_I \tilde{f}(t) dt]^r) \in \sigma(x, |I| \cdot \text{conv} [f(I)]^r) \). Since \(|I| \cdot \text{conv} \tilde{f}(1) \) is a closed convex set, we have

\[
\int_I \tilde{f}(t) dt \in |I| \cdot \text{conv} \tilde{f}(1). \quad \square
\]

**Theorem 3.12** ([24]). Let \( D \) be a closed convex subset of a Banach space \( X \), and let \( F \) be a weakly sequentially continuous map of \( D \) into itself. If for some \( x \in D \) the implication

\[
\nabla = \text{conv} ((x) \cup F(V)) \implies V \text{ is relatively weakly compact}, \quad (3.6)
\]

holds for every subset \( V \) of \( D \), then \( F \) has a fixed point.

For any bounded subset \( A \) of Banach space \( X \) we denote \( \mu(A) \) the measure of weak non-compactness of \( A \), i.e.,

\[
\mu(A) = \inf \{ t > 0 : \text{there exists } C \in \mathcal{K} \text{ such that } A \subset C + tB_0 \},
\]

where \( \mathcal{K} \) is the set of weakly compact subsets of \( X \) and \( B_0 \) is the norm unit ball in \( X \). For the properties of the weak non-compactness \( \mu(\cdot) \), we refer to [1] for example.

**Lemma 3.13** ([1]). Let \( H \subset C(I_\alpha, X) \) be a family of strong equicontinuous functions. Then \( \mu(H(I_\alpha)) = \sup \mu(H(t)) \) and the function \( t \to \mu(H(t)) \) is continuous.

Let a closed and convex \( C(x_0, \alpha) = \{ x \in C(I_\alpha, \mathbb{E}^n) | x(0) = x_0, D(x, \bar{0}) \leq D(x_0, \bar{0}) + b \} \) and let the sequence of functions \( G = \{ \bar{f}_x | x \in C(x_0, \alpha) \} \). Define the operator \( \bar{f}_x \) by the following:

\[
\bar{f}_x(t) = x_0 + \int_0^t \bar{f}(s, x(s)) ds \quad \text{or} \quad \bar{f}_x(t) = x_0 + (-1) \cdot \int_0^t \bar{f}(s, x(s)) ds
\]

for \( t \in I_\alpha, x \in C(x_0, \alpha) \).

**Theorem 3.14.** Suppose that a function \( x : I_\alpha \to \mathbb{E}^n \) is ACG*. \( \tilde{f}(\cdot, x(\cdot)) \) is (FHP)-integrable and \( \tilde{f}(t, \cdot) \) is weak continuous about first variable \( t \) and

\[
\mu(\sigma(x, [\tilde{f}(I \times X)]^r)) \leq c \cdot \mu(\sigma(x, X)), \quad 0 \leq c \cdot \alpha < 1 \quad (3.7)
\]

for each bounded subset \( X \subset \mathbb{E}^n \) and for each subinterval \( I \) of \( I_\alpha \). Assume that the set \( G = \{ \bar{f}_x | x \in C(x_0, \alpha) \} \) is
strong equicontinuous and weak uniformly ACG* on $I_\alpha$. Then there exists at least one weak solution of problem (3.5) on $I_\beta$, for some number $0 < \beta \leq \alpha$.

Proof. We will prove, in fact, the existence of a solution for the following problem:

$$x(t) = x_0 + \int_0^t \tilde{f}(s, x(s))ds, \quad t \in I_\alpha. \quad (3.8)$$

By Theorem 3.9 each solution of problem (3.8) is a solution of problem (3.5). Fix an arbitrary $b \geq 0$. By the equicontinuity of $G$, there exists a number $\beta, 0 < \beta \leq \alpha$, such that

$$D(\int_0^t \tilde{f}(s, x(s))ds, 0) \leq b$$

for all $t \in I_\alpha$ and $x \in C(x_0, \alpha)$.

Next, we prove $\tilde{f}$ is sequentially continuous. In fact, for every $t \in I_\beta$, there exist a sequence $x_n(t)$ convergent to $x(t)$ on $C(I_\beta, \mathbb{R}^n)$. That is $\tilde{f}(t, x_n(t)) \longrightarrow \tilde{f}(t, x(t))$. By the Controlled Convergence Theorem 3.4, we have

$$\lim_{n \to \infty} \int_0^t \tilde{f}(s, x_n(s))ds = \int_0^t \tilde{f}(s, x(s))ds.$$

So, $\tilde{f}_{x_n} \longrightarrow \tilde{f}_x$. That is to say $\tilde{f}$ is continuous.

Assume that $V \subset C(x_0, \beta)$ satisfies the condition $V = \overline{\text{conv}}(\tilde{f}(V) \cup \{x\})$. We shall prove $V$ is relatively compact, thus (3.6) is satisfied. In fact, let

$$\tilde{f}(V(t)) = \{\tilde{f}_x|x \in V\} = \{x_0 + \int_0^t \tilde{f}(s, x(s))ds, x \in V\}.$$

By properties of the measure of weak non-compactness and the assumption (3.7), we have

$$\mu(\sigma(x, [\tilde{f}(V(t))]^\tau)) = \mu(\sigma(x, [x_0 + \int_0^t \tilde{f}(s, x(s))ds]^\tau))$$

$$\leq \mu(\sigma(x, [\int_0^t \tilde{f}(s, x(s))ds]^\tau))$$

$$\leq \mu(\sigma(x, t \cdot \overline{\text{conv}}([\tilde{f}([0, t] \times V([0, t]))]^\tau)))$$

$$\leq t \cdot \mu(\sigma(x, [\tilde{f}([0, t] \times V([0, t]))]^\tau))$$

$$\leq \beta \cdot \mu(\sigma(x, [\tilde{f}(I_\beta \times V(I_\beta))]^\tau))$$

$$\leq \beta \cdot c \cdot \mu(\sigma(x, V(I_\beta))).$$

Hence $\mu(\sigma(x, [\tilde{f}(V(t))]^\tau)) \leq \beta \cdot c \cdot \mu(\sigma(x, V(I_\beta)))$ for each $t \in I_\beta$.

Because $V = \overline{\text{conv}}(\tilde{f}(V) \cup \{x\})$ then

$$\mu(\sigma(x, V(t))) \leq \mu(\sigma(x, [\tilde{f}(V(t))]^\tau)) \leq \beta \cdot c \cdot \mu(\sigma(x, V(I_\beta))).$$

By Lemma 3.13, we have

$$\mu(\sigma(x, V(I_\beta))) \leq \beta \cdot c \cdot \mu(\sigma(x, V(I_\beta))) \leq \alpha \cdot c \cdot \mu(\sigma(x, V(I_\beta))).$$

So, $\mu(\sigma(x, V(I_\beta))) = 0$ and $\mu(\sigma(x, V(t))) = 0$ for each $t \in I_\beta$. By Arzelá-Ascoli theorem $V$ is relatively compact in $C(I_\beta, \mathbb{R}^n)$. Using Theorem 3.12 there exists a fixed point of the operator $F$ which is a weak solution of problem (3.5).

Example 3.15. Consider the following discontinuous system

$$\begin{cases} \dot{x}(t) = \tilde{f}(t, x) + \tilde{h}(t), \\
\dot{x}(0) = x_0, \end{cases} \quad (3.9)$$
where \( \bar{f}(t, x) \) is (FHP)-integrable and \( \bar{f}(t, \cdot) \) is weak continuous about first variable \( t \), \( h(t) = \chi_{\{g(t)\}} + \bar{A} \) is a fuzzy-number-valued function and
\[
g(t) = \begin{cases} 
2t \sin \frac{1}{t^2} - \frac{2}{t} \cos \frac{1}{t^2}, & t \neq 0, \\
0, & t = 0,
\end{cases}
\]
and
\[
\bar{A}(s) = \begin{cases} 
s, & 0 \leq s \leq 1, \\
2 - s, & 1 < s \leq 2, \\
0, & \text{others}.
\end{cases}
\]

Since \( \bar{x}(t) \) is a generalized solution of the initial value problem (3.9) if and only if the integral equation
\[
\bar{x}(t) = \bar{x}_0 + \text{FHP} \int_0^t (\bar{f}(s, \bar{x}(s)) + \bar{h}(s)) \, ds,
\]
holds true. That is to say
\[
\bar{x}(t) = \bar{H}(t) + \text{FHP} \int_0^t \bar{f}(s, \bar{x}(s)) \, ds,
\]
where \( \bar{H} = \bar{x}_0 + \chi_{\{G(t)\}} + \bar{A} \cdot t \) and
\[
G(t) = \begin{cases} 
t^2 \sin \frac{1}{t^2}, & t \neq 0, \\
0, & t = 0.
\end{cases}
\]

Let \( \bar{f}(s, \bar{x}) \) to be satisfied (3.7) in Theorem 3.14 and \( H = \{ \bar{H}_x | x \in C(x_0, \alpha) \} \) is strong equicontinuous and weak uniformly \( ACG^* \) on \( I_\alpha \). Since \( \bar{x}(t) \) is a generalized solution of the initial value problem (3.9), we have \( \bar{x}(t) \) is continuous. In fact, for all \( \lambda \in [0, 1] \), \( x^-_\lambda \) and \( x^+_-\lambda \) are continuous. Therefore, for \( t_0 \in [0, a] \) and for all \( \varepsilon > 0 \), there exists \( \delta > 0 \), we have
\[
|x^-_\lambda(t) - x^-_\lambda(t_0)| < \varepsilon, \quad |x^+_\lambda(t) - x^+_\lambda(t_0)| < \varepsilon.
\]

For above \( t \), there exists a fuzzy number \( \bar{A} \) such that \( \bar{x}(t) = \bar{x}(t_0) + \bar{A} \). Then, we have \( |A^-_0| < \varepsilon \) and \( |A^+_0| < \varepsilon \). We notice that
\[
D(\bar{x}(t), \bar{x}(t_0)) = \sup_{\lambda \in [0, 1]} \max \{|x^-_\lambda(t) - x^-_\lambda(t_0)|, |x^+_\lambda(t) - x^+_\lambda(t_0)|\}
= \sup_{\lambda \in [0, 1]} \max \{|A^-_\lambda|, |A^+_\lambda|\} = \max \{|A^-_0|, |A^+_0|\} < \varepsilon.
\]

So, \( \bar{x}(t) \) is continuous on \([0, a] \). Then, by Theorem 3.14, the initial value problem (3.9) has at least one weak solution \( \bar{x}(t) \).

4. Conclusions and future works

In this paper, we study the Henstock-Pettis integral of compact convex set-valued functions and fuzzy-number-valued function and the convergence theorem of fuzzy Henstock-Pettis integrals. In addition, we deal with the Cauchy problem of discontinuous fuzzy systems involving the weak fuzzy Henstock integral in fuzzy number space. The function governing the equations is supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrability. Our result improves the result given in [3, 4, 6, 7, 20, 22, 23, 39] and [9] (where uniform continuity was required), as well as those referred therein. In the future research, we shall deals with a new derivative and Hestock-Pettis-\( \Delta \)-integral for fuzzy-number-valued functions on time scales. Also, we shall study and investigate fuzzy differential equations and fuzzy integral equations with \( \Delta_H \)-derivative and FHP-\( \Delta \)-integral on time scales.
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