Fixed points for multivalued contractions with respect to a Pompeiu type metric

Iulia Coroian

Department of Land Measurements and Exact Sciences, University Of Agricultural Sciences and Veterinary Medicine, 3-5 Manastur St., 400372 Cluj-Napoca, Romania.

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Abstract

The purpose of this paper is to present a fixed point theory for multivalued $H^+$-contractions from the following perspectives: existence/uniqueness of the fixed and strict fixed points, data dependence of the fixed point set, sequence of multivalued operators and fixed points, Ulam-Hyers stability of a multivalued fixed point equation, well-posedness of the fixed point problem, limit shadowing property for a multivalued operator, set-to-set operatorial equations and fractal operator theory. ©2017 All rights reserved.

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1. Introduction

Let $(X, d)$ be a metric space and $\mathcal{P}(X)$ be the set of all subsets of $X$. We denote

$$\mathcal{P}(X) := \{ Y \in \mathcal{P}(X) \mid Y \neq \emptyset \},$$

$$\mathcal{P}_{cl}(X) := \{ Y \in \mathcal{P}(X) \mid Y \text{ is closed} \},$$

$$\mathcal{P}_{b,cl}(X) := \{ Y \in \mathcal{P}(X) \mid Y \text{ is bounded and closed} \},$$

$$\mathcal{P}_{cp}(X) := \{ Y \in \mathcal{P}(X) \mid Y \text{ is compact} \}.$$

By $B(x, r)$ and respectively $\overline{B}(x, r)$ we will denote the open and respectively the closed ball centered at $x \in X$ with radius $r > 0$.

The following (generalized) functionals are used in the main sections of the paper.

1. The gap functional generated by $d$:

$$D_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad D_d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

2. The diameter generalized functional:

$$\delta : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}, \quad \delta(A, B) = \sup\{d(a, b) \mid a \in A, b \in B\}.$$

Email address: coroian.iulia@gmail.com (Iulia Coroian)

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Lemma 2.1. Extend the results given in [13], by presenting several properties of the fixed point set of multivalued continuity notions for multivalued operators are also given. The second purpose of this paper is to study different properties of the Pompeiu functional (i.e., $H$-contractions). Several other fixed point results and applications of it will be also given.

Remark. Let $(X,d)$ be a metric space. A multivalued mapping $T:X\to P_{b,c1}(X)$ is called $H^+$-contraction with constant $\alpha$, if

1. there exists a fixed real number $\alpha$, $0<\alpha<1$ such that for every $x,y\in X$
   \[H^+(T(x),T(y)) \leq \alpha d(x,y);\]

2. for every $x\in X$, $y\in T(x)$ and for every $\epsilon>0$ there exists $z\in T(y)$ such that
   \[d(y,z) \leq H^+(T(x),T(y)) + \epsilon.\]

Lemma 1.1 ([13]). The following conclusions take place:

(a) $H^+$ is a metric on $P_{b,c1}(X)$;

(b) $H^+$ is a generalized metric (in the sense that it can take also infinite values) on $P_{c1}(X)$.

Using the Pompeiu type functional $H^+$, the following notion was introduced in [13], see also [12].

Definition 1.2 ([13]). Let $(X,d)$ be a metric space. A multivalued mapping $T:X\to P_{b,c1}(X)$ is called $H^+$-contraction with constant $\alpha$, if

1. there exists a fixed real number $\alpha$, $0<\alpha<1$ such that for every $x,y\in X$
   \[H^+(T(x),T(y)) \leq \alpha d(x,y);\]

2. for every $x\in X$, $y\in T(x)$ and for every $\epsilon>0$ there exists $z\in T(y)$ such that
   \[d(y,z) \leq H^+(T(x),T(y)) + \epsilon.\]

Remark 1.3. Let $(X,d)$ be a metric space. A multivalued mapping $T:X\to P_{c1}(X)$ is called $(H^+,\alpha)$-Lipschitz if $\alpha>0$ and

\[H^+(T(x),T(y)) \leq \alpha d(x,y), \quad \forall x,y\in X.\]

If $0<\alpha<1$, then $T$ is called a multivalued $(H^+,\alpha)$-contraction.

The purpose of this paper is to study different properties of the Pompeiu functional $H^+$ and of the multivalued operators satisfying a Lipschitz condition with respect to $H^+$. The connections with some continuity notions for multivalued operators are also given. The second purpose of this paper is to extend the results given in [13], by presenting several properties of the fixed point set of multivalued $H^+$-contractions. Several other fixed point results and applications of it will be also given.

2. Properties of the Pompeiu type functional

Concerning the functional $H^+$ defined above, we have some nice properties.

Lemma 2.1 ([13]). We have the following relations:

\[\frac{1}{2}H(A,B) \leq H^+(A,B) \leq H(A,B),\]

(i.e., $H$ and $H^+$ are strong equivalent metrics).

Proposition 2.2 ([13]). Let $(X,\|\cdot\|)$ be a normed linear space. For any $\lambda$ (real or complex), $A, B \in P_{b,c1}(X)$

1. $H^+(\lambda A, \lambda B) = |\lambda| H^+(A,B)$;

2. $H^+(A + a, B + a) = H^+(A,B)$.
Theorem 2.3 ([13]). If $a, b \in X$ and $A, B \in P_{b,c}(X)$, then the following relations hold:

1. $d(a, b) = H^+(a, b)$.
2. $A \subset \bar{S}(B, r_1), B \subset \bar{S}(A, r_2) \Rightarrow H^+(A, B) \leq r$ where $r = \frac{r_1 + r_2}{2}$.

Theorem 2.4 ([13]). If the metric space $(X, d)$ is complete, then $(P_{cp}(X), H^+), (P_{b,c}(X), H^+)$ and $(P_{c}(X), H^+)$ are complete too.

The following concept was introduced by Nadler jr. as follows.

Definition 2.5. Let $(X, d)$ be a metric space. A mapping $T : X \to P_{c}(X)$ is called a multivalued $\alpha$-contraction if $\alpha \in (0,1)$ and

$$H(T(x), T(y)) \leq \alpha d(x, y), \quad \forall x, y \in X.$$  

Notice that any multivalued $\alpha$-contraction is an $(H^+, \alpha)$-contraction, but the reverse implication does not hold.

We will now introduce a similar concept. For this purpose, we recall now the concept of (strong) comparison function.

Definition 2.6. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$.

As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and $\varphi$ is continuous in $0$.

Definition 2.7. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strong comparison function if it is a comparison function and $\sum_{k=0}^{\infty} \varphi^k(t) < \infty$, for any $t > 0$.

With respect to the Pompeiu type functional $H^+$, we define the following concept.

Definition 2.8. Let $(X, d)$ be a metric space. Then, the multivalued operator $T : X \to P_{b,c}(X)$ is called a $\varphi$-contraction w.r.t. $H^+$, if

1. $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function;
2. for all $x, y \in X$, we have that

$$H^+(T(x), T(y)) \leq \varphi(d(x, y)).$$

In particular, if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\varphi(t) := kt$ (for some $k \in [0,1]$), then $\varphi$ is a strong comparison function and the multivalued operator $T$ is an $(H^+, k)$-contraction.

We recall now some useful concepts in the theory of multivalued operators.

Definition 2.9 (see, for example, [1, 14]). Let $(X, d)$ be a metric space and $T : X \to P_{b,c}(X)$. Then, $T$ is called upper semi-continuous (briefly u.s.c.) in $x \in X$, if for any open subset $U$ of $X$ with $F(x) \subset U$, there exists $\eta > 0$ such that $T(B(x; \eta)) \subset U$. $T$ is u.s.c. on $X$ if it is u.s.c. in each $x \in X$.

Definition 2.10 ([1, 14]). Let $(X, d)$ be a metric space and $T : X \to P_{b,c}(X)$. Then $T$ is called lower semi-continuous (briefly l.s.c.) in $x \in X$, if for all $(x_n)_{n \in \mathbb{N}} \subset X$ such that $\lim_{n \to \infty} x_n = x$ and for all $y \in T(x)$, there exists a sequence $(y_n)_{n \in \mathbb{N}} \subset X$ such that $y_n \in T(x_n)$, for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} y_n = y$. $T$ is l.s.c. on $X$ if it is l.s.c. in each $x \in X$.

Definition 2.11 ([1, 14]). Let $(X, d)$ be a metric space. $T : X \to P_{b,c}(X)$ is called $H$-upper semi-continuous in $x_0 \in X$ (H-u.s.c.) respectively $H$-lower semi-continuous (H-l.s.c.), if for each sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that

$$\lim_{n \to \infty} x_n = x_0,$$

we have

$$\lim_{n \to \infty} \rho(T(x_n), T(x_0)) = 0,$$

respectively

$$\lim_{n \to \infty} \rho(T(x_0), T(x_n)) = 0.$$
It is well-known that if T is u.s.c. in \( x \in X \), then T is H-u.s.c. in \( x \in X \), while if T is H-l.s.c. in \( x \in X \) implies that T is l.s.c. in \( x \in X \).

**Definition 2.12 ([1, 14]).** Let \((X, d)\) be a metric space and \( T : X \to P(X) \). Then T is said to be with closed graph, if for each \( x \in X \) and for all \((x_n)_{n \in \mathbb{N}^*} \subset X\) such that

\[
\lim_{n \to \infty} x_n = x,
\]

and for all \((y_n)_{n \in \mathbb{N}^*} \subset X\) with \( y_n \in T(x_n) \), for all \( n \in \mathbb{N}^* \) and

\[
\lim_{n \to \infty} y_n = y,
\]

we have \( y \in T(x) \).

Some properties of a multivalued \((H^+, \alpha)\)-Lipschitz operators are given now.

**Theorem 2.13.** Let \((X, d)\) be a metric space and \( T : X \to P_{b,c1}(X) \) be \((H^+, \alpha)\)-Lipschitz. Then

1. T has closed graph in \( X \times X \);
2. T is H-l.s.c. on \( X \);
3. T is H-u.s.c. on \( X \);
4. If, additionally T has compact values, then T is l.s.c.

**Proof.**

(1) Let \((x_n, y_n) \subset X \times X\) such that \((x_n, y_n) \xrightarrow{d} (x, y)\), when \( n \to \infty \) and \( y_n \in T(x_n) \), for all \( n \in \mathbb{N} \). It follows that

\[
D(y, T(x)) \leq d(y, y_n) + D(y_n, T(x)) \\
\leq d(y, y_n) + H(T(x_n), T(x)) \\
\leq d(y, y_n) + 2H^+(T(x_n), T(x)) \\
\leq d(y, y_n) + 2kd(x_n, x), \quad n \in \mathbb{N}.
\]

Let us consider \( n \to \infty \) and we obtain

\[
D(y, T(x)) \leq 0 \Rightarrow y \in \overline{T(x)} = T(x).
\]

(2) Let \( x \in X \) such that \( x_n \to x \). We have

\[
\rho(T(x), T(x_n)) \leq H(T(x), T(x_n)) \\
\leq 2 \cdot H^+(T(x), T(x_n)) \\
\leq 2k \cdot d(x, x_n) \to 0.
\]

In conclusion, T is H-l.s.c. on X.

(3) Using the relation

\[
\rho(T(x_n), T(x)) \leq H(T(x_n), T(x)) \\
\leq 2 \cdot H^+(T(x_n), T(x)) \\
\leq 2k \cdot d(x, x_n) \to 0,
\]

we obtain that T is H-u.s.c. on X.

(4) The conclusion follows by the fact that any H-l.s.c. multivalued operator with compact values is l.s.c. (see [11]).

\[ \square \]

**Lemma 2.14.** Let \((X, d)\) be a metric space and \( T : X \to P_{cp}(X) \) such that

\[
H^+(T(x), T(y)) < d(x, y), \quad \forall x, y \in X, \quad x \neq y.
\]

Then T is u.s.c. on X.
Proof. Let $Z \subset Y$ be a closed set. We will prove that $T^{-}(Z)$ is closed in $X$. Let $x \in T^{-}(Z) \setminus T^{-}(Z)$ and $(x_{n})_{n \in \mathbb{N}} \subset X$ such that $x_{n} \to x$, when $n \to \infty$, $x_{n} \neq x$, for all $n \in \mathbb{N}$ and $x_{n} \in T^{-}(Z)$, for all $n \in \mathbb{N}$. It follows $T(x_{n}) \cap H \neq \emptyset$, for all $n \in \mathbb{N}$. Let $(y_{n})_{n \in \mathbb{N}} \in T(x_{n}) \cap Z$, $n \in \mathbb{N}$. Then

$$D(y_{n}, T(x)) \leq H_{d}(T(x_{n}), T(x)) \leq 2H^{+}(T(x_{n}), T(x)) < 2d(x_{n}, x).$$

If $n \to \infty$ we get that

$$\lim_{n \to \infty} D(y_{n}, T(x)) = 0.$$

But

$$D(y_{n}, T(x)) = \inf_{y \in T} d(y_{n}, y) = d(y_{n}, x_{n}),$$

(using the compactness of the set $T(x)$).

When $n \to \infty$ we have $d(y_{n}, x') \to 0$. Because $(x_{n})_{n \in \mathbb{N}} \subset T(x)$, we obtain that there exists a subsequence $(x'_{n_{k}})_{k \in \mathbb{N}}$ which converges to an element $x \in T(x)$. Then

$$d(y_{n_{k}}, x) \leq d(y_{n_{k}}, x_{n_{k}}) + d(x_{n_{k}}, x)$$

when $k \to \infty$.

Hence, $y_{n_{k}} \to x \in T(x)$, $n \to \infty$.

Because $(y_{n_{k}})_{k \in \mathbb{N}} \subset Z$ and $Z$ is closed, we obtain that $x \in Z$. So $T(x) \cap Z \neq \emptyset$, which implies $x \in T^{-}(Z)$, a contradiction. In conclusion, $T^{-}(Z) = T^{-}(Z)$ and hence $T^{-}(Z)$ is closed in $X$. \qed

3. MWP operators and multivalued $\alpha$-contractions w.r.t. $H^{+}$

The following concepts appeared in [15].

**Definition 3.1.** Let $(X, d)$ be a metric space. Then, $T : X \to P(X)$ is called a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_{n})_{n \in \mathbb{N}}$ in $X$ such that:

1. $x_{0} = x$, $x_{1} = y$;
2. $x_{n+1} \in T(x_{n})$, for all $n \in \mathbb{N}$;
3. the sequence $(x_{n})_{n \in \mathbb{N}}$ is convergent and its limit is a fixed point of $T$.

**Definition 3.2.** Let $(X, d)$ be a metric space and $T : X \to P(X)$ be an MWP operator. Then we define the multivalued operator $T^{\infty} : \text{Graph}(T) \to P(\text{Graph}(T))$ by the formula $T^{\infty}(x, y) = \{ z \in F_{T} \}$ there exists a sequence of successive approximations of $T$ starting from $(x, y)$ that converges to $z$.

**Definition 3.3.** Let $(X, d)$ be a metric space and $T : X \to P(X)$ an MWP operator. Then $T$ is said to be a $c$-multivalued weakly Picard operator (briefly $c$-MWP operator) if and only if there exists a selection $t^{\infty}$ of $T^{\infty}$ such that

$$d(x, t^{\infty}(x, y)) \leq cd(x, y), \quad \forall (x, y) \in \text{Graph}(T).$$

We recall now the notion of multivalued Picard operator.

**Definition 3.4.** Let $(X, d)$ be a complete metric space and $T : X \to P(X)$. By definition, $T$ is called a multivalued Picard operator (briefly MP operator) if and only if

1. $(\text{SF})_{T} = F_{T} = \{ x^{*} \}$;
2. $T^{n}(x) \xrightarrow{H} \{ x^{*} \}$ as $n \to \infty$, for each $x \in X$.

Recall that, by definition, for $(A_{n})_{n \in \mathbb{N}} \subset P_{c}(X)$, we will write $A_{n} \xrightarrow{H} A^{*}$ as $n \to \infty$ if and only if $H(A_{n}, A^{*}) \to 0$ as $n \to \infty$. Notice also that

$$A_{n} \xrightarrow{H} A^{*} \in P_{c}(X) \text{ as } n \to \infty \text{ if and only if } A_{n} \xrightarrow{H^{+}} A^{*} \in P_{c}(X) \text{ as } n \to \infty.$$
The purpose of this section is to study some properties of the fixed point set of $H^+$-contraction with constant $\alpha$ from the MWP operator theory point of view.

We will start by presenting some auxiliary results.

**Lemma 3.5** (see, for example, [14]). Let $(X,d)$ be a metric space and $A, B \in P_{cl}(X)$. Suppose that there exists $\eta > 0$ such that for each $a \in A$ there exists $b \in B$ such that $d(a,b) \leq \eta$ and for each $b \in B$ there exists $a \in A$ such that $d(a,b) \leq \eta$. Then $H(A, B) \leq \eta$.

**Lemma 3.6** ([14]). Let $(X,d)$ be a metric space, $A, B \in P(X)$ and $q > 1$. Then, for every $a \in A$ there exists $b \in B$ such that $d(a,b) \leq qH(A, B)$.

**Lemma 3.7** ([11]). Let $(X,d)$ be a metric space and $A, B \in P_{cp}(X)$. Then for every $a \in A$ there exists $b \in B$ such that $d(a,b) \leq H(A, B)$.

**Lemma 3.8** ([16]). Let $(X,d)$ be a metric space. If $A, B \in P(X)$ and $\varepsilon > 0$ then for every $a \in A$ there exists $b \in B$ such that

$$d(a,b) \leq H(A, B) + \varepsilon.$$

**Lemma 3.9.** Let $(X,d)$ be a metric space, $A, B \in P_{cl}(X)$ and $\varepsilon > 0$. If $H^+(A, B) < \varepsilon$, then

1. for all $a \in A$ there exists $b \in B$ such that $d(a,b) < \varepsilon$; or
2. for all $b \in B$ there exists $a \in A$ such that $d(a,b) < \varepsilon$.

**Proof.** Suppose, by reductio ad absurdum, that

(i) there exists $a_0 \in A$, for all $b \in B$ such that $d(a_0,b) \geq \varepsilon$;

(ii) there exists $b_0 \in B$, for all $a \in A$ such that $d(a,b_0) \geq \varepsilon$.

Then, taking $\inf_{b \in B}$ in (i) and $\inf_{a \in A}$ in (ii), we obtain $D(a_0,B) \geq \varepsilon$. Since $\rho(A,B) \geq D(a_0,B)$, we get

$$\rho(A,B) \geq \varepsilon.$$

On the other hand, we also have $D(b_0,A) \geq \varepsilon$. Since $\rho(B,A) \geq D(b_0,A)$, we get

$$\rho(B,A) \geq \varepsilon.$$

Adding the above relations and then dividing by 2, we obtain $H^+(A,B) \geq \varepsilon$, which is a contradiction with $H^+(A,B) < \varepsilon$. \hfill \Box

**Lemma 3.10** (Cauchy, see [17]). Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two sequences of non-negative real numbers, such that

$$\sum_{k=0}^{+\infty} a_k < +\infty, \quad \text{and} \quad \lim_{n \to +\infty} b_n = 0.$$

Then,

$$\lim_{n \to \infty} \sum_{k=0}^{n} a_{n-k}b_k = 0.$$

**Theorem 3.11.** Let $(X,d)$ be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued $H^+$-contraction with constant $\alpha$. Then we have

(i) $F_T \neq \emptyset$.

(ii) $T$ is a $\frac{1}{1-\alpha}$-MWP operator.
(iii) Let $S : X \to P_{cl}(X)$ be an $H^+$-contraction with constant $\alpha$ and $\eta > 0$ such that $H^+(S(x), T(x)) \leq \eta$, for each $x \in X$. Then $H^+(F_S, F_T) \leq \frac{2 \cdot \eta}{1 - \alpha}$.

(iv) Let $T_n : X \to P_{cl}(X), n \in \mathbb{N}$ be a sequence of multivalued $H^+$-contraction with constant $\alpha$ such that $T_n(x) \stackrel{H^*}{\longrightarrow} T(x)$ as $n \to \infty$, uniformly with respect to $x \in X$. Then, $F_{T_n} \stackrel{H^*}{\rightarrow} F_T$ as $n \to \infty$.

If, additionally $T(x) \in P_{cp}(X)$ for each $x \in X$, then we also have

(v) (Ulam-Hyers stability of the inclusion $x \in T(x)$) Let $\epsilon > 0$ and $x \in X$ be such that $D(x, T(x)) \leq \epsilon$. Then there exists $x^* \in F_T$ such that $d(x, x^*) \leq \frac{\epsilon}{1 - \alpha}$.

(vi) The fractal operator $\hat{T} : P_{cp}(X) \to P_{cp}(X), \hat{T}(Y) := \bigcup_{x \in Y} T(x)$ is a $2\alpha$-contraction.

(vii) If, additionally, $\alpha \in [0, \frac{1}{2}]$, then $F_T = \{A^*_T\}$ and $T^n(x) \stackrel{H^*}{\longrightarrow} A^*_T$ as $n \to \infty$, for each $x \in X$. Moreover, $F_T \subset A^*_T$, $F_T$ is compact and

$$A^*_T = \bigcup_{n \in \mathbb{N}^*} T^n(x), \quad \forall x \in F_T.$$ 

Proof.

(i) Let $\epsilon > 0$ be given. Let $x_0 \in X$ be arbitrary. Fix an element $x_1 \in T(x_0)$. From the definition of $H^+$-contraction with constant $\alpha$ it follows that we can choose $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq H^+(T(x_0), T(x_1)) + \epsilon.$$ 

In general, if $x_n$ is chosen, then we choose $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \leq H^+(T(x_{n-1}), T(x_n)) + \epsilon.$$ 

Suppose $H^+(T(x_{n-1}), T(x_n)) > 0$ for each $n \in \mathbb{N}^*$ (if not, i.e., if there is $k \in \mathbb{N}^*$ such that

$$H^+(T(x_{k-1}), T(x_k)) = 0,$$

then $x_k \in T(x_{k-1}) = T(x_k)$ is a fixed point for $T$ and we are done). Let $1 < q < \frac{1}{\alpha}$ and set

$$\epsilon_n := (q - 1)H^+(T(x_{n-1}), T(x_n)).$$

Then, from the above relation it follows that

$$d(x_n, x_{n+1}) \leq qH^+(T(x_{n-1}), T(x_n)).$$

Thus, if we set $\beta := q\alpha < 1$, we have

$$d(x_n, x_{n+1}) \leq qH^+(T(x_{n-1}), T(x_n)) \leq q\alpha d(x_{n-1}, x_n) = \beta d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. Repeating the same argument $n$-times we get a sequence of successive approximations for $T$ starting from $(x_0, x_1) \in \text{Graph}(T)$ such that, for each $n \in \mathbb{N}$, we have

$$d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1).$$

Then,

$$d(x_n, x_{n+p}) \leq \beta^n \frac{1 - \beta^p}{1 - \beta} d(x_0, x_1), \quad \forall n \in \mathbb{N}^*, \ p \in \mathbb{N}^*. \quad (3.1)$$

This implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence convergent in $(X, d)$ to some $x^* \in X$. Notice that, by the contraction condition, we immediately get that $\text{Graph}(T)$ is closed in $X \times X$. Hence $x^* \in F_T$. 
(ii) By (3.1), letting \( p \to \infty \), we get that
\[
d(x_n, x^*) \leq \beta^n \frac{1}{1 - \beta} d(x_0, x_1), \quad \forall n \in \mathbb{N}^*.
\]
For \( n = 1 \) we get
\[
d(x_1, x^*) \leq \frac{\beta}{1 - \beta} d(x_0, x_1).
\]
Then
\[
d(x_0, x^*) \leq d(x_0, x_1) + d(x_1, x^*) \leq \frac{1}{1 - \beta} d(x_0, x_1) = \frac{1}{1 - q\alpha} d(x_0, x_1).
\]
Letting \( q \nrightarrow 1 \) we get that for each \( (x_0, x_1) \in \text{Graph}(T) \), there exists \( x^* := t^\infty(x_0, x_1) \in F_T \) such that
\[
d(x_0, t^\infty(x_0, x_1)) \leq \frac{1}{1 - \alpha} d(x_0, x_1),
\]
proving that \( T \) is a \( \frac{1}{1 - \alpha} \)-multivalued weakly Picard operator.

(iii) Let \( x_0 \in S(x_0) \) and \( q > 1 \). Then, by Lemma 3.6, there exists \( x_1 \in T(x_0) \) such that
\[
d(x_0, x_1) \leq qH(S(x_0), T(x_0)) \leq 2qH^+(S(x_0), T(x_0)) \leq 2q\eta.
\]
Then, by (ii) and the above relation, we have proved that for each \( x_0 \in F_S \) there exists \( t^\infty(x_0, x_1) \in F_T \) such that
\[
d(x_0, t^\infty(x_0, x_1)) \leq \frac{1}{1 - \alpha} d(x_0, x_1) \leq \frac{1}{1 - \alpha} 2q\eta.
\]
Now Lemma 3.5 tells us that
\[
\rho(F_S, F_T) \leq \frac{2q\eta}{1 - \alpha}. \quad (3.2)
\]
By a similar procedure we can prove that for each \( y_0 \in T(y_0) \) there exists \( y_1 \in S(y_0) \) such that
\[
d(y_0, y_1) \leq qH(T(y_0), S(y_0)) \leq 2qH^+(T(y_0), S(y_0)) \leq 2q\eta.
\]
Thus, we have proved that for each \( y_0 \in F_T \) there exists \( s^\infty(y_0, y_1) \in F_S \) such that
\[
d(y_0, s^\infty(y_0, y_1)) \leq \frac{1}{1 - \alpha} 2q\eta.
\]
Again, Lemma 3.5 gives that
\[
\rho(F_S, F_T) \leq \frac{2q\eta}{1 - \alpha}. \quad (3.3)
\]
Adding (3.2) and (3.3), and then dividing by 2, we get
\[
H^+(F_S, F_T) \leq \frac{2q\eta}{1 - \alpha}, \quad \forall q > 1.
\]
Letting \( q \nrightarrow 1 \), we get the conclusion.

(iv) Let \( \epsilon > 0 \) be given and choose \( N_\epsilon \in \mathbb{N} \) such that for \( n \geq N_\epsilon \) we have
\[
\sup_{x \in X} H^+(T_n(x), T(x)) < \epsilon, \quad n \geq N_\epsilon.
\]
Then, from (iii), we have
\[
H^+(F_{T_n}, F_T) < \frac{2\epsilon}{1 - \alpha}, \quad \text{for all } n \geq N_\epsilon.
\]
Thus, \( F_{T_n} \xrightarrow{H^+} F_T \) as \( n \to \infty \).
(v) Let $\varepsilon>0$ and $x \in X$ be such that $D(x,T(x)) \leq \varepsilon$. Then, since $T(x)$ is compact, there exists $y \in T(x)$ such that $d(x,y) \leq \varepsilon$. By the proof of (i), we have that

$$d(x,t^\infty(x,y)) \leq \frac{1}{1-\alpha}d(x,y).$$

Since $x^* := t^\infty(x,y) \in F_T$, we get the conclusion $d(x,x^*) \leq \frac{\varepsilon}{1-\alpha}$.

(vi) By the contraction condition with respect to $H^+$, one obtains (see Theorem 2.13) that the operator $T$ is $H$-u.s.c. Since $T(x)$ is compact, for each $x \in X$, we obtain that $T$ is upper semicontinuous. Thus $T$ is u.s.c.

We will prove now that

$$H^+(T(A),T(B)) \leq 2\alpha H^+(A,B).$$

For this purpose, let $u \in T(A)$. Then there exists $a \in A$ such that $u \in T(a)$. From Lemma 3.7 there exists $b \in T(B)$ such that

$$d(a,b) \leq H_d(A,B).$$

Since

$$D(u,T(b)) \leq D(u,T(b)) \leq \rho(T(a),T(b)),$$

taking $\sup_{u \in T(A)}$, we get

$$\rho(T(A),T(B)) \leq \rho(T(a),T(b)).$$

Interchanging the roles of $A$ and $B$, we get

$$\rho(T(B),T(A)) \leq \rho(T(b),T(a)).$$

Adding the above relations and then dividing by 2, we get

$$H^+(T(A),T(B)) \leq H^+(T(a),T(b)).$$

Thus,

$$H(T(A),T(B)) \leq 2H^+(T(A),T(B)) \leq 2\alpha d(a,b) \leq 2\alpha H(A,B), \quad \forall A,B \in P_{cp}(X).$$

(vii) By (vi) it follows that $\hat{T}$ is a self-contraction (with constant $2\alpha < 1$) on the complete metric space $(P_{cp}(X),H)$. By the contraction principle, we obtain that

$$F_T = \{A^+_T\} \text{ and } \hat{T}^n(A) \rightharpoonup A^+_T, \text{ as } n \to \infty, \quad \text{for each } A \in P_{cp}(X).$$

As a consequence of Lemma 2.1, we also get that $\hat{T}^n(A) \rightharpoonup A^+_T$ as $n \to \infty$, for each $A \in P_{cp}(X)$. In particular, if $A := \{x\}$, we get that $T^n(x) = \hat{T}^n(x) \rightharpoonup A^+_T$ as $n \to +\infty$, for each $x \in X$. Let $x \in F_T$ be arbitrary. Then $x \in T(x) \subseteq T^2(x) \subseteq \cdots \subseteq T^n(x) \subseteq \cdots$. Hence $x \in T^n(x)$, for each $n \in \mathbb{N}^*$. Moreover,

$$\lim_{n \to +\infty} T^n(x) = \bigcup_{n \in \mathbb{N}^*} T^n(x).$$

By (vi), we immediately get that $A^+_T = \bigcup_{n \in \mathbb{N}^*} T^n(x)$. Hence

$$F_T \subseteq \bigcup_{n \in \mathbb{N}^*} T^n(x) = A^+_T.$$

Since $F_T$ is closed subset of the compact $A^+_T$, it follows that $F_T$ is compact, too. \hfill \Box

Some new conclusions with respect to the fixed point and the strict fixed point sets are given in our next result.

**Theorem 3.12.** Let $(X,d)$ be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued $(H^+,\alpha)$-contraction with $(SF)_T \neq \emptyset$. Then, the following assertions hold:
(i) \((\text{SF})_T = \{x^*\}\).

If additionally, \(\alpha \in [0, \frac{1}{2}]\), then

(ii) \(F_T = (\text{SF})_T = (\text{SF}_T^n) = \{x^*\}\), for each \(n \in \mathbb{N}^*\).

(iii) \(T^n(x) \xrightarrow{H^+} \{x^*\}\) as \(n \to \infty\), for each \(x \in X\).

(iv) Let \(S : X \to P_{cl}(X)\) be a multivalued operator such that \(F_S \neq \emptyset\) and suppose there exists \(\eta > 0\) such that

\[
H^+(S(x), T(x)) \leq \eta, \quad \forall x \in X.
\]

Then

\[
H^+(F_S, F_T) \leq \frac{2\eta}{1 - 2\alpha}.
\]

(v) (Well-posedness of the fixed point problem w.r.t. to \(H^+\)) If \((x_n)_{n \in \mathbb{N}}\) is a sequence in \(X\) such that

\[
H^+(x_n, T(x_n)) \to 0 \text{ as } n \to \infty,
\]

then \(x_n \xrightarrow{d} x^*\) as \(n \to \infty\).

(vi) (Limit shadowing property of the multivalued operator) If \((y_n)_{n \in \mathbb{N}}\) is a sequence in \(X\) such that

\[
D(y_{n+1}, T(y_n)) \to 0 \text{ as } n \to \infty,
\]

then there exists a sequence \((x_n)_{n \in \mathbb{N}} \subset X\) of successive approximations for \(T\), such that \(d(x_n, y_n) \to 0\) as \(n \to \infty\).

\textbf{Proof.}

(i) Let \(x^* \in (\text{SF})_T\). Notice first that \((\text{SF})_T = \{x^*\}\). Indeed, if \(z \in (\text{SF})_T\) with \(z \neq x^*\), then \(0 < d(x^*, z) = H^+(T(x^*), T(z)) \leq \alpha d(x^*, z)\), which is a contradiction. Thus \((\text{SF})_T = \{x^*\}\).

(ii) Suppose that \(y \in F_T\). Then,

\[
d(y, x^*) = D(y, T(x^*))
\leq \rho(T(y), T(x^*))
\leq H(T(y), T(x^*))
\leq 2H^+(T(y), T(x^*))
\leq 2\alpha d(y, x^*).
\]

Hence, \(y = x^*\) and \(F_T \subset (\text{SF})_T\). Since \((\text{SF})_T \subset F_T\), we get that \((\text{SF})_T = F_T\).

Notice now that \(x^* \in (\text{SF})_T^n\), for each \(n \in \mathbb{N}^*\). Consider \(y \in (\text{SF})_T^n\), for arbitrary \(n \in \mathbb{N}^*\). Then

\[
d(x^*, y) = H(T^n(x^*), T^n(y))
\leq 2\alpha H(T^{n-1}(x^*), T^{n-1}(y))
\leq (2\alpha)^2 H(T^{n-2}(x^*), T^{n-2}(y))
\]
\[
\vdots
\]
\[
\leq (2\alpha)^n d(x^*, y).
\]

Thus, \(y = x^*\) and hence \((\text{SF})_T^n = \{x^*\}\).
(iii) Let \( x \in X \) be arbitrarily chosen. Then we have
\[
H^+(T^n(x), x^*) = H^+(T^n(x), T^n(x^*)) \\
\leq H(T^n(x), T^n(x^*)) \\
\leq (2\alpha)H(T^{n-1}(x), T^{n-1}(x^*)) \\
\vdots \\
\leq (2\alpha)^n d(x, x^*) \to 0 \text{ as } n \to \infty.
\]

(iv) Let \( y \in F_S \). Then
\[
d(y, x^*) \leq H(S(y), x^*) \\
\leq 2H^+(S(y), x^*) \\
\leq 2(H^+(S(y), T(y)) + H^+(T(y), x^*)) \\
\leq 2(\eta + \alpha d(y, x^*)).
\]

Thus, \( d(y, x^*) \leq \frac{2\eta}{1-2\alpha} \). The conclusion follows by the relations
\[
H^+(F_S, F_T) \leq \sup_{y \in F_S} d(y, x^*) \leq \frac{2\eta}{1-2\alpha}.
\]

(v) Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( X \) such that \( H^+(x_n, T(x_n)) \to 0 \) as \( n \to \infty \). Then,
\[
d(x_n, x^*) \leq D(x_n, T(x_n)) + H_d(T(x_n), T(x^*)) \\
\leq H_d(x_n, T(x_n)) + 2H^+(T(x_n), T(x^*)) \\
\leq 2H^+(x_n, T(x_n)) + 2k d(x_n, x^*).
\]

Then
\[
d(x_n, x^*) \leq \frac{2}{1-2k} \to 0 \text{ as } n \to \infty.
\]

(vi) Let \( (y_n)_{n \in \mathbb{N}} \) be a sequence in \( X \) such that
\[
D(y_{n+1}, T(y_n)) \to 0 \text{ as } n \to \infty.
\]

Then, there exists \( u_n \in T(y_n), n \in \mathbb{N} \) such that
\[
d(y_{n+1}, u_n) \to 0 \text{ as } n \to \infty.
\]

We shall prove that \( d(y_n, x^*) \to 0 \) as \( n \to \infty \).

We successively have
\[
d(x^*, y_{n+1}) \leq H(x^*, T(y_n)) + D(y_{n+1}, T(y_n)) \\
\leq 2H^+(x^*, T(y_n)) + D(y_{n+1}, T(y_n)) \\
\leq 2\alpha d(x^*, y_n) + D(y_{n+1}, T(y_n)) \\
\leq 2\alpha [2\alpha d(x^*, y_{n-1}) + D(y_{n}, T(y_{n-1}))] + D(y_{n+1}, T(y_n)) \\
\vdots \\
\leq (2\alpha)^{n+1} d(x^*, y_0) + (2\alpha)^n D(y, T(y_0)) + \cdots + D(y_{n+1}, T(y_n)).
\]

By Lemma 3.10, the right hand side tends to 0 as \( n \to \infty \). Thus, \( d(x^*, y_{n+1}) \to 0 \) as \( n \to \infty \).
On the other hand, by the fact that $T$ is an MWP operator, we know that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of successive approximations for $T$ starting from arbitrary $(x_0, x_1) \in \text{Graph}(T)$ which converges to a fixed point $x^* \in X$ of the operator $T$. Since the fixed point is unique, we get that $d(x_n, x^*) \to 0$ as $n \to \infty$. Hence, for such a sequence $(x_n)_{n \in \mathbb{N}}$, we have

$$d(y_n, x_n) \leq d(y_n, x^*) + d(x^*, x_n) \to 0, \text{ as } n \to \infty.$$ 

□

**Remark 3.13.** Similar results can be given for the case of multivalued $\varphi$-contraction w.r.t. $H^+$. The results of this type can be viewed as generalizations of some theorems given in [7].

We now give an application of the above results to the continuous dependence of the solution set for a Cauchy problem associated to a differential inclusion, with respect to the initial condition. The existence of a solution to the initial value problem

$$\begin{align*}
\begin{cases}
\dot{x}(t) &\in T(t, x(t)), \\
x(0) & = b,
\end{cases}
\end{align*}$$

was proved by Filippov [3] and Castaing [2] under certain conditions on $T$.

In [10], Markin proved a stability theorem on the set of solutions to (3.4) using the $L^2$ norm, while Lim [9] proved a stability result in terms of the Hausdorff-Pompeiu functional. We will prove now a similar theorem using the sup norm and the Pompeiu functional generated by it.

We recall first the concept of solution.

**Definition 3.14.** Let $D = [0, a] \times \mathbb{R}^n$ and $T : D \to \mathcal{P}(\mathbb{R}^n)$ be a continuous operator. Then, a mapping $x : [0, a] \to \mathbb{R}^n$ is said to be a solution of the differential inclusion (3.4), if $x$ is an absolutely continuous mapping and $x'(t) \in T(t, x(t))$, a.e. on $[0, a]$.

Let $B$ be an origin-centered closed ball in $\mathbb{R}^n$ and $P_{cl,cv}(B)$ endowed with the $H^+$ metric generated by the Euclidean norm $\| \cdot \|$ of $\mathbb{R}^n$. Let $C[0, a]$ be the set of the continuous maps of $[0, a]$ into $\mathbb{R}^n$ with the sup norm $\| \cdot \|_C$.

Assume that $T$ is a continuous map of $[0, a] \times B$ into $P_{cl,cv}(B)$ satisfying, for some $k > 0$, the condition

$$H^+(T(t, u), T(t, v)) \leq k\|u - v\|_C, \quad \forall t \in [0, a], \quad u, v \in B.$$ 

For $b \in B$, we will denote $S(b)$ the set of solutions of (3.4) on $[0, a]$. $S(b)$ is nonempty and compact, by [3] and [2].

**Theorem 3.15.** If the following conditions hold:

1. $T : [0, a] \times B \to P_{cl,cv}(B)$ is continuous;
2. there exists $k > 0$ such that
   $$H^+(T(t, u), T(t, v)) \leq k\|u - v\|_C, \quad \forall t \in [0, a], \quad \forall u, v \in B \subseteq \mathbb{R}^n;$$
3. $2ka < 1$,

then $S(b)$ is continuous from $B$ into the family of nonempty compact subsets of $C[0, a]$ equipped with the $H^+$ metric.

**Proof.** Suppose $b_n \to b_0$. For $x \in C[0, a]$, define

$$F(b, x) = \{ y \in [0, a] : y(t) = b + \int_0^t z(s)ds, \quad z(s) \in T(s, x(s)) \}.$$ 

Let $F_n(x) = F(b_n, x)$, $n = 0, 1, 2, \ldots$. Since $F_0(x) = b_n - b_0 + F_n(x)$ it is obvious that $F_n(x)$ converges uniformly to $F_0(x)$. $F_n(x)$ is compact convex for each $x$ and $n$. Given any pair $x_1, x_2 \in C[0, a]$ and $y_1 \in F(b, x_1)$, let

$$y_1(t) = b + \int_0^t r_1(s)ds, \quad r_1(s) \in T(s, x_1(s)).$$
Define \( r_2(s) \) to be the point in \( T(s, x_2(s)) \) nearest to \( r_1(s), \) i.e., \( r_2 \in T(s, x_2(s)) \) and
\[
||r_1(s) - r_2(s)|| = \min(||r_1(s) - z|| \mid z \in T(s, x_2(s))).
\]

It follows from the measurability of \( r_1(s) \) and the continuity of \( T(s, x_2(s)) \) and the nearest point projection that \( r_2(s) \) is measurable.

Setting
\[
y_2(t) = b + \int_0^t r_2(s) ds, \quad r_2(s) \in T(s, x_2(s)),
\]
we have
\[
||y_2 - y_1|| \leq \int_0^a ||r_1(s) - r_2(s)|| ds
\]
\[
= \int_0^a \min(||r_1(s) - z|| \mid z \in T(s, x_2(s)))
\]
\[
= \int_0^a D ||r_1(s), T(s, x_2(s))|| ds
\]
\[
\leq \int_0^a H_d \left( T(s, x_1(s)), T(s, x_2(s)) \right) ds
\]
\[
\leq \int_0^a 2H^+ \left( T(s, x_1(s)), T(s, x_2(s)) \right) ds \leq 2k \int_0^a ||x_1(s) - x_2(s)|| ds
\]
\[
\leq 2ka||x_1 - x_2||_{C}.
\]

Thus \( F_n \) are \( \lambda \)-contraction with \( \lambda = 2ka < 1. \)

By (iv) of Theorem 3.11, \( F(T_n) \xrightarrow{H^+} F(T_0) \) i.e., \( S(b_n) \xrightarrow{H^+} S(b_0) \).

More generally, we have the following result.

**Theorem 3.16.** For each \( n = 0, 1, 2, \cdots \), let \( T_n \) be a continuous map of \([0, a] \times B \) into \( C(B) \) satisfying, for some \( k > 0, \) the condition
\[
H^+(T_n(t, u), T_n(t, v)) \leq k||u - v||_C, \quad \forall t \in [0, a], \quad \forall u, v \in B.
\]
Assume that \( T_n \rightarrow T_0 \) uniformly on \([0, a] \times B. \) For each \( b \in B \) and \( n = 0, 1, 2, \cdots \). Let \( S_n(b) \) be the set of solutions of
\[
\begin{cases}
\dot{x}(t) \in T_n(t, x(t)), \\
x(0) = b.
\end{cases}
\]
If \( 2ka < 1 \) and \( b_n \rightarrow b_0 \) in \( B, \) then \( S_n(b_n) \rightarrow S_0(b_0). \)

**Proof.** Let \( b_n \rightarrow b_0. \) For \( x_n \in C[0, a], \) define
\[
F(b, x_n) = \{ y_n \in [0, a] : y_n(t) = b + \int_0^t z_n(s) ds, \ z_n(s) \in T_n(s, x(s))\}.
\]
Let \( F_n(x_n) = F(b_n, x_n), \) \( n = 0, 1, 2, \cdots. \) Since \( F_0(x) = b_n - b_0 + F_n(x) \) it is obvious that \( F_n(x) \) converges uniformly to \( F_0(x). \) \( F_n(x) \) is compact convex for each \( x_n \) and \( n. \)

Given any pair \( x_n^{(1)}, x_n^{(2)} \in C[0, a] \) and \( y_n^{(1)} \in F(b, x_n^{(1)}), \) let
\[
y_n^{(1)}(t) = b + \int_0^t r_n^{(1)}(s), \ r_n^{(1)}(s) \in T_n(s, x_n^{(1)}(s)) ds.
\]

Define \( r_n^{(2)}(s) \) to be the point in \( T_n(s, x_n^{(2)}(s)) \) nearest to \( r_n^{(1)}(s), \) i.e., \( r_n^{(2)} \in T_n(s, x_n^{(2)}(s)) \) and
\[
||r_n^{(1)}(s) - r_n^{(2)}(s)|| = \min(||r_n^{(1)}(s) - z_n|| \mid z_n \in T_n(s, x_n^{(2)}(s))).
\]
It follows from the measurability of \( r_n^{(1)}(s) \) and the continuity of \( T_n(s, x_n^{(2)}(s)) \) and the nearest point pro-
Definition 3.17. The notion of Ulam-Hyers stability for a differential inclusion is defined as follows.

Thus \( F_n \) are \( \lambda \)-contraction with \( \lambda = 2ka < 1 \).

By (iv) of Theorem 3.11, we have \( F(T_n) \xrightarrow{H^+} F(T_0) \), i.e., \( S_n(b_n) \xrightarrow{H^+} S_n(b_0) \).

Next, an application to Ulam-Hyers stability of the inclusion \( x \in T(x) \) ((v) of Theorem 3.11) is given. The notion of Ulam-Hyers stability for a differential inclusion is defined as follows.

Definition 3.18. Let \( F : [0, a] \rightarrow P_{cl, cv}(\mathbb{R}^n) \) be a measurable multivalued operator. If \( L^1([0, a], \mathbb{R}^n) \) denotes the set of all measurable and integrable mappings from \([0, a]\) to \( \mathbb{R}^n \), then \( S_F \) will denote the set of all integrable selections of \( F \), i.e.,

\[
S_F := \{ f \in L^1([0, a], \mathbb{R}^n) | f(t) \in F(t), \text{ a.e. } t \in [0, a] \}.
\]

Remark 3.19. In particular, if \( x : [0, a] \rightarrow \mathbb{R}^n \) and \( T : [0, a] \times \mathbb{R}^n \rightarrow P_{cl}(\mathbb{R}^n) \), then the set of all integrable selections of \( T \) will be denoted by

\[
S_{T(x(t))} := \{ f \in L^1([0, a], \mathbb{R}^n) | f(t) \in T(t, x(t)) \text{ a.e. } t \in [0, a] \}.
\]

Theorem 3.20. Let us consider the inclusion (3.5). We assume:

(a) \( T : [0, a] \times \mathbb{R}^n \rightarrow P_{cl, cv}(\mathbb{R}^n) \) is a continuous, measurable and integrably bounded multivalued operator.

(b) There exists \( L > 0 \) such that

\[
H^+(T(t, u_1), T(t, u_2)) \leq L\|u_1 - u_2\|, \quad \forall (t, u_1), (t, u_2) \in [0, a] \times \mathbb{R}^n.
\]

Then the differential inclusion (3.5) with initial condition \( x(0) = x^0 \) has at least one solution. Moreover the differential inclusion (3.5) is Ulam-Hyers stable.

Proof. Let us define \( U : C[0, a] \rightarrow P(C[0, a]) \), \( u \rightarrow Ux \) and \( Ux(t) := b + \int_0^t T(s, x(s))ds, \ t \in [0, a] \). We notice that, since \( T \) is u.s.c., (3.5) is equivalent with the fixed point problem

\[
x \in Ux. \tag{3.6}
\]

We will show that the fixed point problem (3.6) is Ulam-Hyers stable.

Let \( y, z \in C[a, b] \) and \( u_1 \in Ux \). Then \( u_1 \in C[0, a] \) and

\[
u_1(t) \in x(0) + \int_0^t T(s, x(s))ds \text{ a.e. on } [0, a].
\]
It follows that there is a mapping $k_y \in S_{T(\cdot,y(\cdot))}$ such that
\[ u_1(t) = x(0) + \int_0^t k_x(s)ds \text{ a.e. on } [0,a]. \]

Since
\[ H^+(T(t,x(t)), T(t,y(t))) \leq L|x(t) - y(t)|, \]
on one obtains that there exists $w \in T(t,y(t))$ such that
\[ \|k_x(t) - w\| \leq H(T(t,x(t)), T(t,y(t))) \leq 2H^+(T(t,x(t)), T(t,y(t))) \leq 2L|x(t) - y(t)|. \]
Thus the multivalued operator $G$ defined by $G(t) = T_y(t) \cup K(t)$ (where $T_y(t) = T(t,y(t))$ and
\[ K(t) = \{w | \|k_x(t) - w\| \leq 2L|x(t) - y(t)|\}, \]
has nonempty values and is measurable.

Let $k_z$ be a measurable selection for $G$ (which exists by Kuratowski and Ryll Nardzewski’s selection theorem). Then $k_y(t) \in T(t,y(t))$ and
\[ \|k_x(t) - k_y(t)\| \leq 2L|x(t) - y(t)|, \text{ a.e. on } [0,a]. \]

Define $u_2 = x(0) + \int_0^t k_y(s)ds$. It follows that $u_2 \in Uy$ and
\[
\|u_1(t) - u_2(t)\| \leq \int_0^t \|k_x(s) - k_y(s)\|ds \\
\leq 2L \int_0^t \|x(s) - y(s)\|ds \\
= 2L \int_0^t \|x(s) - y(s)\| \cdot e^{-\tau(s-a)} \cdot e^{\tau(s-a)}ds \\
\leq 2L \|x - y\|B \int_0^t e^{\tau(s-a)}ds \\
\leq \frac{2L}{\tau} e^{\tau(s-a)}\|x - y\|B.
\]

Here $\| \cdot \|_B$ denotes the Bielecki-type norm on $C[0,a]$. Finally, we have that
\[ \|u_1 - u_2\| \leq \frac{2L}{\tau} \|x - y\|_B. \]

From this and the analogous inequality obtained by interchanging the roles $x$ and $y$ and adding them and then dividing by 2 we get that
\[ H^+(Ux, Uy) \leq \frac{2L}{\tau} \|x - y\|_B, \quad \forall x, y \in C[0,a]. \]

Taking $\tau > 2L$, it follows that $U$ is multivalued $(H^+, \alpha)$-contraction.

4. Continuation results for multivalued $(H^+, \alpha)$-contractions

In this section, we present a local result and a continuation result for a special kind of multivalued $(H^+, \alpha)$-contractions. Following Kirk and Shahzad ([6]), we will replace the second condition of the Definition 1.2:
for every \( x \in X, y \in T(x) \) and for every \( \epsilon > 0 \) there exists \( z \in T(y) \) such that
\[
d(y, z) \leq H^+(T(x), T(y)) + \epsilon,
\]
with the following one:
\[
(**) \text{ for every } x \in X \text{ and every } y \in T(x) \text{ we have that}
\[
D(y, T(y)) \leq H^+(T(x), T(y)).
\]
Notice that (**) implies (*). Moreover if we consider the following condition:
\[
(***) \text{ for every } \epsilon > 0, \text{ for every } x \in X, y \in T(x) \text{ there exists } z \in T(y) \text{ such that}
\[
d(y, z) \leq H^+(T(x), T(y)) + \epsilon.
\]
Then it is easy to see that (**) is equivalent with (***) . In this last case, we also notice that for each \( x \in X \) and every \( y \in T(x) \) we have \( \rho(T(x), T(y)) \leq H^+(T(x), T(y)) \). As a consequence, \( \rho(T(x), T(y)) \leq \rho(T(y), T(x)) \) and so
\[
H^+(T(x), T(y)) \leq \rho(T(y), T(x)), \quad \forall x \in X, \; y \in T(x).
\]
Homotopy results for multivalued operators of contractive types are well-known in the literature, see [4,5,8]. This approach is applied in all cases on a local fixed point theorem. The first result of this section is the following local fixed point theorem.

**Theorem 4.1.** Let \( (X,d) \) be a complete metric space, \( x_0 \in X, \; r > 0 \) and \( T : \bar{B}(x_0, r) \to P_{cl}(X) \) is a multivalued operator. We suppose that:

(i) \( T \) is a multivalued \((H^+, \alpha)\)-contraction, i.e., \( \alpha \in ]0,1[ \) and
\[
H^+(T(x), T(y)) \leq \alpha d(x, y), \quad \forall x, y \in X;
\]
(ii) for every \( x \in X \) and every \( y \in T(x) \) we have that \( D(y, T(y)) \leq H^+(T(x), T(y)) \);
(iii) \( D(x_0, T(x_0)) < (1 - \alpha)r \).

Then, there exists \( x^* \in \bar{B}(x_0, r) \) such that \( x^* \in T(x^*) \).

**Proof.** Notice first that, by (iii), we can find an element \( x_1 \in T(x_0) \) such that \( d(x_0, x_1) < (1 - \alpha)r \). Clearly \( x_1 \in \bar{B}(x_0, r) \). Now, for arbitrary \( \epsilon > 0 \), by (ii) and (i), it follows that we can choose \( x_2 \in T(x_1) \) such that
\[
d(x_1, x_2) < D(x_1, T(x_1)) + \epsilon \leq H^+(T(x_0), T(x_1)) + \epsilon \leq \alpha d(x_0, x_1) + \epsilon.
\]
If we take \( \epsilon := \alpha(1 - \alpha)(d(x_0, x_1)) > 0 \), then we get that
\[
d(x_1, x_2) < \alpha(1 - \alpha)r.
\]
Moreover \( d(x_2, x_0) \leq d(x_0, x_1) + d(x_1, x_2) < (1 - \alpha)r + \alpha(1 - \alpha)r = (1 - \alpha^2)r \), proving that \( x_2 \in \bar{B}(x_0, r) \).

Using this procedure (taking at each step \( k \geq 2 \), for the construction of \( x_k \), the value of \( \epsilon \) as \( \epsilon_k := \alpha(\alpha^{k-2}(1 - \alpha)r - d(x_{k-2}, x_{k-1})) \)), we obtain a sequence \( (x_n)_{n \in \mathbb{N}} \) having the properties:

1. \( d(x_0, x_n) < (1 - \alpha^n)r \), for each \( n \in \mathbb{N}^* \) (i.e., \( x_n \in \bar{B}(x_0, r) \), for each \( n \in \mathbb{N}^* \));
2. \( x_{n+1} \in T(x_n) \), for each \( n \in \mathbb{N} \);
3. \( d(x_n, x_{n+1}) < \alpha^n(1 - \alpha)r \), for each \( n \in \mathbb{N} \).
Then, by (3), we get that the sequence \((x_n)_{n \in \mathbb{N}}\) is Cauchy in \((X, d)\) and hence converges in \((X, d)\) to some \(x^* \in \bar{B}(x_0, r)\). By Theorem 2.13, we have that \(T : \bar{B}(x_0, r) \rightarrow P_c(X)\) has closed graph, thus we immediately get, by (2), that \(x^* \in T(x^*)\) as \(n \to \infty\).

\(\square\)

**Theorem 4.2.** Let \((X, d)\) be a complete metric space. Let \(U\) be an open subset of \((X, d)\). Let \(G : U \times [0, 1] \rightarrow P(X)\) be a multivalued operator such that the following conditions are satisfied:

1. \(x \neq G(x, t)\) for each \(x \in \partial B\) and each \(t \in [0, 1]\);
2. there exists \(\alpha \in [0, 1]\) such that for each \(t \in [0, 1]\) and each \(x, y \in U\) we have
   \[H^+(G(x, t), G(y, t)) \leq \alpha d(x, y);\]
3. there exists a continuous increasing function \(\phi : [0, 1] \rightarrow \mathbb{R}\) such that
   \[H^+(G(x, t), G(x, s)) \leq |\phi(t) - \phi(s)|, \quad \forall t, s \in [0, 1], \ t \neq s, \ \forall x \in U;\]
4. \(G : U \times [0, 1] \rightarrow P((X, d))\) is closed.

Then \(G(\cdot, 0)\) has a fixed point if and only if \(G(\cdot, 1)\) has a fixed point.

**Proof.** Let us consider the set \(Q = \{(t, x) \in [0, 1] \times U : x \in G(x, t)\}\). Clearly \(Q \neq \emptyset\), since \((0, z) \in Q\) where \(z \in G(z, 0)\). On \(Q\) we define the partial order

\[(t, x) \leq (s, y) \quad \text{if and only if} \quad t \leq s \quad \text{and} \quad d(x, y) \leq \frac{2}{1 - \alpha}(\phi(s) - \phi(t)).\]

Let \(P\) be a totally ordered subset of \(Q\). Define \(t^* = \sup\{t : (t, x) \in P\}\).

Taking a sequence \(\{(t_n, x_n)\}\) in \(P\) such that

\[(t_n, x_n) \leq (t_{n+1}, x_{n+1}) \quad \text{and} \quad t_n \to t^* \text{ as } n \to \infty.

We have

\[d(x_m, x_n) \leq \frac{2}{1 - \alpha}(\phi(t_m) - \phi(t_n)), \quad \text{for } m > n, \ m, n \in \mathbb{N}^*.

Thus, \((x_n)\) is a Cauchy sequence, and hence converges to some \(x \in \bar{U}\). Since \(x_n \in G(x_n, t_n)\), \(n \in \mathbb{N}^*\) and \(G\) is closed, we have \(x^* \in G(x^*, t^*)\). Thus \((t^*, x^*) \in Q\).

Since \(P\) is totally ordered we get \((t, x) \leq (t^*, x^*)\) for each \((t, x) \in P\). That means that \((t^*, x^*)\) is a bound of \(P\). It follows from Zorn’s Lemma that \(Q\) admits a maximal element \((t_0, x_0)\).

To complete the proof, we have to show that \(t_0 = 1\). Suppose this is false. Then, we can choose \(r > 0\) and \(t \in (t_0, 1]\) such that \(\bar{B}(x_0, r)\) and \(r := \frac{2}{1 - \alpha}(\phi(t) - \phi(t_0))\). It follows that

\[D(x_0, G(x_0, t)) \leq \rho(G(x_0, t_0), G(x_0, t)) \leq H(G(x_0, t_0), G(x_0, t)) \leq 2H^+(G(x_0, t_0), G(x_0, t)) < \phi(t) - \phi(t_0) = (1 - \alpha)r.

Since \(\bar{B}(x_0, r) \subset U\), the multivalued operator \(G(\cdot, t) : \bar{B}(x_0, r) \rightarrow P_c(X)\) satisfies, for all \(t \in [0, 1]\) the assumptions of Theorem 4.1. Hence, for all \(t \in [0, 1]\), there exists \(x \in \bar{B}(x_0, r)\) such that \(x \in G(x, t)\). Thus \((t, x) \in Q\).

Since \(d(x_0, x) \leq r = \frac{2}{1 - \alpha}(\phi(t) - \phi(t_0))\) we immediately get \((t_0, x_0) < (t, x)\). This is a contradiction with the maximality of \((t_0, x_0)\).

Conversely, if \(G(\cdot, 1)\) has a fixed point, then putting \(t := 1 - t\) and using the first part of the proof we get the conclusion. \(\square\)
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References