A mass-conservative characteristic splitting mixed element method for saltwater intrusion problem

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Abstract

A new characteristic mixed finite element method is developed for solving saltwater intrusion problem. In this algorithm, the splitting mixed finite element (SMFE) method is applied for solving the parabolic-type water head equation, and the mass-conservative characteristic (MCC) finite element method is applied for solving the convection-diffusion type concentration equation. The application of the splitting mixed element method results in a symmetric positive definite coefficient matrix of the mixed element system and separating the flux equation from the water head equation. While the mass-conservative characteristic finite element method does well in handling convection-dominant diffusion problem and keeps mass balance. The convergence of this method is considered and the optimal $L^2$-norm error estimate is also derived. ©2017 All rights reserved.

Keywords: Method of characteristics, mass-conservative, splitting mixed finite element, error estimate, saltwater intrusion problem.


1. Introduction

In this paper, we will consider the following coupled system composed of the water head equation and the concentration (of Cl$^-$) equation as a mathematical model of seawater intrusion problems (see [2, 19]):

\begin{equation}
\begin{aligned}
(a) \quad & S_s \frac{\partial H}{\partial t} - \nabla \cdot (\kappa (\nabla H - \eta \mathbf{e}_3)) = -\phi \eta \frac{\partial c}{\partial t} + \frac{\rho}{\rho_0} q,
(b) \quad & \phi \frac{\rho_0}{\rho} \frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u} c - \phi D \nabla c) = \tilde{c} q,
\end{aligned}
\end{equation}

with the initial-boundary conditions:

\begin{equation}
\begin{aligned}
\mathbf{u} \cdot \mathbf{n} &= 0, & D \nabla c \cdot \mathbf{n} &= 0, & \text{on } \partial \Omega,
H(x, 0) &= H^0(x), & c(x, 0) &= c^0(x), & x \in \Omega,
\end{aligned}
\end{equation}

where $\Omega$ is a convex bounded domain in $\mathbb{R}^3$ with the boundary $\partial \Omega$, $S_s$ is the specific retention, $H = \frac{p}{\rho g} - z$.

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The structure of this paper is organized as follows: First, we present our method for saltwater intrusion problems. In this paper, in order to solve saltwater intrusion problems, the splitting mixed element method is used. The splitting mixed element method has three advantages: 1. it can obtain more accurate approximation value to the flux like classical mixed element methods; 2. the coefficient matrix of the new mixed system is symmetric positive definite so that LBB condition required by classical mixed element methods is not necessary; 3. the flux equation is separated from the water head equation, so that no coupled problem is solved. While the mass-conservative characteristic finite element method not only keeps the advantages of traditional MMOC and does well in handling convection-dominant diffusion problem, but also maintains the global mass conservation. Here, the convergence of this combined method is considered and an optimal L2-norm error estimate is also derived.
problem and give the convergence theorem in Section 2. And then, we give some preliminaries, which will be used to prove our convergence theorem in Section 3. Next, we consider the error estimates for the concentration and the flux in Sections 4 and 5, respectively. Finally, we complete the proof of the convergence theorem in Section 6.

2. Formulation of the method

Throughout this paper, the notations of standard Sobolev spaces are adopted as in [1]. Let \((\cdot, \cdot)\) be the inner product in \(L^2(\Omega)\). Introduce the space \(H(\text{div}; \Omega) = \{ v \in [L^2(\Omega)]^3 ; \nabla \cdot v \in L^2(\Omega) \}, \) \(V = \{ v \in H(\text{div}; \Omega) ; v \cdot v = 0 \) on \(\partial \Omega\) and \(W = L^2(\Omega)\). For convenience of analysis, we assume that the problem (1.1) is \(\Omega\)-periodic, i.e., all functions will be assumed to be spatially \(\Omega\)-periodic throughout the rest of this paper. This assumption is physically reasonable, because no-flow condition (1.2) is generally treated by reflection, and boundary effects are of less interest than interior flow patterns.

2.1. The MCC for the concentration

Define the differentiation along the characteristic curves of the transport \(\frac{\phi}{\beta} \frac{\partial}{\partial t} + u \cdot \nabla,\)

\[
\psi(x, c, u) \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \frac{\phi}{\beta} + u \cdot \nabla,
\]

where \(\beta = \rho(c)/\rho_0, \psi(x, c, u) = \sqrt{\phi^2/\beta^2(c) + |u|^2}.\) Note that the characteristic direction \(\tau\) depends on \(x\), the concentration \(c\), and Darcy velocity \(u\), which vary in space and time. It follows easily that the concentration equation can be rewritten in the equivalent form

\[
\frac{\partial c}{\partial t} + \nabla \cdot uc - \nabla \cdot (\phi D \nabla c) = \tilde{c}q.
\] (2.1)

Define the time partition for the concentration \(0 := t^c_0 < t^c_1 < \cdots < t^c_n < \cdots < t^c_{N-1} < t^c_N := T, \) with \(\Delta t_n^c := t^c_n - t^c_{n-1}.\) The characteristic derivative is approximated by

\[
\psi \frac{\partial c}{\partial t} \approx \psi \frac{c^n - c^{n-1} \circ X^n}{\sqrt{(X^n - x)^2 + (t^n - t^{n-1})^2}} = \frac{\phi}{\beta^{n-1}} \frac{c^n - c^{n-1} \circ X^n}{\Delta t_n^c},
\]

where

\[
c^{n-1} \circ X^n = c^{n-1}(X^n), \quad X^n(x) = x - \frac{u^{n-1} \beta^{n-1}}{\phi} \Delta t_n^c \quad \beta^{n-1} = \frac{\rho(c^{n-1})}{\rho_0}.
\]

And then the traditional continuous-in-space characteristic procedure of the concentration equation (2.1) can be given by

\[
\frac{\phi}{\beta^{n-1}} \frac{c^n - c^{n-1} \circ X^n}{\Delta t_n^c} + \nabla \cdot u^n c^n - \nabla \cdot (\phi D \nabla c^n) = \tilde{c}^n q^n.
\]

As we know, this procedure can not keep the mass balance. In [12], a new mass-conservative characteristic algorithm was established. By use of the similar technique, the continuous-in-space MCC procedure of the concentration equation (2.1) can be written as follows

\[
\frac{1}{\Delta t_n^c} \left( \frac{\phi}{\beta^{n-1}} c^n - \left( \frac{\phi}{\beta^{n-1}} c^{n-1} \circ X^n \delta^n \right) - \nabla \cdot (\phi D \nabla c^n) = \tilde{c}^n q^n, \right.
\]

where \(\delta^n\) is the Jacobian of the transformation \(X^n\), that is,

\[
\delta^n = \det \left( \frac{\partial X^n}{\partial x} \right) = \det \left( \delta_{ij} - \Delta t_n^c \frac{\partial u^n_i}{\partial x_j} \right).
\]

Let \(T_{h_c}\) be one quasi-regular finite element partition of the domain \(\Omega\), such that the elements in the partitions have the diameters bounded by \(h_c\). Let \(M_{h_c} \subset H^1(\Omega)\) be \(k\)-degree polynomial finite element spaces defined on the partition \(T_{h_c}\). The MCC finite element procedure can be given as follows:
**MCC Scheme.** For given \( u_h^{n-1} \in \mathcal{V}_h \), seek by \( c_h^n \in \mathcal{M}_h \) such that

\[
\frac{1}{\Delta t_h} \left( \frac{\phi c_h^n}{\beta_h^-} - \left( \frac{\phi c_h^{n-1}}{\beta_h^-} \right) \circ X^n \delta^n, z_h \right) + \left( \phi D \nabla c_h^n, \nabla z_h \right) = \left( \tilde{c}_h^n q^n, z_h \right), \quad \forall z_h \in \mathcal{M}_h,
\]

where \( \beta_h^-= \rho(c_h^{n-1})/\rho_0 \), \( X^n(x) = x - \Delta t_h u_h^{n-1} \beta_h^-/\phi \).

As in [12], we will show the mass conservation of the MCC Scheme.

As we know, the water head equation is a parabolic type equation. Now we deal it with the splitting positive definite mixed finite element method. Define the flux \( \sigma \) as follows:

\[
\sigma = -\kappa (\nabla H - \eta e_3) = \frac{g \rho^2}{\rho_0 \mu} \cdot u.
\]

So we have \( u = a(c) \sigma, a(c) = \rho_0 \mu / g \rho^2 \). A mixed weak form of the system (1.1)-(a) is given by:

\[
\begin{align*}
(a) \quad & \frac{\partial H}{\partial t}(w) + (B \nabla \cdot \sigma, w) = (B \beta q, w) - (B \phi \eta \frac{\partial c}{\partial t}, w), \quad \forall w \in \mathcal{W}, \\
(b) \quad & (a(c) \sigma, v) - (H, \nabla \cdot v) = -(\eta \sigma e_3, v), \quad \forall v \in \mathcal{V},
\end{align*}
\]

where \( a(c) = 1/\kappa \) and \( B = 1/S_\sigma \). From (2.3)-(b) we derive

\[
\frac{\partial}{\partial t}(a(c) \sigma), v) - (\frac{\partial H}{\partial t}, \nabla \cdot v) = -\left( \eta \frac{\partial c}{\partial t} e_3, v \right).
\]

Taking \( w = \nabla \cdot v \) in (2.3)-(a) and substituting it into (2.4), we get the mixed system

\[
\begin{align*}
(a) \quad & \frac{\partial}{\partial t}(a(c) \sigma), v) + (B \nabla \cdot \sigma, \nabla \cdot v) = (B \beta q, \nabla \cdot v) - (B \phi \eta \frac{\partial c}{\partial t}, \nabla \cdot v) - (\eta \frac{\partial c}{\partial t} e_3, v), \quad \forall v \in \mathcal{V}, \\
(b) \quad & \frac{\partial H}{\partial t}(w) = (B [\beta q - \nabla \cdot \sigma], w) - (B \phi \eta \frac{\partial c}{\partial t}, w), \quad \forall w \in \mathcal{W}.
\end{align*}
\]

From system (2.5) we can see that the flux equation is separated from the water head equation and then the water head function \( H \), if required, can be obtained from (2.5)-(b) straightly.

We define a temporal partition on the time interval \([0, T]\) for the head water and flux grid by \( 0 = t_0 < t_1 < \cdots < t_m < \cdots < t_{M-1} < t_M := T \), with \( \Delta t_m := t_m - t_{m-1} \). Each water head step is also
a concentration step, i.e., for each \( m \) there exists \( n \) such that \( t_n^m = t_m^H \). Let \( \mathcal{T}_{h_\sigma}, \mathcal{T}_{h_\tau} \) be two families of quasi-regular finite element partitions of the domain \( \Omega \) which may be the same one or not, such that the elements in the partitions have the diameters bounded by \( h_\sigma, h_\tau \), respectively. Let \( \mathcal{V}_h \subset \mathcal{V}, \mathcal{W}_h \subset \mathcal{W} \) be \( r \) and \( l \)-degree polynomial finite element spaces defined on the partitions \( \mathcal{T}_{h_\sigma} \) and \( \mathcal{T}_{h_\tau} \), respectively. A splitting mixed element procedure for water head and flux can be given as follows:

**SMFE Scheme.** Seek \( (\sigma^m_h, H^m_h) \subset \mathcal{V}_h \times \mathcal{W}_h \) for given concentration \( c_h \in \mathcal{M}_h \) such that

\[
(a) \quad \left( \frac{\alpha(c^m_h)\sigma^m_h - \alpha(c^{m-1}_h)\sigma^{m-1}_h}{\Delta t^m_h}, v_h \right) + (B \nabla \cdot \sigma^m_h, \nabla \cdot v_h) = (B\beta^m_h q^m - \nabla \cdot \sigma^m_h, v_h) - (B\phi c^m_h - c^{m-1}_h e_3, v_h), \quad \forall v_h \in \mathcal{V}_h,
\]

\[
(b) \quad \left( \frac{H^m_h - H^{m-1}_h}{\Delta t^m_h}, w_h \right) = (B\beta^m_h q^m - \nabla \cdot \sigma^m_h, w_h) - (B\phi c^m_h - c^{m-1}_h e_3, w_h), \quad \forall w_h \in \mathcal{W}_h.
\]

**2.3. The combined approximation procedure**

Next, we will present a new mass-conservative characteristic splitting mixed finite element (MCC-SMFE) method for solving saltwater intrusion problem. For convenience, we define a uniform time partition: \( 0 = t_0 < t_1 < \cdots < t_n = n\Delta t < \cdots < t_{n-1} < t_N := T \), with \( \Delta t := t_n - t_{n-1} \). Combined the method of mass-conservative characteristics with the splitting mixed element procedure, a new numerical method can be established:

**MCC-SMFE Algorithm.** Given an initial approximation \((c^0_h, H^0_h, \sigma^0_h) = (P_h c^0, Q_h H^0, \Pi_h \sigma^0) \in \mathcal{M}_h \times \mathcal{W}_h \times \mathcal{V}_h, \) for \( n = 1, 2, \ldots, N \), seek \((c^n_h, \sigma^n_h, H^n_h) \in \mathcal{M}_h \times \mathcal{V}_h \times \mathcal{W}_h \) such that

\[
(a) \quad \left( \frac{1}{\Delta t} \left( \frac{\phi c^n_h - \phi c^{n-1}_h}{\beta_h^{n-1}}, X^n \delta^n, z_h \right) + (B \nabla c^n_h, \nabla z_h) = (\bar{c} q^n, z_h), \quad \forall z_h \in \mathcal{M}_h,
\]

\[
(b) \quad \left( \frac{\alpha(c^n_h)\sigma^n_h - \alpha(c^{n-1}_h)\sigma^{n-1}_h}{\Delta t}, v_h \right) + (B \nabla \cdot \sigma^n_h, \nabla \cdot v_h) = (B\beta^n_h q^n - \nabla \cdot \sigma^n_h, v_h) - (B\phi c^n_h - c^{n-1}_h e_3, v_h), \quad \forall v_h \in \mathcal{V}_h,
\]

\[
(c) \quad \left( \frac{H^n_h - H^{n-1}_h}{\Delta t}, w_h \right) = (B\beta^n_h q^n - \nabla \cdot \sigma^n_h, w_h) - (B\phi c^n_h - c^{n-1}_h e_3, w_h), \quad \forall w_h \in \mathcal{W}_h,
\]

where the definitions of the projection operators \( P_h, \Pi_h \), and \( Q_h \) can be found in Section 3.

For convenience of analysis, we assume that \( K \) and \( \varepsilon \) indicate a generic constant and a small positive constant independent of mesh parameters \( h_\sigma, h_\tau, h_c \), and time increment \( \Delta t \), which may be different at their occurrences. We assume that the diffusion matrix \( D \) is independent of the concentration \( c \) as in \([11, 19]\), and make the following hypotheses

\[
0 < \phi_s \leq \phi \leq \phi^*, \quad 0 < D_s \leq D \leq D^*, \quad 0 < S_s \leq S \leq S^*,
\]

\[
0 < \alpha_s \leq \alpha \leq \alpha^*, \quad 0 < \beta_s \leq \beta \leq \beta^*, \quad 0 < \alpha_s \leq \alpha \leq \alpha^*.
\]

\[
\frac{\partial \alpha(c)}{\partial c} + \frac{\partial \beta(c)}{\partial c} + \frac{\partial \alpha(c)}{\partial c} + \frac{\partial^2 \alpha(c)}{\partial c^2} \leq K^*.
\]

Moreover, we also assume the regularities of the solution of (1.1)-(1.2) as follows:

\[
c \in L^\infty(H^{k+1}) \cap L^2(W^1), \quad \frac{\partial c}{\partial t} \in L^2(H^{k+1}) \cap L^\infty(L^2),
\]

\[
\frac{\partial^2 c}{\partial t^2} \in L^2(L^2), \quad H \in L^\infty(H^{l+1}) \cap H^2(L^2),
\]

\[
\sigma \in L^\infty(H^{r+1}) \cap W^1, \quad \frac{\partial \sigma}{\partial t} \in L^2(H^{r+1}) \cap L^\infty(L^2), \quad \frac{\partial^2 \sigma}{\partial t^2} \in L^2(L^2).
\]
For MCC-SMFE Algorithm, we have the following main result.

**Theorem 2.1.** Assume that the hypotheses (2.6) hold and the solution of system (1.1)-(1.2) has the regular properties (2.7). If the mesh parameters $h_c$, $h_\sigma$, and $\Delta t$ satisfy the relations

$$
\Delta t \leq o(h_c^3) = o(h_\sigma^3),
$$

then there hold the priori error estimates

$$
(a) \quad \max_n \|c^n - c_h^n\|_{L^2} + \max_n \|\sigma^n - \sigma_h^n\|_{L^2} \leq K \left\{ h_c^{k+1} + h_\sigma^{r+1} + h_\sigma^{r_1+1} + \Delta t \right\},
$$

$$
(b) \quad \max_n \|H^n - H_h^n\|_{L^2} \leq K \left\{ h_c^{k+1} + h_\sigma^{r+1} + h_\sigma^{r_1} + h_\sigma^{l+1} + \Delta t \right\},
$$

where $K$ is a constant independent of the parameters $\Delta t$, $h_H$, $h_\sigma$, $h_c$; $r$, $k > 0$, $l \geq 0$ denote some integers, $r_1 = r$ in cases of BDDM, BDM, and BDFM elements, or $r_1 = r + 1$ in cases of RT and Nedelec elements.

3. Some preliminaries

We assume that finite element spaces $V_h$ and $W_h$ have the approximate properties (see [5]) that there exist some integers $r$, $r_1 > 0$ and $l \geq 0$, such that, for $1 \leq q \leq \infty$,

$$
\inf_{v_h \in V_h} \|v - v_h\|_q \leq K_1 h_\sigma^{r+1} \|v\|_{W^{r+1,q}}, \quad \forall v \in H(div; \Omega) \cap W^{r+1,q}(\Omega),
$$

$$
\inf_{v_h \in V_h} \|\nabla \cdot (v - v_h)\|_q \leq K_1 h_\sigma^{r_1} \|\nabla \cdot v\|_{W^{r_1,q}}, \quad \forall v \in H(div; \Omega) \cap W^{r_1+1,q}(\Omega),
$$

$$
\inf_{w_h \in W_h} \|w - w_h\|_q \leq K_1 h_\sigma^{l+1} \|w\|_{W^{l+1,q}}, \quad \forall w \in L^2(\Omega) \cap W^{l+1,q}(\Omega).
$$

It is well-known that, in any one of the classical mixed finite element spaces, there exists an operator $\Pi_h$ from $V$ onto $V_h$, see [5], such that, for any $1 \leq q \leq +\infty$,

$$
(a) \quad (\nabla \cdot (v - \Pi_h v), \nabla \cdot v_h) = 0, \quad \forall v_h \in V_h,
$$

$$
(b) \quad \|v - \Pi_h v\|_q \leq K h_\sigma^{r+1} \|v\|_{W^{r+1,q}},
$$

$$
(c) \quad \|\nabla \cdot (v - \Pi_h v)\|_q \leq K h_\sigma^{r_1} \|\nabla \cdot v\|_{W^{r_1,q}}.
$$

And we define a projection operator $P_h$ from $H^1(\Omega)$ onto $M_h$ such that, for all $z_h \in M_h$ and $c \in H^1$,

$$
(\phi D\nabla c, \nabla z_h) = (\phi D\nabla P_h c, \nabla z_h).
$$

By use of the inverse property and approximate properties of the finite element space $M_h$ (see [5]), the following error bounds were given: for some integer $k > 0$,

$$
(a) \quad \|c - P_h c\|_{L^2} + h_c \|\nabla(c - P_h c)\|_{L^2} \leq K h_c^{k+1} \|c\|_{H^{k+1}},
$$

$$
(b) \quad \|\nabla P_h c\|_{L^\infty} \leq K(c) < +\infty,
$$

$$
(c) \quad \|\frac{\partial (c - P_h c)}{\partial t}\|_{L^2} \leq K h_c^{k+1} \|c\|_{H^{k+1}} + \|\frac{\partial c}{\partial t}\|_{H^{k+1}}.
$$

Meanwhile, we also introduce the $L^2$ projection operator $Q_h$ from $L^2(\Omega)$ onto $W_h$ such that

$$
(H - Q_h H, w_h) = 0, \quad \forall w_h \in W_h.
$$

It is well-known that the a priori error estimate: for $l \geq 0$

$$
\|H - Q_h H\|_{L^2} \leq K h_H^{l+1} \|H\|_{H^{l+1}}, \quad \forall w \in H^{l+1}(\Omega)
$$

holds.

Next, we will give a lemma which is important to prove our theoretical result in the following sections.
Lemma 3.1 ([17]). Assume that the finite element space $V_h$ is any one of the classical mixed finite element spaces defined in [5]. The super-approximation,

$$
\langle \varphi \nabla \cdot (v - \Pi_h v), \nabla \cdot v_h \rangle 
\leq K h_n^2 \| \nabla \cdot v_h \|_{L^2} \min(\| \varphi \|_{H^1}, \| \nabla \cdot (v - \Pi_h v) \|_{L^\infty} , \min(\| \varphi \|_{W^{1,\infty}}, h_n^{-2} \| \nabla \cdot (v - \Pi_h v) \|_{L^2})
$$

holds, for each function $\varphi \in W^{1,\infty}, v \in V$ and $v_h \in V_h$.

4. Error estimate for the concentration

In order to derive the error estimate for the concentration, we make an induction hypothesis as follows:

$$
\max_n \| u^n_h \|_{\infty} \leq \kappa h_n^{-1} \frac{h_n^2}{\Delta t^2}.
$$

(4.1)

Obviously, from MCC-SMFE Algorithm, we know that the induction hypothesis (4.1) holds when $n = 0$. Moreover, we assume that (4.1) holds until $n = N - 1$. Set $\xi^n_c = c^n_h - P_h c^n$ and $z^n_c = c^n - P_h c^n$. By the definition of projection operator $P_h$, we have the error residual equation:

$$
\frac{\phi \xi^n_c - \xi^{n-1}_c}{h_n^{-1} \Delta t}, z_h) + (\phi D \nabla \xi^n_c, \nabla z_h)
= \left( \frac{\phi}{h_n^{-1} \Delta t} \frac{\partial c}{\partial t} + u^{n-1} \cdot \nabla c^n + \nabla \cdot u^{n-1} c^n - \frac{1}{\Delta t} \left( \frac{\phi}{h_n^{-1}} c^n - \left( \frac{\phi}{h_n^{-1}} c^{n-1} \right) \circ X^n \delta^n \right) , z_h \right)
$$

$$
+ \left( \frac{\phi}{h_n^{-1} \Delta t} \frac{\partial c}{\partial t} , z_h \right) + \left( \frac{1}{\Delta t} \left( \frac{\phi}{h_n^{-1}} c^{n-1} - \left( \frac{\phi}{h_n^{-1}} c^{n-1} \right) \circ X^n \delta^n \right) , z_h \right)
$$

$$
+ \left( \frac{\phi}{h_n^{-1} \Delta t} (z^n_c - z^{n-1}_c) , z_h \right) + \left( \frac{1}{\Delta t} \left( \frac{\phi}{h_n^{-1}} z^{n-1}_c - \left( \frac{\phi}{h_n^{-1}} z^{n-1}_c \right) \circ X^n \delta^n \right) , z_h \right)
$$

(4.2)

We have the following approximate result.

Lemma 4.1. Assume that $\beta, \beta', a, a', \phi, \delta$, and $\phi'$ are bounded, then there exists an estimate

$$
\sum_{n=1}^{N} \| d_t \xi^n_c \|_{L^2}^2 \Delta t + \| \nabla \xi^n_c \|_{L^2}^2 
\leq \kappa \left\{ \sum_{n=1}^{N-1} \left[ \| \xi^n_c \|_{L^2}^2 + \| \xi^{n-1}_c \|_{L^2}^2 + \| \nabla \xi^n_c \|_{L^2}^2 \right] \Delta t + h_n^{2k+2} + h_n^{2r+2} + (\Delta t)^2 \right\}.
$$

(4.3)

Proof. Denote $d_t f^n := (f^n - f^{n-1})/\Delta t$. Taking $z_h = d_t \xi^n_c$ in (4.2), then we have

$$
\frac{\phi}{h_n^{-1} \Delta t} (\xi^n_c - \xi^{n-1}_c), d_t \xi^n_c) + (\phi D \nabla \xi^n_c, \nabla d_t \xi^n_c)
$$

$$
= \left( \frac{\phi}{h_n^{-1} \Delta t} \frac{\partial c}{\partial t} + u^{n-1} \cdot \nabla c^n + \nabla \cdot u^{n-1} c^n - \frac{1}{\Delta t} \left( \frac{\phi}{h_n^{-1}} c^n - \left( \frac{\phi}{h_n^{-1}} c^{n-1} \right) \circ X^n \delta^n \right) , d_t \xi^n_c \right)
$$

$$
+ \left( \frac{\phi}{h_n^{-1} \Delta t} \frac{\partial c}{\partial t} , d_t \xi^n_c \right) + \left( \frac{1}{\Delta t} \left( \frac{\phi}{h_n^{-1}} c^{n-1} - \left( \frac{\phi}{h_n^{-1}} c^{n-1} \right) \circ X^n \delta^n \right) , d_t \xi^n_c \right)
$$

$$
+ \left( \frac{\phi}{h_n^{-1} \Delta t} (z^n_c - z^{n-1}_c) , d_t \xi^n_c \right) + \left( \frac{1}{\Delta t} \left( \frac{\phi}{h_n^{-1}} z^{n-1}_c - \left( \frac{\phi}{h_n^{-1}} z^{n-1}_c \right) \circ X^n \delta^n \right) , d_t \xi^n_c \right)
$$

(4.4)

$$
+ (\nabla \cdot ((u^n - u^{n-1}) c^n), d_t \xi^n_c) + (q^n \xi^n_c, d_t \xi^n_c) - (q^n \xi^n_c, z_h).
$$
The left-hand side of (4.4) is bigger than the quantity

\[ \frac{\alpha^*}{\beta} ||d_t \xi^c||^2_{L^2} + \frac{1}{2} d_t((\Phi D)\nabla \xi^c, \nabla \xi^c) - \frac{1}{2} (d_t(\Phi D)\nabla \xi^{n-1}_c, \nabla \xi^{n-1}_c), \]

and

\[ \frac{1}{2} (d_t(\Phi D)\nabla \xi^{n-1}_c, \nabla \xi^{n-1}_c) \leq K(1 + ||d_t \beta^{n-1}||^2_{0,\infty}) ||\nabla \xi^{n-1}_c||^2_{L^2}. \]

Now we multiply the relation (4.4) by \( \Delta t \), sum over \( 1 \leq n \leq N \) and denote the resulting right-hand terms by \( T_1, T_2, \ldots, T_8 \). We turn to analyze these terms one by one.

\[
T_1 = \sum_{n=1}^N \left( \frac{\Phi}{\beta_{n-1}} \frac{d c}{d t} + u_{n-1}^c \cdot \nabla c^n + \nabla \cdot u_{n-1}^c \nabla c^n - \frac{1}{\Delta t} \left( \frac{\Phi}{\beta_{n-1}} c^n - \left( \frac{\Phi}{\beta_{n-1}} c^{n-1} \right) \circ X^n \delta^n \right), d_t \xi^c \right) \Delta t
\]

\[
= \sum_{n=1}^N \left( \frac{\Phi}{\beta_{n-1}} \frac{d c}{d t} + u_{n-1}^c \cdot \nabla c^n - \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \Delta t, d_t \xi^c \right) \Delta t
\]

\[
+ \sum_{n=1}^N \left( \nabla \cdot u_{n-1}^c \nabla c^n + \frac{1}{\Delta t} \left( \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \delta^n - \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \right), d_t \xi^c \right) \Delta t
\]

\[
+ \sum_{n=1}^N \left( \frac{1}{\Delta t} \left( \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n - \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \right), d_t \xi^c \right) \Delta t
\]

\[= T_{11} + T_{12} + T_{13}. \]

For \( T_{11} \) we have the estimate

\[ T_{11} \leq K || \frac{\partial^2 c}{\partial t^2} ||^2_{L^2((0,T];L^2(\Omega))} \Delta t^2 + \varepsilon \sum_{n=1}^N ||d_t \xi^c||^2_{L^2} \Delta t. \] (4.6)

Based on the definite of \( \delta^n \) and Taylor expansion, \( T_{12} \) can be decomposed as

\[
T_{12} = \sum_{n=1}^N \left( \nabla \cdot u_{n-1}^c \nabla c^n + \frac{1}{\Delta t} \left( \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \delta^n - \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \right), d_t \xi^c \right) \Delta t
\]

\[
= \sum_{n=1}^N \left( \nabla \cdot u_{n-1}^c (c^n - c^{n-1} \circ X^n), d_t \xi^c \right) \Delta t
\]

(4.7)

\[
+ \sum_{n=1}^N \left( \nabla \cdot u_{n-1}^c \nabla c^n + \frac{1}{\Delta t} \left( \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \delta^n - \frac{\Phi}{\beta_{n-1}} c^{n-1} \circ X^n \right), d_t \xi^c \right) \Delta t
\]

\[
\leq K \Delta t^2 ||c||^2_{H^1(L^2(\Omega)) \cap C^0(H^2)} + \varepsilon \sum_{n=1}^N ||d_t \xi^c||^2_{L^2} \Delta t,
\]

where we have used the induction hypothesis (4.1). Substituting (4.6) and (4.7) into (4.5), we have

\[ T_1 \leq K \left( ||c||^2_{H^1(L^2)} + ||c||^2_{C^0(H^2)} + ||\frac{\partial^2 c}{\partial t^2}||^2_{L^2((0,T];L^2(\Omega))} \right) \Delta t^2 + \varepsilon \sum_{n=1}^N ||d_t \xi^c||^2_{L^2} \Delta t.
\]

For \( T_2, T_7 \), we have

\[ T_2 + T_7 \leq K \sum_{n=1}^N \left( ||\xi^n||^2_{L^2} + \Delta t \left| \frac{d c}{d t} \right|^2_{L^2((0,T];L^2(\Omega))} \right) \Delta t + \varepsilon \sum_{n=1}^N ||d_t \xi^c||^2_{L^2} \Delta t + Kh^2 \]
For $T_3$, using the fact that $\xi^0_c = 0$, we have

$$T_3 = \sum_{n=1}^{N} \left( \frac{\phi_{n-1}^{\xi^0_c} \circ \hat{X}^n - \phi_{n-1}^{\xi^0_c}}{\Delta t}, dt \xi^0_c \right) \Delta t$$

$$= \frac{1}{\Delta t} \sum_{n=1}^{N} \left( \frac{\phi_{n-1}^{\xi^0_c} \circ \hat{X}^n - \phi_{n-1}^{\xi^0_c}}{\Delta t}, dt \xi^0_c \right) \Delta t$$

$$- \frac{1}{\Delta t} \sum_{n=1}^{N} \left( \frac{\phi_{n-1}^{\xi^0_c} \circ \hat{X}^n - \phi_{n-1}^{\xi^0_c}}{\Delta t}, dt \xi^0_c \right) \Delta t$$

$$= -\sum_{n=2}^{N} \left( dt \left( \frac{\phi_{n-1}^{\xi^0_c} \circ \hat{X}^n - \phi_{n-1}^{\xi^0_c}}{\Delta t} \right)^2 \right) \Delta t = T_{31} + T_{32}.$$

Note that

$$T_{31} = \sum_{n=2}^{N} \left( dt \left( \frac{\phi_{n-1}^{\xi^0_c} \circ \hat{X}^n - \phi_{n-1}^{\xi^0_c}}{\Delta t} \right)^2 \right) \Delta t$$

where we have used the fact in the third equation that

$$\left( \frac{\phi_{n-1}^{\xi^0_c} \circ \hat{X}^n}{\Delta t} \right)^2 = \left( \frac{\phi_{n-1}^{\xi^0_c}}{\Delta t} \right)^2,$$

$$\left( \frac{\phi_{n-2}^{\xi^0_c} \circ \hat{X}^n}{\Delta t} \right)^2 = \left( \frac{\phi_{n-2}^{\xi^0_c}}{\Delta t} \right)^2.$$

For fixed $\bar{z}$, considering the transformation

$$y = f_\bar{z}(x) = x - \frac{u^{n-1}b^{n-1}}{\phi} \Delta t \bar{z},$$

we can find that $\text{det}(Df_\bar{z}) = O(\delta^n)$ with $0 \leq \bar{z} \leq 1$. For all $z(x)$

$$\|\frac{\phi^n \circ X^n - \phi^n}{\Delta t} \|_{(\delta^n)^{1/2}}^2 = (\Delta t^{-2}) \int_\Omega (\phi^n(x) - \phi^n(x - \frac{u^{n-1}b^{n-1}}{\phi}) \Delta t)^2 \delta^n dx$$

$$= (\Delta t^{-2}) \int_\Omega (\int_{x}^{x - \frac{u^{n-1}b^{n-1}}{\phi}} \frac{\partial \phi^n}{\partial z} dz)^2 \delta^n dx.$$
then we get

\[ T_{311} \leq \varepsilon \sum_{n=2}^{N} \| d_t \xi_{c, n+1} \|^2_{L^2} \Delta t + K \sum_{n=2}^{N} \| \nabla \xi_{n+1} \|^2_{L^2} \Delta t, \quad T_{312} \leq K \sum_{n=2}^{N} (\| \nabla \xi_{n+1} \|^2_{L^2} + \| \xi_{n+1} \|^2_{L^2}) \Delta t. \]

For \( T_{32} \), we have

\[ T_{32} = \left( \frac{\phi}{\beta_h} \xi_{c, n+1} - \frac{\phi}{\beta_h} \xi_{c, n} \right) \Delta t \]

\[ = \left( \frac{\phi}{\beta_h} \xi_{c, n+1} - \frac{\phi}{\beta_h} \xi_{c, n} \right) \Delta t \]

\[ \leq K \| \nabla \xi_{n+1} \|^2_{L^2} + \varepsilon \| \nabla \xi_{n+1} \|^2_{L^2} \]

it implies that

\[ T_3 \leq \varepsilon \sum_{n=2}^{N} \| d_t \xi_{n+1} \|^2_{L^2} \Delta t + K \| \xi_{n+1} \|^2_{L^2} + \varepsilon \| \nabla \xi_{n+1} \|^2_{L^2} + K \sum_{n=2}^{N} (\| \nabla \xi_{n+1} \|^2_{L^2} + \| \xi_{n+1} \|^2_{L^2}) \Delta t. \]

Now, we consider \( T_5 \). We have

\[ T_5 = \sum_{n=1}^{N} \left( \frac{\phi}{\beta_h} \xi_{c, n+1} - \frac{\phi}{\beta_h} \xi_{c, n} \right) \Delta t \]

\[ = \sum_{n=1}^{N} \left( \frac{\phi}{\beta_h} \xi_{c, n+1} - \frac{\phi}{\beta_h} \xi_{c, n} \right) \Delta t \]

\[ \leq K \| \nabla \xi_{n+1} \|^2_{L^2} + \varepsilon \| \nabla \xi_{n+1} \|^2_{L^2} \]

With the same argument as \( T_3 \), we can get

\[ T_5 \leq K \sum_{n=1}^{N} (\| d_t \xi_{c, n+1} \|^2_{L^2} + \| \nabla \xi_{n+1} \|^2_{L^2} + \| \xi_{c, n+1} \|^2_{L^2}) \Delta t + k \| \xi_{c, n+1} \|^2_{L^2} + \varepsilon \| \nabla \xi_{c, n+1} \|^2_{L^2} \]

\[ \leq K(h_{c, n+1}^2 + \sum_{n=1}^{N} \| \nabla \xi_{c, n+1} \|^2_{L^2} + \varepsilon \| \nabla \xi_{c, n+1} \|^2_{L^2} \]

\( T_4 \) can be easily bounded as follows

\[ T_4 = \sum_{n=1}^{N} \left( \frac{\phi}{\beta_h} \xi_{c, n+1} - \frac{\phi}{\beta_h} \xi_{c, n} \right) \Delta t \]

\[ \leq K \sum_{n=1}^{N} (\| d_t \xi_{c, n+1} \|^2_{L^2} + \varepsilon \sum_{n=1}^{N} \| d_t \xi_{c, n+1} \|^2_{L^2} \Delta t \leq K \| \xi_{c, n+1} \|^2_{H^1(\Omega)} h_{c, n+1}^2 + \varepsilon \sum_{n=1}^{N} \| d_t \xi_{c, n+1} \|^2_{L^2} \Delta t. \]
And it is easy to see
\[T_6 = \sum_{n=1}^{N} \left( \nabla \cdot \left( (\mathbf{u}^n - \mathbf{u}_h^{n-1}) \mathbf{c}^n \right), \mathbf{d}_t \mathbf{\xi}_c^n \right) \Delta t \]
\[= \sum_{n=1}^{N} - (\mathbf{u}^n - \mathbf{u}_h^{n-1}, \mathbf{c}^n \nabla \mathbf{d}_t \mathbf{\xi}_c^n) \Delta t \]
\[\leq \sum_{n=1}^{N} \left\{ \int_{t_{n-1}}^{t_n} \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{c}^n, \nabla \mathbf{d}_t \mathbf{\xi}_c^n \right\} \Delta t \]
\[+ \| (\alpha(c_h^n) - \alpha(c_h^{n-1})) \sigma^n - \alpha(c_h^{n-1}) \cdot \mathbf{c}^{n-1}, \nabla \mathbf{d}_t \mathbf{\xi}_c^n \| \right\} \Delta t \]
\[\leq K \sum_{n=1}^{N} \left\{ \| \alpha(c_h^n) - \alpha(c_h^{n-1}) \sigma^n \|_2 + \| \alpha(c_h^{n-1}) - \alpha(c_h^{n-1}) \cdot \mathbf{c}^{n-1} \|_2 \right\} \Delta t \]
\[+ \sum_{n=1}^{N} \left( \| \nabla \mathbf{\xi}_c^n \|_2^2 - \| \nabla \mathbf{\xi}_c^{n-1} \|_2^2 \right) \]
\[\leq K \sum_{n=1}^{N} \| \mathbf{\xi}_c^n \|_2^2 \Delta t + \varepsilon \sum_{n=1}^{N} \| \mathbf{d}_t \mathbf{\xi}_c^n \|_2^2 \Delta t \leq K \| \mathbf{c} \|_{H^1(H^2)}^2 h_h^{k+2} + \varepsilon \sum_{n=1}^{N} \| \mathbf{d}_t \mathbf{\xi}_c^n \|_2^2 \Delta t. \]

Combining these above estimates, we can easily get the inequality (4.3). \qed

5. Error estimate for the flux

Set \( \mathbf{\xi}_\sigma^n = \mathbf{\sigma}_h^n - \Pi_h \mathbf{\sigma}^n, \mathbf{\xi}_H^n = \mathbf{\sigma}^n - \Pi_h \mathbf{\sigma}^n, \mathbf{\xi}_H^n = \mathbf{H}_h^n - \Pi_h \mathbf{H}_h^n, \) and \( \mathbf{\xi}_H^n = \mathbf{H}_h^n - \Pi_h \mathbf{H}_h^n. \) We have to estimate bounds of \( \mathbf{\xi}_\sigma \) and \( \mathbf{\xi}_H, \) which satisfy the error residual equations:
\[
(\alpha(c_h^n) \mathbf{\xi}_\sigma^n - \alpha(c_h^{n-1}) \mathbf{\xi}_\sigma^{n-1}) \Delta t, \mathbf{v}_h) + (B \nabla \cdot \mathbf{\xi}_\sigma^n, \nabla \cdot \mathbf{v}_h)
\]
\[= \left( \frac{\partial}{\partial t} (\alpha(c)) \mathbf{\sigma}^n - \alpha(c) \mathbf{\sigma}^n - \alpha(c) \mathbf{\sigma}^{n-1}, \mathbf{v}_h \right) + (B \mathbf{\eta} (\frac{\partial c}{\partial t} - \mathbf{\sigma}^n - \mathbf{\sigma}^{n-1}), \nabla \cdot \mathbf{v}_h)
\]
\[+ (\eta)(\frac{\partial c}{\partial t} - \mathbf{\sigma}^n - \mathbf{\sigma}^{n-1}) \mathbf{e}_3, \mathbf{v}_h \) + \left( \alpha(c_h^n) \mathbf{\xi}_H^n - \alpha(c_h^{n-1}) \mathbf{\xi}_H^{n-1}, \mathbf{\mathbf{v}_h} \right)
\]
\[+ \left( \frac{(\alpha(c^n) - \alpha(c^{n-1})) \mathbf{\sigma}^n - \alpha(c^{n-1}) \mathbf{\sigma}^{n-1}}{\Delta t}, \mathbf{\xi}_H^n \right) \right\}
\[\Delta t \right\} \]
\[\leq K \sum_{n=1}^{N} \| \mathbf{\xi}_H^n \|_2^2 \Delta t + \varepsilon \sum_{n=1}^{N} \| \mathbf{d}_t \mathbf{\xi}_H^n \|_2^2 \Delta t \]
and
\[
\frac{(\xi^n_H - \xi^{n-1}_H)}{\Delta t}, w_h) = (Bq^n(\beta^n_H - \beta^n), w_h) - (B \nabla \cdot (\sigma^n_H - \sigma^n), w_h)
\]
\[
+ (B \phi \eta \frac{\partial c}{\partial t} - \frac{c^n - c^{n-1}}{\Delta t}, w_h) + (B \phi \eta \frac{\xi^n_c - \xi^{n-1}_c}{\Delta t}, \nabla \cdot v_h)
\]
\[
- (B \phi \eta \frac{\xi^n_c - \xi^{n-1}_c}{\Delta t}, w_h), \forall w_h \in W_h.
\]

(5.2)

To obtain the error estimate for the flux, we also make other induction hypotheses as follows:

\[
\max_n ||c^n||_\infty \leq kh_c^{-1} (h_{\sigma}^2 + \frac{3}{\Delta t}),
\]

(5.3)

\[
\max_n ||\xi^n_0||_L^2 + \max_n ||\xi^n_c||_L^2 = \mathcal{O}\left(\max(h_{\sigma}^3, h_{\sigma}^3)\right).
\]

(5.4)

From MCC-SMFE Algorithm, we can easily show that (5.3) and (5.4) hold when \( n = 0 \). Now, we assume that (5.3) and (5.4) hold for \( n = 1, 2, \ldots, N - 1 \), we have the following results.

**Lemma 5.1.** Assume that \( \alpha, \alpha', \) and \( \alpha'' \) are bounded, then the priori estimate

\[
\frac{[\alpha(c^n) - \alpha(c^n_H)]\sigma^n - [\alpha(c^{n-1}) - \alpha(c^{n-1}_H)]\sigma^{n-1}}{\Delta t}, v_h)
\]

\[
\leq K\{||\xi^n_c||_L^2 + ||\xi^{n-1}_c||_L^2 + ||\xi^n_{\sigma}||_L^2 + ||\xi^{n-1}_{\sigma}||_L^2
\]

\[
+ ||\xi^{n-2}_c||_L^2 + ||\nabla \cdot v_h||_L^2 + h_{\sigma}^{2k+2} + h_{\sigma}^{2r+2} + (\Delta t)^2\} + \delta ||v_h||_L^2
\]

(5.5)

holds, for any \( v_h \in V_h \).

**Proof.** Note that

\[
\sigma^{n-1} = \sigma^n - \int_{t_{n-1}}^{t_n} \frac{\partial \sigma}{\partial t} dt,
\]

\[
\alpha(c^n) - \alpha(c^{n-1}) = \int_0^1 \alpha'(c^{n-1} + s(c^n - c^{n-1})) ds (c^n - c^{n-1}),
\]

\[
\alpha(c^n_H) - \alpha(c^{n-1}_H) = \int_0^1 \alpha'(c^{n-1} + s(c^n_H - c^{n-1}_H)) ds (c^n_H - c^{n-1}_H).
\]

So we have

\[
[\alpha(c^n) - \alpha(c^{n-1})] - [\alpha(c^n_H) - \alpha(c^{n-1}_H)]
\]

\[
= \int_0^1 \alpha'(c^{n-1} + s(c^n - c^{n-1})) ds (c^n_c - c^{n-1}_c)
\]

\[
- \int_0^1 \alpha'(c^{n-1} + s(c^n_H - c^{n-1}_H)) ds (c^n_c - c^{n-1}_c)
\]

\[
+ \int_0^1 [\xi^{n-1}_c - \xi^{n-1}_c] \alpha'' ds \int_{t_{n-1}}^{t_n} \frac{\partial c}{\partial t} dt - \int_0^1 [\xi^{n-1}_c - \xi^{n-1}_c] \alpha'' ds \int_{t_{n-1}}^{t_n} \frac{\partial c}{\partial t} dt.
\]

Utilizing this equation, we can easily get

\[
\frac{[\alpha(c^n) - \alpha(c^n_H)]\sigma^n - [\alpha(c^{n-1}) - \alpha(c^{n-1}_H)]\sigma^{n-1}}{\Delta t}, v_h)
\]

\[
= \frac{[\alpha(c^n) - \alpha(c^n_H) - \alpha(c^n_H) - \alpha(c^{n-1}_H)]\sigma^n}{\Delta t}, v_h) + \frac{[\alpha(c^{n-1}) - \alpha(c^{n-1}_H)]}{\Delta t} \int_{t_{n-1}}^{t_n} \frac{\partial \sigma}{\partial t} dt, v_h)
\]
Using Lemma 4.1, we can derive

\[
\begin{align*}
F_1 + F_3 + F_4 + F_5 & \leq K\left\{ ||\mathbf{\xi}_c^n||^2_{L^2} + ||\mathbf{\xi}_e^n||^2_{L^2} + ||\mathbf{\xi}_e^{n-1}||^2_{L^2} + ||\mathbf{\xi}_\sigma^{n-1}||^2_{L^2} \right. \\
& \quad + ||\mathbf{\xi}_\sigma^{n-2}||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2} + h_c^{2k+2} + h_\sigma^{2r+2} + (\Delta t)^2 \} + \delta||\mathbf{v}_h||^2_{L^2}.
\end{align*}
\] (5.7)

For $F_2$, we have

\[
F_2 = \left(\frac{\xi^n_c - \xi^{n-1}_c}{\Delta t}, \int_0^1 \alpha'(c_h^n - c_h^{n-1})ds\sigma^n \cdot \mathbf{v}_h, \beta_h^{n-1}/\phi\right)
\]

\[
= \left(\frac{\xi^n_c - \xi^{n-1}_c}{\Delta t}, R_M\left[\int_0^1 \alpha'(c_h^n - c_h^{n-1})ds\sigma^n \cdot \mathbf{v}_h, \beta_h^{n-1}/\phi\right]\right)
\]

\[
\leq K\left\{ ||\mathbf{\xi}_c^n||^2_{L^2} + ||\mathbf{\xi}_e^n||^2_{L^2} + ||\mathbf{\xi}_e^{n-1}||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2} + h_c^{2k+2} + h_\sigma^{2r+2} + (\Delta t)^2 \} + \delta||\mathbf{v}_h||^2_{L^2},
\]

where $R_M$ is a weighted $L^2$-projection operator from $\mathbf{L}^2(\Omega)$ onto $\mathbf{M}_h$ such that

\[
\left(\frac{\xi^n_c - \xi^{n-1}_c}{\Delta t}, \mathbf{z}_h\right) = 0, \quad \forall \mathbf{z} \in \mathbf{L}^2(\Omega), \mathbf{z}_h \in \mathbf{M}_h.
\]

Substituting (5.7) and (5.8) into (5.6), we get the estimate (5.5).

**Lemma 5.2.** Under the conditions of Lemmas 4.1 and 5.1, we have the following estimate

\[
\left(\frac{\partial}{\partial t}(\mathbf{\alpha}(c^n)\mathbf{\sigma}^n) - \frac{\alpha(c^n)\mathbf{\sigma}^n - \alpha(c^{n-1})\mathbf{\sigma}^{n-1}}{\Delta t}, \mathbf{v}_h\right) + (B\nabla \cdot \mathbf{\xi}_c^n, \nabla \cdot \mathbf{v}_h)
\]

\[
\leq K\left\{ ||\mathbf{\xi}_c^n||^2_{L^2} + ||\mathbf{\xi}_e^n||^2_{L^2} + ||\mathbf{\xi}_e^{n-1}||^2_{L^2} + ||\mathbf{\xi}_\sigma^{n-1}||^2_{L^2} + ||\mathbf{\xi}_\sigma^{n-2}||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2} + h_c^{2k+2} + h_\sigma^{2r+2} + (\Delta t)^2 \} + \delta||\mathbf{v}_h||^2_{L^2}.
\] (5.9)

**Proof.** It is easily seen that

\[
\left(\frac{\partial}{\partial t}(\mathbf{\alpha}(c^n)\mathbf{\sigma}^n) - \frac{\alpha(c^n)\mathbf{\sigma}^n - \alpha(c^{n-1})\mathbf{\sigma}^{n-1}}{\Delta t}, \mathbf{v}_h\right) + (B\nabla \cdot \mathbf{\xi}_c^n, \nabla \cdot \mathbf{v}_h)
\]

\[
\leq K\left(\Delta t\left[ ||\mathbf{\xi}_c^n||^2_{L^2} + ||\mathbf{\xi}_e^n||^2_{L^2} + ||\mathbf{\xi}_e^{n-1}||^2_{L^2} + ||\mathbf{\xi}_\sigma^{n-1}||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2} \right] + \delta||\mathbf{v}_h||^2_{L^2},
\]

\[
(B\nabla \cdot (\beta^n_h - \beta^n), \nabla \cdot \mathbf{v}_h) \leq K\left(||\mathbf{\xi}_c^n||^2_{L^2} + ||\mathbf{\xi}_e^n||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2}\right),
\]
As noted above, we know that

\[
\frac{\xi_n - \xi_{n-1}}{\Delta t}, \nabla \cdot \mathbf{v}_h + (\eta - \xi_{n-1} \mathbf{e}_3, \mathbf{v}_h) \leq K\left\{ ||\nabla \cdot \mathbf{v}_h||^2_{L^2} + h_c^{2k+2}\right\} + \delta \mathbf{v}_h^2.
\]

As noted above, we know that

\[
\alpha (c_h^{n-1}) = \alpha (c_h^n) - \int_0^1 \alpha' (c_h^{n-1} + s(c_h^n - c_h^{n-1})) ds (c_h^n - c_h^{n-1})
\]

\[
= \alpha (c_h^n) - \int_0^1 \alpha' (c_h^{n-1} + s(c_h^n - c_h^{n-1})) ds (\xi^n - \xi_{n-1})
\]

\[
+ \int_0^1 \alpha' (c_h^{n-1} + s(c_h^n - c_h^{n-1})) ds (\xi^n - \xi_{n-1}) - \int_0^1 \alpha' (c_h^{n-1} + s(c_h^n - c_h^{n-1})) ds \int_{t_{n-1}}^{t_n} \frac{\partial c}{\partial t} dt.
\]

So we have

\[
\left( \frac{\alpha (c_h^n) \xi^n_n - \alpha (c_h^{n-1}) \xi_{n-1}^n}{\Delta t}, \mathbf{v}_h \right) = \left( \frac{\alpha (c_h^n) \xi^n_n - \alpha (c_h^{n-1}) \xi_{n-1}^n}{\Delta t}, \mathbf{v}_h \right) + \left( \int_0^1 \alpha' (c_h^{n-1} + s(c_h^n - c_h^{n-1})) ds \xi_{n-1}^n - \xi_{n-1}^n, \mathbf{v}_h \right)
\]

\[
- \left( \int_0^1 \alpha' (c_h^{n-1} + s(c_h^n - c_h^{n-1})) ds \xi_{n-1}^n - \xi_{n-1}^n, \mathbf{v}_h \right) + \left( \int_0^1 \alpha' (c_h^{n-1} + s(c_h^n - c_h^{n-1})) ds \xi_{n-1}^n - \xi_{n-1}^n, \mathbf{v}_h \right)
\]

\[
\leq \left( \frac{\Phi}{\beta_h^{n-1}} \frac{\xi_n - \xi_{n-1}}{\Delta t}, \mathbf{v}_h \right) \leq K\left\{ \left( \frac{\xi_n - \xi_{n-1}}{\Delta t} \right)^2 + ||\xi_{n-1}^n||^2_{L^2} + ||\xi_{n-1}^n||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2} + h_r^2 + h_c^{2k+2} + (\Delta t)^2 \right\} + \delta \mathbf{v}_h^2.
\]

By Lemma 3.1 and the inverse property of the finite element space \( \mathcal{V}_h \), we have the estimate

\[
(B \nabla \cdot \xi_n, \nabla \cdot \mathbf{v}_h) \leq K\left\{ ||\nabla \cdot \xi_n||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2} \right\} \leq K\left\{ h_{\beta}^2 + \delta ||\nabla \cdot \mathbf{v}_h||^2_{L^2} \right\}.
\]

Utilizing the similar technique as in (5.8), we can get the following inequality

\[
||B \Phi_{\beta_h^{n-1}} \xi_{n-1}^n, \nabla \cdot \mathbf{v}_h || + ||\xi_{n-1}^n, \nabla \cdot \mathbf{v}_h || \leq K\left\{ \left( \frac{\xi_n - \xi_{n-1}}{\Delta t} \right)^2 + ||\xi_{n-1}^n||^2_{L^2} + ||\nabla \cdot \mathbf{v}_h||^2_{L^2} + h_r^2 + h_c^{2k+2} + (\Delta t)^2 \right\} + \delta \mathbf{v}_h^2.
\]

Substituting these estimates into (5.1), we complete the proof of Lemma 5.2.

\[\Box\]

**Lemma 5.3.** Under the conditions of Lemma 5.2, we have the estimate

\[
\sum_{n=1}^N ||d_t \xi_{\sigma}^n||^2_{L^2} \Delta t + ||\nabla \cdot \xi_{\sigma}^n||^2_{L^2}
\]

\[
\leq K\left\{ \sum_{n=1}^{N-1} \left( ||\xi^2_{\sigma}||^2_{L^2} + ||\xi^2_{\sigma}||^2_{L^2} + ||\nabla \cdot \xi_{\sigma}^2||^2_{L^2} \right) \Delta t + h_{\beta}^2 + h_{\beta}^2 + h_{\beta}^2 + (\Delta t)^2 \right\}.
\]
Proof. Taking $v_h = d_t \xi_h^n$ in (5.9), and noting that
\[
\left( \frac{\alpha(c_h^n) \xi_h^n - \alpha(c_h^{n-1}) \xi_h^{n-1}}{\Delta t}, \xi_h^n \right) = (\alpha(c_h^n) d_t \xi_h^n, d_t \xi_h^n) + (d_t (\alpha(c_h^n)) \xi_h^{n-1}, d_t \xi_h^n),
\]
and
\[
(B \nabla \cdot \xi_h^n, \nabla \cdot d_t \xi_h^n) = \frac{1}{2} d_t (B \nabla \cdot \xi_h^n, \nabla \cdot \xi_h^n) - \frac{1}{2} (d_t B \nabla \cdot \xi_h^{n-1}, \nabla \cdot \xi_h^{n-1}) \geq \frac{1}{2} d_t (B \nabla \cdot \xi_h^n, \nabla \cdot \xi_h^n) - K \|\nabla \cdot \xi_h^{n-1}\|^2.
\]
Under the inductive induction hypothesis (5.4), using the similar technique as in (5.8) and Lemma 4.1, we get
\[
\left[(\alpha(c_h^n) d_t \xi_h^n, d_t \xi_h^n) - (d_t (\alpha(c_h^n)) \xi_h^{n-1}, d_t \xi_h^n)\right] + \frac{1}{\Delta t} \left(\|\nabla \cdot \xi_h^n\|^2 - \|\nabla \cdot \xi_h^{n-1}\|^2\right) - K \|\nabla \cdot \xi_h^{n-1}\|
\]
\[
\leq K \left[\|\xi_h^n\|^2 + \|\xi_h^{n-1}\|^2 + \|\xi_h^{n-2}\|^2 + \|\xi_h^{n-3}\|^2 + \|\xi_h^{n-4}\|^2 + \|\xi_h^{n-5}\|^2 + \|\xi_h^{n-6}\|^2 + \|\xi_h^{n-7}\|^2 + \|\xi_h^{n-8}\|^2 + \|\xi_h^{n-9}\|^2 + \|\xi_h^{n-10}\|^2\right]
\]
\[
+ \|\nabla \cdot v_h\|^2 + h_\sigma^{2r+2} + h_\sigma^{3r+2} + h_\sigma^{4k+2} + (\Delta t)^2 \right\} + \delta \|v_h\|^2.
\]
Multiplying (5.11) by $2\Delta t$ and summing it over $n$, for sufficiently small $\delta$, we get the estimate (5.10). \qed

6. The proof of Theorem 2.1

Now, we can complete the proof of Theorem 2.1.

Proof. Note the fact that
\[
\|q^N\|_{L^2}^2 - \|q^0\|_{L^2}^2 \leq K \sum_{n=1}^{N} \|q^n\|_{L^2}^2 \Delta t + \varepsilon \sum_{n=1}^{N} \|d_t q^n\|_{L^2}^2 \Delta t.
\]
Using Lemmas 4.1 and 5.3, we can get
\[
\|\xi_h^n\|_{L^2}^2 + \|\xi_h^N\|_{L^2}^2 + \|\nabla \cdot \xi_h^N\|_{L^2}^2 + \|\nabla \xi_h^N\|_{L^2}^2 + \frac{\Delta t}{\sum_{n=1}^{N} \|d_t \xi_h^n\|_{L^2}^2 \Delta t} + \frac{\sum_{n=1}^{N} \|d_t \xi_h^n\|_{L^2}^2 \Delta t}{\sum_{n=1}^{N} \|d_t \xi_h^n\|_{L^2}^2 \Delta t}
\]
\[
\leq K \left\{h_\sigma^{2r+2} + h_\sigma^{3r+2} + h_\sigma^{4k+2} + (\Delta t)^2 \right\}.
\]
Applying the discrete Gronwall’s inequality, we have
\[
\|\xi_h^n\|_{L^2}^2 + \|\xi_h^N\|_{L^2}^2 + \|\nabla \cdot \xi_h^N\|_{L^2}^2 + \|\nabla \xi_h^N\|_{L^2}^2 + \Delta t \sum_{i=1}^{N} \|d_t \xi_h^n\|_{L^2}^2 + \|d_t \xi_h^N\|_{L^2}^2
\]
\[
\leq K \left\{h_\sigma^{2r+2} + h_\sigma^{3r+2} + h_\sigma^{4k+2} + (\Delta t)^2 \right\}.
\]
It is clear that the optimal error estimate (6.2) is derived under the inductive hypotheses (4.1), (5.3), and (5.4). Now we have to check it when $n = N$. Under the condition (2.8), for the integers $r, k, r_1 > 0$, using (6.2) we know that
\[
\|u_h^N\|_{\infty} = \|a(c_h^N)\|_{\infty} \leq K \left\{\|\Pi_h \sigma^N\|_{\infty} + \|\xi_h^N\|_{\infty}\right\}
\]
\[
\leq K \left\{\|\Pi_h \sigma^N\|_{\infty} + \|u_h^N\|_{\infty}\right\}.
\]
easily get

So, we know that the inductive hypotheses (4.1), (5.3), and (5.4) are true for \( n \). Finally, we consider the bound of \( \xi_H \). Taking \( \alpha_h = \xi_H^\alpha \) in (5.2) and using the estimate (6.2), we can easily get

\[
\max_n \| \xi_H^n \|^2_{L^2} \leq K \{ h_c^{2k+2} + h_\sigma^{2r+2} + h_\sigma^{2r} + h_\sigma^{2r+1} + (\Delta t)^2 \}.
\]

This ends the proof of Theorem 2.1.

\[ \square \]

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**References**


