On some extensions of Nadler’s fixed point theorem

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Abstract

In this paper, we give the notion of the pseudo-fixed point for multi-valued mappings which enable us to extend Nadler’s theorem and other well-known results in the literature. ©2017 All rights reserved.

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1. Introduction and preliminaries

Banach’s contraction principle (1922) was the starting point of a growing field which is the fixed point theory. The interest of this field comes from its application to several branches in mathematics, for example the resolution of nonlinear problems involving integral equations and integro-differential equations and the convergence of some iterative processes. Banach’s contraction principle has been the subject of several extensions in various types of metric spaces and using some generalized contraction conditions on the self-mappings. In 1969, Nadler [6] generalized this contraction principle to the case of multi-valued mappings. This contribution is of great importance due to the fact that this class of mappings plays a central role in applied sciences (optimization, equilibrium problems, games theory, differential and partial differential equations involving integral inclusions, ···). For a good reading concerning theses mappings, we can quote for example [1, 3, 4].

Nadler’s result is given as follows:

Theorem 1.1 ([6]). Let \((X, d)\) be a complete metric space and \(f\) be a multi-valued mapping on \(X\) such that \(f(x)\) is a nonempty closed bounded subset for any \(x \in X\). If there exists \(\alpha \in (0, 1)\) such that

\[
H(f(x), f(y)) \leq \alpha d(x, y), \quad \forall x, y \in X,
\]

then there exists \(z \in X\) such that \(z \in f(z)\) (here \(H\) is the Hausdorff metric distance defined on the family of the collection of all nonempty closed subsets of \(X\)).
Let \((X, \delta)\) be a complete metric space, \(\text{N}(X)\) denotes the collection of all nonempty subsets of \(X\), \(\text{C}(X)\) is the collection of all nonempty closed subsets of \(X\), \(\text{CB}(X)\) is the collection of all nonempty closed bounded subsets of \(X\) and \(\text{K}(X)\) is the collection of all nonempty compact subsets of \(X\).

For \(A, B \subset \text{C}(X)\), let
\[
d(A, B) = \inf\{ d(x, z) | x \in A, z \in B \},
\]
and
\[
\text{H}(A, B) = \max\{ \sup_{x \in B} d(x, A), \sup_{y \in A} d(y, B) \}.
\]

\(H\) is called the generalized Hausdorff distance induced by \(d\).

Let \((X, \delta)\) be a complete metric space and \(f\) be a multi-valued mapping on \(X\) such that \(f(x)\) is a nonempty closed bounded subset for any \(x \in X\) and let \(\{A_x, x \in X\} \subset \text{K}(X)\) be a family of compact subsets of \(X\) such that \(x \in A_x\) for all \(x \in X\). Let \(x \in X\), we denote by \(Z^x = \{ z \in X \mid A_z \cap f(x) \neq \emptyset \}\). It is easy to show that for all \(x \in X\), \(Z^x\) is nonempty since \(Z^x\) contains the nonempty subset \(\{z \in f(x)\}\). In the following assume that the family \(\{A_x, x \in X\}\) satisfies the following two assumptions:

(i) \(d(A_x, f(x)) = \inf_{z \in Z^x} \{ H(A_x, f(x) \cap A_z) \}\).

(ii) If \(x_n \rightarrow x\), then \(d(A_x, f(x)) \leq \liminf_{n \to \infty} d(A_{x_n}, f(x_n))\).

For a positive real number \(0 < b < 1\), we define the set \(\Gamma_b^x \subset X\) as follows:
\[
\Gamma_b^x = \{ z \in Z^x | b \text{H}(A_x, f(x) \cap A_z) \leq d(A_x, f(x)) \}.
\]

Thus \(\Gamma_b^x \neq \emptyset\). Indeed, if we assume that for all \(z \in Z^x\), \(d(A_x, f(x)) < b \text{H}(A_x, f(x) \cap A_z)\), it follows by (i) that \(d(A_x, f(x)) \leq b \inf_{z \in Z^x} \{ H(A_x, f(x) \cap A_z) \} < d(A_x, f(x))\) which is a contradiction.

In the following, assume that for every \(\epsilon > 0\) the set
\[
A^\epsilon_x = \{ y \in X | d(x, y) - \epsilon \leq d(A_x, f(x)) \},
\]
is nonempty.

**Definition 1.2.** Under the above notations, we say that \(f : X \rightarrow \text{N}(X)\) has a pseudo-fixed point if there exists \(x_0 \in X\) such that \(A_{x_0} \cap f(x_0) \neq \emptyset\).

**Remark 1.3.** Noting that Definition 1.2 is new and extends strictly the classical one in the case of single-valued mappings or multi-valued mappings at the same time, indeed, here \(x_0\) is not necessarily an element of the set \(A_{x_0} \cap f(x_0)\).

**Remark 1.4.** If \(A_x = \{ x \}\) for all \(x \in X\), the precedent definition is reduced to the well-known definition of fixed points concerning multi-valued mappings.

**Example 1.5.** Let \(X = [0, +\infty[\) and \(f : [0, +\infty[ \rightarrow \text{N}([0, +\infty[\) be a mapping defined by \(f(x) = [0, \frac{x}{2}]\), we denote by \(F_{\text{p}}(f)\) the set of its pseudo-fixed points. Thus

1. If \(A_x = [\frac{x}{2}, x]\), then \(F_{\text{p}}(f) = [0, +\infty[\) since \(f(x) \cap A_x = [\frac{x}{2}, x]\) for all \(x \in [0, +\infty[\).
2. If \(A_x = \{ x \}\), then \(F_{\text{p}}(f) = \{ 0 \}\) since \(f(x) \cap A_x \neq \emptyset\) is just satisfied for \(x = 0\).

**2. Main results**

**Theorem 2.1.** Let \((X, \delta)\) be a complete metric space and \(f : X \rightarrow \text{C}(X)\) a multi-valued mapping. Assume that the assumptions (i) and (ii) are satisfied. Let \(0 < b < 1\). If there exists \(0 < c < 1\) such that for all \(x \in X\) and for all \(\epsilon > 0\), there exists \(y \in A^\epsilon_x \cap \Gamma_b^x\) satisfying that
\[
d(A_y, f(y)) \leq c \text{H}(A_x, A_y \cap f(x)),
\]
then \(f\) has a pseudo-fixed point provided that \(c < b\).
Proof. For an initial point \( x_0 \in X \), there exists \( x_1 \in \Lambda_{x_0}^{\frac{1}{2}} \cap \Gamma_b^{x_0} \) such that
\[
d(A_{x_1}, f(x_1)) \leq cH(A_{x_0}, A_{x_1} \cap f(x_0)).
\]
Thus, the fact that the set \( \Lambda_{x_0}^{\frac{1}{2}} \) is nonempty and by assumptions, there exists \( x_2 \in \Lambda_{x_1}^{\frac{1}{2}} \cap \Gamma_b^{x_1} \) for which we have the following inequalities
\[
d(x_1, x_2) - \frac{1}{2} \leq d(A_{x_1}, f(x_1)) \leq cH(A_{x_0}, A_{x_1} \cap f(x_0)) \leq \frac{c}{b} d(A_{x_0}, f(x_0)),
\]
and consequently by induction on \( k \), we can deduce that there exists \( x_k \in \Lambda_{x_{k-1}}^{\frac{1}{2}} \cap \Gamma_b^{x_k} \) such that
\[
d(x_k, x_{k+1}) - \frac{1}{2^k} \leq d(A_{x_k}, f(x_k)) \leq cH(A_{x_{k-1}}, A_{x_k} \cap f(x_{k-1})) \leq \frac{c^k}{b^k} d(A_{x_0}, f(x_0)).
\]
It follows that
\[
\sum_{k=1}^{\infty} d(x_k, x_{k+1}) \leq 1 + \left( \sum_{k=1}^{\infty} \frac{c^k}{b^k} \right) d(A_{x_0}, f(x_0)).
\]
Hence the series \( \sum_{k=1}^{\infty} d(x_k, x_{k+1}) \) is convergent and consequently by Cauchy’s criterion, the sequence \( \{x_n\}_n \) is a Cauchy sequence in \( X \). The completeness of \( X \) implies the existence of \( x \in X \) such that \( x_n \rightarrow x \).
Moreover, it is easy to observe that \( \lim_{k \rightarrow \infty} d(A_{x_k}, f(x_k)) = 0 \). Then by (\( \text{(i)} \)), we get that
\[
d(A_x, f(x)) = \liminf_{k \rightarrow \infty} d(A_{x_k}, f(x_k)) = 0.
\]
Hence \( d(A_x, f(x)) = 0 \). The fact that \( f(x) \) is closed and \( A_x \) is compact implies that \( A_x \cap f(x) \neq \emptyset \) which gives the result.

Corollary 2.2 (see [2, Theorem 3.1]). Let \((X, d)\) be a complete metric space, \( f : X \rightarrow C(X) \) a multi-valued mapping. Let \( 0 < b < 1 \). If there exists \( 0 < c < 1 \) such that for any \( x \in X \), there exists \( y \in \Gamma_b^x \) satisfying
\[
d(y, f(y)) \leq cd(x, y),
\]
and the function \( h(x) = d(x, f(x)) \) is lower semi-continuous. Then there exists \( x_0 \in X \) such that \( x_0 \in f(x_0) \) provided that \( c < b \).

Proof. This result follows from Theorem 2.1 by taking \( A_x = \{x\} \) for all \( x \in X \). Indeed, in this case we have \( z^x = f(x) \) and \( H(A_x, A_y \cap f(x)) = d(x, y) \) for any \( x, y \in X \). Moreover, we have \( A_x^c \neq \emptyset \) for all \( x \in X \) and \( \epsilon > 0 \) by the definition of the infimum and \( A_x^c \subseteq \Gamma_b^x \) (\( \epsilon \leq \frac{1-bd(x, f(x))}{b} \)). On the other hand, the fact that the family \( A_x = \{x\}, x \in X \) satisfies (\( \text{(i)} \)) follows from the lower semi-continuity of the function \( h(x) = d(x, f(x)) \) [2].

Corollary 2.3 (see [2, Theorem 3.1]). Let \((X, d)\) be a complete metric space, \( f : X \rightarrow C(X) \) a multi-valued mapping. If there exists \( 0 < c < 1 \) such that for any \( x, y \in X \), we have
\[
H(f(x), f(y)) \leq cd(x, y).
\]
Then there exists \( x_0 \in X \) such that \( x_0 \in f(x_0) \).

Proof. By [2, Remark 1(2)], this result follows directly from Corollary 2.2. Hence it can be seen as a particular case of our general framework given by Theorem 2.1.
Example 2.4. Let \( X = \{ \frac{1}{2}, \frac{1}{4}, \cdots \} \cup \{0, 1\} \), \( d(x, y) = |x - y| \), for \( x, y \in X \). It is easy to observe that \((X, d)\) is a complete metric space. Let \( f : X \to C(X) \) defined as follows:

\[
f(x) = \begin{cases} 
\frac{1}{2n+1}, & x = \frac{1}{2n}, \\
\frac{1}{2}, & x = 0,
\end{cases} \quad n = 0, 1, 2, \ldots,
\]

and let \( A_x = \{ x \} \) for any \( x \in X \). It is easy to observe that \( A_x \cap f(x) = \emptyset \) for any \( x \in X \). Hence necessarily \( f \) is not a contraction mapping. This fact can be deduced by observing that \( f \) is not continuous at 0.

2.1. Extension to the Hardy-Rogers mappings

We start this subsection by giving the following main result which can be regarded as an extension of Hardy-Rogers result and consequently Nadler’s result.

Theorem 2.5. Let \((X, d)\) be a complete metric space, \( f : X \to CB(X) \) a multi-valued mapping. Let \( \{A_x, x \in X\} \subset K(X) \), be a family of compact subsets of \( X \) such that \( x \in A_x \) for all \( x \in X \) satisfying that:

1. \( \{A_x, x \in X\} \) is closed in \( (CB(X), H) \).

2. If \( A, B \in CB(X) \), then for all \( x \) such that \( A_x \cap A \neq \emptyset \) and for all \( \epsilon > 0 \), there exists \( A_y \subset B \) such that

\[
H(A_x, A_y) \leq H(A, B) + \epsilon.
\]

3. If \( A_{x_n} \to A_x \) in \( (CB(X), H) \), then there exists \( \mu > 0 \) such that \( d(A_x, f(x)) \leq \mu \liminf_{n \to \infty} d(A_{x_n}, f(x_n)) \).

If \( f \) satisfies that

\[
H(f(x), f(y)) \leq \alpha_1 H(A_{x}, A_y) + \alpha_2 d(A_x, f(x)) + d(A_y, f(y))
\]

with \( \alpha_1 + 2\alpha_2 + 2\alpha_3 < 1 \), then there exists \( z \in X \) such that \( z \) is a pseudo-fixed point for \( f \).

Proof. Let \( x_0 \in X \) and \( A_{x_1} \) such that \( A_{x_1} \cap f(x_0) \neq \emptyset \). We define \( \gamma = \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - (\alpha_1 + \alpha_3)} \). If \( \gamma = 0 \), then the result is trivial. Now, assume that \( \gamma > 0 \), thus \( 0 < \gamma < 1 \). Following the assumption (11), there exists \( A_{x_2} \) such that \( A_{x_2} \subseteq f(x_1) \) with

\[
H(A_{x_1}, A_{x_2}) \leq H(f(x_0), f(x_1)) + \gamma.
\]

By (11), we obtain a family of compact subsets \( \{A_{x_n}\}_n \) such that \( A_{x_n} \subseteq f(x_{n-1}) \) and

\[
H(A_{x_n}, A_{x_{n+1}}) \leq H(f(x_{n-1}), f(x_n)) + \gamma^n.
\]

Hence, we have

\[
H(A_{x_n}, A_{x_{n+1}}) \leq H(f(x_{n-1}), f(x_n)) + \gamma^n \leq \alpha_1 H(A_{x_{n-1}}, A_{x_n}) + \alpha_2 d(A_{x_{n-1}}, f(x_n)) + d(A_{x_{n-1}}, f(x_{n+1})) + \alpha_3 d(A_{x_n}, f(x_{n-1})) + \gamma^n, \quad \forall n \in \mathbb{N}.
\]

Also by (11), we get

\[
H(A_{x_n}, A_{x_{n+1}}) \leq \alpha_1 H(A_{x_{n-1}}, A_{x_n}) + \alpha_2 H(A_{x_{n-1}}, A_{x_{n+1}}) + \alpha_3 H(A_{x_{n-1}}, A_{x_n}) + \gamma^n, \quad \forall n \in \mathbb{N}.
\]
Consequently,
\[ H(A_{x_n}, A_{x_{n+1}}) \leq \gamma H(A_{x_{n-1}}, A_{x_n}) + \frac{\gamma^n}{1 - (\alpha_2 + \alpha_3)}, \quad \forall n \in \mathbb{N}. \]

Thus,
\[ H(A_{x_n}, A_{x_{n+1}}) \leq \gamma^n H(A_{x_0}, A_{x_1}) + \frac{n \gamma^n}{1 - (\alpha_2 + \alpha_3)}, \quad \forall n \in \mathbb{N}. \]

Since \( \gamma < 1 \), then \( \sum_{n=1}^{\infty} H(A_{x_n}, A_{x_{n+1}}) < \infty \). This shows that the sequence \( (A_{x_n})_n \) is a Cauchy sequence in \((\text{CB}(X), H)\). The completeness of \((\text{CB}(X), H)\) (see [5]) and the closeness of the set \( \{A_x, x \in X\} \) implies that there exists \( \bar{x} \in X \) such that \( A_{x_n} \to A_{\bar{x}} \) in \((\text{CB}(X), H)\). To prove that \( A_{\bar{x}} \cap f(\bar{x}) \neq \emptyset \), by the assertion (11), it suffices to get that \( \liminf_{n \to \infty} d(A_{x_n}, f(x_n)) = 0 \). We have
\[ d(A_{x_n}, f(x_n)) \leq H(A_{x_n}, A_{x_{n+1}}), \quad \forall n \in \mathbb{N}, \]
which gives that
\[ \liminf_{n \to \infty} d(A_{x_n}, f(x_n)) \leq \liminf_{n \to \infty} H(A_{x_n}, A_{x_{n+1}}) = 0. \]

By the condition (11) and the fact that \( A_{x_n} \to A_{\bar{x}} \) in \((\text{CB}(X), H)\), it follows that \( d(A_{\bar{x}}, f(\bar{x})) = 0 \) which implies that \( A_{\bar{x}} \cap f(\bar{x}) \neq \emptyset \) and achieves the result. \( \square \)

**Proposition 2.6.** Let \((X, d)\) be a complete metric space and let \( f : X \to \text{CB}(X) \) a multi-valued mapping with the following condition:
\[ H(f(x), f(y)) \leq \alpha_1(d(x, f(x)) + d(y, f(y))) + \alpha_2(d(x, f(y)) + d(y, f(x))) + \alpha_3 d(x, y), \quad \forall x, y \in X, \]
with \( 2\alpha_1 + 2\alpha_2 + \alpha_3 < 1 \). Then, there exists \( z \in X \) such that \( z \in f(z) \).

**Proof.** This result follows immediately from Theorem 2.5 by taking \( A_x = \{x\} \) for all \( x \in X \). In this case, it is easy to check that assumption (i) is satisfied, for the assumption (ii), (see [6]) while the assumption (iii) is deduced from the following:

If \( x_n \to x \) then we have for all \( n \in \mathbb{N} \)
\[ d(x, f(x)) \leq d(x, x_n) + d(x_n, f(x_n)) + H(f(x_n), f(x)) \leq d(x, x_n) + d(x_n, f(x_n)) + \alpha_1 d(x_n, f(x_n)) + \alpha_1 d(x, f(x)) + \alpha_2 d(x_n, f(x_n)) + \alpha_2 d(x, x_n) \leq d(x, x_n) + (1 + \alpha_1) d(x_n, f(x_n)) + \alpha_1 d(x, f(x)) + \alpha_2 d(x_n, x) + \alpha_2 d(x, f(x)) + \alpha_2 d(x_n, x_n) + \alpha_2 d(x, x_n) + \alpha_3 d(x, x_n). \]

Hence
\[ (1 - (\alpha_1 + \alpha_2)) d(x, f(x)) \leq (1 + 2\alpha_2 + \alpha_3) d(x, x_n) + (1 + \alpha_1 + \alpha_2) d(x_n, f(x_n)). \]

By taking the limit \( n \to +\infty \), we get
\[ d(x, f(x)) \leq \liminf_{n \to \infty} d(x_n, f(x_n)). \]

Thus (iii) is satisfied by taking \( \mu = \frac{1 + \alpha_1 + \alpha_2}{1 - (\alpha_1 + \alpha_2)} \).

**Corollary 2.7** (see [6]). Let \((X, d)\) be a complete metric space and let \( f : X \to \text{CB}(X) \) a multi-valued mapping satisfying that
\[ H(f(x), f(y)) \leq \alpha d(x, y), \]
for all \( x, y \in X \), where \( 0 \leq \alpha < 1 \). Then there exists \( z \in X \) such that \( z \in f(z) \).
Proof. The result follows from Proposition 2.6 by taking \( \alpha_1 = \alpha_2 = 0 \).

By the same principle, we can also derive the following results:

Corollary 2.8 (see [7, 8]). Let \((X, d)\) be a complete metric space and let \(f\) be a mapping from \((X, d)\) into \((\text{CB}(X), H)\) satisfying

\[
H(f(x), f(y)) \leq \alpha(d(x, f(x)) + d(y, f(y)))
\]

for all \(x, y \in X\), where \(0 \leq \alpha < \frac{1}{2}\). Then there exists \(z \in X\) such that \(z \in f(z)\).

Corollary 2.9. Let \((X, d)\) be a complete metric space and let \(f\) be a mapping from \((X, d)\) into \((\text{CB}(X), H)\) satisfying

\[
H(f(x), f(y)) \leq \alpha_1 d(x, y) + \alpha_2 (d(x, f(x)) + d(y, f(y)))
\]

for all \(x, y \in X\), where \(0 \leq \alpha < \frac{1}{2}\). Then there exists \(z \in X\) such that \(z \in f(z)\).

Corollary 2.10. Let \((X, d)\) be a complete metric space and let \(f\) be a mapping from \((X, d)\) into \((\text{CB}(X), H)\) satisfying

\[
H(f(x), f(y)) \leq \alpha_1 d(x, y) + \alpha_2 (d(x, f(x)) + d(y, f(y)))
\]

for all \(x, y \in X\), where \(\alpha_1 + 2\alpha_2 < 1\). Then there exists \(z \in X\) such that \(z \in f(z)\).

For any nonempty subsets \(A, B\) of \(X\), we define

\[
\delta(A, B) = \sup\{d(a, b) | a \in A, b \in B\}.
\]

In the following, we denote by \(g : X \to \text{CB}(X)\) for which \(g(x) = A_y\) \((A_y \subseteq f(x)\) given in the assumption (ii) of Theorem 2.5). As an extension of Reich and Iseki results (see [3, 8]), we have:

Lemma 2.11. Let \((X, d)\) be a complete metric space. If \(f : X \to \text{CB}(X)\) is a multi-valued function for which the assumptions (i), (ii) and (iii) of Theorem 2.5 are satisfied and

\[
\delta(f(x), f(y)) \leq \alpha_1 (d(A_x, f(x)) + d(A_y, f(y))) + \alpha_2 (d(A_x, f(y)) + d(A_y, f(x)))
\]

\[
+ \alpha_3 H(A_x, A_y)
\]

for every \(x, y \in X\), where \(\alpha_1, \alpha_2, \alpha_3\) are nonnegative and \(2\alpha_1 + 4\alpha_2 + \alpha_3 < 1\), then \(g : X \to \text{CB}(X)\) satisfies a contraction condition of the form (2.1).

Proof. By the definition of \(g\), since \(g(x) \subseteq f(x)\) and \(g(y) \subseteq f(y)\), we have

\[
H(g(x), g(y)) \leq \delta(f(x), f(y))
\]

\[
\leq \alpha_1 (d(A_x, f(x)) + d(A_y, f(y))) + \alpha_2 (d(A_x, f(y)) + d(A_y, f(x))) + \alpha_3 H(A_x, A_y)
\]

\[
\leq \alpha_1 (d(A_x, g(x)) + d(A_y, g(y))) + \alpha_2 (d(A_x, g(y)) + d(A_y, g(x))) + \alpha_3 H(A_x, A_y)
\]

\[
+ \alpha_3 H(A_x, A_y)
\]

The fact that \(2\alpha_1 + 4\alpha_2 + \alpha_3 < 1\) gives the result.

\[
\square
\]

Theorem 2.12. Let \((X, d)\) be a complete metric space. Under the assumptions of Lemma 2.11, if the mapping \(g\) satisfies (iii) of Theorem 2.5, then there exists \(\bar{x} \in X\) such that \(\bar{x}\) is a pseudo-fixed point for \(f\).

Proof. By taking account of Theorem 2.5, the mapping \(g\) has a pseudo-fixed point \(\bar{x} \in X\), then \(A_{\bar{x}} \cap g(\bar{x}) \neq \emptyset\). But by definition, we have \(g(\bar{x}) \subseteq f(\bar{x})\), hence we obtain that \(A_{\bar{x}} \cap f(\bar{x}) \neq \emptyset\) which gives the result.

\[
\square
\]
Remark 2.13. Noting that the uniqueness of fixed points is not ensured in general in the case of multi-valued contraction or generalized contraction mappings, however we have the following result of existence and uniqueness.

**Corollary 2.14.** Let \((X, d)\) be a complete metric space. If \(f : X \rightarrow \text{CB}(X)\) is a multi-valued mapping satisfying that
\[
\delta(f(x), f(y)) \leq \alpha_1(d(x, f(x)) + d(y, f(y))) + \alpha_2(d(x, f(y)) + d(y, f(x))) + \alpha_3d(x, y),
\]
for every \(x, y\) in \(X\), where \(\alpha_1, \alpha_2, \alpha_3\) are nonnegative and \(2\alpha_1 + 4\alpha_2 + \alpha_3 < 1\), then there exists a unique point \(\bar{x} \in X\) such that \(f(\bar{x}) = \{\bar{x}\}\).

**Proof.** The existence can be deduced from Theorem 2.12 by taking \(A_x = \{x\}, x \in X\) and the properties of the single-valued mapping \(g : X \rightarrow \text{CB}(X)\) in this case. For the uniqueness see [3]. □

**References**