Existence and multiplicity of positive solutions for a nonlocal problem

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Abstract

In this work, we are interested in considering the following nonlocal problem

\[
\begin{aligned}
- \left( a - b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u &= f(x)|u|^{p-2}u, \quad \text{in } \Omega, \\
\Delta u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) is a bounded domain with smooth boundary \( \partial \Omega \), \( a, b > 0, 1 \leq p < 2^* \), \( f \in L^{\frac{2^*}{2}}(\Omega) \) is nonzero and nonnegative. By using the variational method, some existence and multiplicity results are obtained. ©2017 All rights reserved.

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1. Introduction and main result

In this paper, we consider the following nonlocal problem

\[
\begin{aligned}
- \left( a - b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u &= f(x)|u|^{p-2}u, \quad \text{in } \Omega, \\
\Delta u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where \( \Omega \subset \mathbb{R}^N \) (\( N \geq 3 \)) is a bounded domain with smooth boundary \( \partial \Omega \), \( a, b > 0, 1 \leq p < 2^* \), the weight function \( f \in L^{\frac{2^*}{2}}(\Omega) \) is nonzero and nonnegative. \( 2^* = \frac{2N}{N-2} \) is the critical Sobolev exponent for the embedding of \( H_0^1(\Omega) \) into \( L^q(\Omega) \) for every \( q \in [1, 2^*] \), where \( H_0^1(\Omega) \) is a Sobolev space equipped with the norm \( ||u|| = \left( \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \), and \( |u|_q = \left( \int_{\Omega} |u|^q \, dx \right)^{\frac{1}{q}} \) denotes the norm of \( L^q(\Omega) \).

When \( 2 < p < 2^* \) and \( f(x) \equiv 1 \), problem (1.1) was considered by [5] for the first time. By using the mountain pass lemma, they obtained the existence of nontrivial solutions for problem (1.1). One of their...
In general, a function \( u \) is called a weak solution of problem (1.1) if \( u \in H^1_0(\Omega) \) and for all \( \varphi \in H^1_0(\Omega) \) it holds

\[
(a - b||u||^2)\int_\Omega (\nabla u, \nabla \varphi)dx - \int_\Omega f(x)|u|^{p-2}u\varphi dx = 0.
\]

Let \( S \) be the best Sobolev constant, namely

\[
S := \inf_{u \in D^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^2 dx)^{\frac{p}{2}}} := \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{(\int_\Omega |u|^2 dx)^{\frac{p}{2}}}.
\]

Now our main result can be described as follows:

**Theorem 1.1.** Assume that \( a, b > 0, 1 \leq p < 2^* \) and \( f \in L^{\frac{2^*}{2^* - p}}(\Omega) \) is nonzero and nonnegative, then

1. when \( 1 \leq p < 2 \), there exists \( T > 0 \) such that for any \( |f|^{\frac{2^*}{2^* - p}} < T \), (1.1) has at least two positive solutions \( u_+, u_{**} \) with \( I(u_+) < 0 \) and \( I(u_{**}) > 0 \);
2. when \( p = 2, |f|^{\frac{2^*}{2^* - p}} < aS \) or \( 2 < p < 2^* \), (1.1) has at least one positive mountain-pass solution \( u_{**} \) with \( I(u_{**}) > 0 \).

**Remark 1.2.** Compared with [3] and [5], we consider (1.1) with \( p = 1,2 \) and obtain the existence of positive solutions by the strong maximum principle. Particular, compared with [5], we study problem (1.1) with \( 1 \leq p \leq 2 \) and obtain the existence and multiplicity of positive solutions. Compared with [3], we generalize the dimension \( N = 3 \) to \( N \geq 3 \).

**2. Proof of Theorem 1.1**

In this part, we will give the proof of Theorem 1.1. Before proving Theorem 1.1, we give the following lemma.

**Lemma 2.1.** Assume \( a, b > 0, 1 \leq p < 2^* \) and \( f \in L^{\frac{2^*}{2^* - p}}(\Omega) \) is nonzero and nonnegative, then \( I \) satisfies the \( (PS)_c \) condition with \( c < \frac{a^2}{4b} \).

**Proof.** Suppose \( \{u_n\} \subset H^1_0(\Omega) \) is a \( (PS)_c \) sequence for \( I \), that is,

\[
I(u_n) \to c, \quad I'(u_n) \to 0, \quad \text{as } n \to \infty.
\]

By the Hölder inequality and (1.3), one has

\[
\int_\Omega f(x)|u|^p dx \leq |f|^{\frac{2^*}{2^* - p}}|u|^{\frac{2^*p}{2^* - p}} \leq |f|^{\frac{2^*}{2^* - p}} S^{-\frac{p}{2}} \|u\|^p.
\]
When \(1 \leq p < 2\), it follows from (2.1) and (2.2) that
\[
1 + c + o(\|u_n\|) \geq I(u_n) - \frac{1}{4}(I'(u_n), u_n)
\]
\[
= \frac{a}{4}\|u_n\|^2 - \frac{4 - p}{4p}\int_\Omega f(x)|u_n|^p\,dx
\]
\[
\geq \frac{a}{4}\|u_n\|^2 - \frac{(4 - p)|f|_{\infty}^{\frac{p}{2}}}{4ps^{\frac{p}{2}}}\|u_n\|^p,
\]
which implies that \(\{u_n\}\) is bounded in \(H^1_0(\Omega)\). When \(2 \leq p < 2^*\), it follows from (2.1) that
\[
1 + c + o(\|u_n\|) \geq I(u_n) - \frac{1}{2}(I'(u_n), u_n)
\]
\[
= \frac{b}{4}\|u_n\|^4 + \frac{p - 2}{2p}\int_\Omega f(x)|u_n|^p\,dx
\]
\[
\geq \frac{b}{4}\|u_n\|^4,
\]
which implies that \(\{u_n\}\) is bounded in \(H^1_0(\Omega)\). Going if necessary to a subsequence, still denoted by \(\{u_n\}\), there exists \(u \in H^1_0(\Omega)\) such that
\[
\begin{aligned}
&u_n \rightharpoonup u, \quad \text{weakly in } H^1_0(\Omega), \\
&u_n \rightarrow u, \quad \text{strongly in } L^s(\Omega), \quad 1 \leq s < 2^*, \\
&u_n(x) \rightarrow u(x), \quad \text{a.e. in } \Omega,
\end{aligned}
\tag{2.3}
\]
as \(n \rightarrow \infty\). Moreover, by the Vitali Theorem, one obtains
\[
\lim_{n \rightarrow \infty} \int_\Omega f(x)|u_n|^p\,dx = \int_\Omega f(x)|u|^p\,dx.
\]
Set \(w_n = u_n - u\), then \(\|w_n\| \rightarrow 0\). Otherwise, there exists a subsequence, still denoted by \(\{w_n\}\), such that
\[
\lim_{n \rightarrow \infty} \|w_n\| = 1 > 0.
\]
From (2.1), for every \(\phi \in H^1_0(\Omega)\), it holds
\[
(a - b\|u_n\|^2)\int_\Omega (\nabla u_n, \nabla \phi)\,dx - \int_\Omega f(x)|u_n|^{p-2}u_n\phi\,dx = o(1).
\]
Letting \(n \rightarrow \infty\), by using (2.3), we have
\[
(a - bl^2 - b\|u\|^2)\int_\Omega (\nabla u, \nabla \phi)\,dx - \int_\Omega f(x)|u|^{p-2}u\phi\,dx = 0. \tag{2.4}
\]
Taking \(\phi = u\) in (2.4), one has
\[
(a - bl^2 - b\|u\|^2)\|u\|^2 - \int_\Omega f(x)|u|^p\,dx = 0. \tag{2.5}
\]
Note that \(\langle I'(u_n), u_n \rangle \rightarrow 0\) as \(n \rightarrow \infty\), it holds
\[
a\|w_n\|^2 + a\|u\|^2 - b\|w_n\|^4 - 2b\|w_n\|^2\|u\|^2 - b\|u\|^4 - \int_\Omega f(x)|u|^p\,dx = o(1). \tag{2.6}
\]
It follows from (2.5) and (2.6) that
\[ a\|w_n\|^2 - b\|w_n\|^4 - b\|w_n\|^2\|u\|^2 = \alpha(1). \] (2.7)

Consequently, one has \( t^2(a - b\|u\|^2 - bt^2) = 0 \), that is,
\[ t^2 = \frac{a}{b} - \|u\|^2. \] (2.8)

On the one hand, from (2.5) and (2.8), we have
\[
I(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p} \int_\Omega f(x)|u|^p \, dx \\
= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p} (a\|u\|^2 - bt^2\|u\|^2 - b\|u\|^4) \\
= \frac{a(p-2)}{2p}\|u\|^2 + \frac{b(4-p)}{4p}\|u\|^4 + \frac{b}{p}\|u\|^2 \frac{a-b\|u\|^2}{b} \\
= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4.
\] (2.9)

On the other hand, by (2.1), (2.7) and (2.8), it follows from \( c < \frac{a^2}{4b} \) that
\[
I(u) = \lim_{n \to \infty} \left[ I(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u\|^2 \right] \\
= \lim_{n \to \infty} \left[ I(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u\|^2 \right] \\
= c - \frac{b}{4}\|u\|^4 \\
= c - \frac{a^2}{4b} + \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 \\
< \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4,
\]
which contradicts (2.9). Hence, \( t = 0 \), that is, \( u_n \to u \) in \( H^1_0(\Omega) \) as \( n \to \infty \). Therefore, I satisfies the \( (PS)_c \) condition for \( c < \frac{a^2}{4b} \). This completes the proof of Lemma 2.1. \( \Box \)

Now, we give the following two important propositions.

**Proposition 2.2.** Assume \( 1 \leq p < 2 \) and \( f \in L^{\frac{2^*}{2-p}}(\Omega) \) is nonzero and nonnegative. There exists \( T > 0 \) such that for any \( |f|_{\frac{2^*}{2-p}} < T \), (1.1) has at least one positive local minimal solution \( u_* \) with \( I(u_*) < 0 \).

**Proof.** We claim that there exist \( T, R, \rho > 0 \) such that for every \( |f|_{\frac{2^*}{2-p}} < T \), I satisfies
\[
I(u|_{u \in S_R}) \geq \rho, \quad \inf_{u \in B_R} I_\lambda(u) < 0,
\]
where \( B_R = \{ u \in H^1_0(\Omega) : \|u\| \leq R \} \) is a closed ball and \( S_R = \{ u \in H^1_0(\Omega) : \|u\| = R \} \). It follows from (2.2) that
\[
I(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{p} \int_\Omega f(x)|u|^p \, dx \\
\geq \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{|f|_{\frac{2^*}{2-p}}^2\|u\|^p}{pS^2} \\
= \|u\|^p \left( \frac{a}{2}\|u\|^{2-p} - \frac{b}{4}\|u\|^{4-p} - \frac{|f|_{\frac{2^*}{2-p}}^2}{pS^2} \right).
\]
For any $t \geq 0$, $g(t)$ is defined by

$$g(t) = \frac{a}{2} t^{2-p} - \frac{b}{4} t^{4-p} - \frac{|f|}{p}^{\frac{2*p}{2-p}},$$

then

$$g'(t) = t^{1-p} \left[ \frac{a(2-p)}{2} - \frac{b(4-p)}{4} t^2 \right].$$

Consequently, let $g'(t) = 0$, we can easily get $t_{\text{max}} = \left[ \frac{2a(2-p)}{b(4-p)} \right]^\frac{1}{2}$ such that

$$\max_{t \geq 0} g(t) = g(t_{\text{max}}) = \frac{a}{4-p} \left[ \frac{2a(2-p)}{b(4-p)} \right]^{\frac{2*p}{2-p}} - \frac{|f|}{p}^{\frac{2*p}{2-p}}.$$

Choosing

$$T = \frac{apS^\frac{p}{2}}{4-p} \left[ \frac{2a(2-p)}{b(4-p)} \right]^{\frac{2*p}{2-p}}, \quad R = \left[ \frac{2a(2-p)}{b(4-p)} \right]^\frac{1}{2},$$

then there exists $\rho > 0$ such that for all $|f|^{\frac{2*p}{2-p}} < T$, one has $I(u)|_{u \in S_R} \geq \rho$. Moreover, fixing $u_0 \in H^1_0(\Omega)$ and $u_0 \neq 0$, one gets

$$\lim_{t \to 0^+} \frac{I(tu_0)}{t^p} = -\frac{1}{p} \int_{\Omega} f(x)|u_0|^p \, dx < 0.$$

Thus, one has $\inf_{u \in B_R} I(u) < 0$. Therefore, our claim is true. Without loss of generality, we denote

$$m = \inf_{u \in B_R} I(u).$$

For this minimization problem, there exists a minimization sequence $\{u_n\}$ such that $\lim_{n \to \infty} I(u_n) = m$. Moreover, by [4, Proposition 9], we can take a subsequence from $\{u_n\}$, still denoted by $\{u_n\}$, such that $\{u_n\}$ is a (PS)$m$ sequence of $I$ in $H^1_0(\Omega)$. Thus, by Lemma 2.1, there exists $u_* \in H^1_0(\Omega)$ such that $u_n \to u_*$ in $H^1_0(\Omega)$ as $n \to \infty$ and $I(u_*) = m < 0$. Consequently, $u_*$ is a nonzero solution of problem (1.1). Since $I(u) = I(|u|)$, we can assume that $u_* \geq 0$ in $\Omega$. From (1.2), choosing $\varphi = u_*$, we have

$$(a - b\|u_*\|^2) \int_{\Omega} |\nabla u_*|^2 \, dx - \int_{\Omega} f(x)|u_*|^p \, dx = 0.$$ 

Consequently, one has

$$(a - b\|u_*\|^2) \int_{\Omega} |\nabla u_*|^2 \, dx = \int_{\Omega} f(x)|u_*|^p \, dx \geq 0,$$

which implies that

$$a - b\|u_*\|^2 \geq 0. \quad (2.10)$$

Obviously, we have

$$-(a - b\|u_*\|^2) \Delta u_* = f(x)u_*^{p-1}.$$

Combining with (2.10), we get

$$-\Delta u_* = \frac{f(x)u_*^{p-1}}{a - b\|u_*\|^2} \geq 0.$$

Hence, by the strong maximum principle, one has $u_* > 0$, that is, $u_*$ is a positive local minimal solution of problem (1.1). Thus, the proof of Proposition 2.2 is completed.

Proposition 2.3. Assume $1 \leq p < 2$, $2 < p < 2^*$ or $p = 2$, $|f|^{\frac{2*p}{2-p}} < aS$, and $f \in L^{\frac{2*p}{2-p}}(\Omega)$ is nonzero and nonnegative. Then (1.1) has at least one positive mountain-pass solution $u_{**}$ with $I(u_{**}) > 0$. 

\[ \square \]
Proof. We claim that I satisfies the mountain-pass geometry in $H_0^1(\Omega)$. Firstly, we should prove that there exists $e \in H_0^1(\Omega)$ with $\|e\| > R$ such that $I(e) < 0$. In fact, fixing $u \in H_0^1(\Omega)$ and $u \neq 0$, as $t \to +\infty$, one has
\[
I(tu) = \frac{at^2}{2} \|u\|^2 - \frac{bt^4}{4} \|u\|^4 - \frac{t^p}{p} \int_{\Omega} f(x)|u|^p \, dx \to -\infty,
\]
which implies that there exists $e \in H_0^1(\Omega)$ with $\|e\| > R$ such that $I(e) < 0$. Secondly, we prove that there exist $R, \rho > 0$ such that $I(u)|_{u \in S_R} \geq \rho$. When $1 < p \leq 2$, from Proposition 2.2, there exist $T, R, \rho > 0$, for every $|f|_{\mathcal{L}^p}^p < T$ such that $I(u)|_{u \in S_R} \geq \rho$. While $p = 2$, $|f|_{\mathcal{L}^p}^p < a\mathcal{S}$, we can easily obtain this conclusion by the similar way. When $2 < p < 2^*$, obviously, 0 is a local minimizer of $I$ with $I(0) = 0$. In fact, fixing $u_0 \in H_0^1(\Omega)$ and $u_0 \neq 0$, we have
\[
\lim_{t \to 0^+} \frac{I(tu_0)}{t^2} = \frac{a}{2} \|u_0\|^2 > 0,
\]
which implies that there exist $R, \rho > 0$ such that $I(u)|_{u \in S_R} \geq \rho$. Hence, our claim is proved to be true. Taking
\[
c_0 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),
\]
where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = e\}$. We claim that $c_0 \leq \frac{a^2}{4b}$. In fact,
\[
\max_{t \in [0,1]} I(te) = \max_{t \in [0,1]} \left( \frac{at^2}{2} \|e\|^2 - \frac{bt^4}{4} \|e\|^4 - \frac{t^p}{p} \int_{\Omega} f(x)|e|^p \, dx \right) \\
\leq \max_{t \in [0,1]} \left( \frac{at^2}{2} \|e\|^2 - \frac{bt^4}{4} \|e\|^4 \right) \\
\leq \frac{a^2}{4b}.
\]
Therefore, by Lemma 2.1, applying the mountain-pass lemma, there exists $u_{**} \in H_0^1(\Omega)$ such that $I(u_{**}) = c_0 > 0$, that is, $u_{**}$ is a nonzero mountain-pass solution of (1.1). Since $I(u) = I(|u|)$, from [1, Theorem 10], one has $u_{**} \geq 0$ in $\Omega$. Similar to Proposition 2.2, by the strong maximum principle, we can prove that $u_{**}$ is a positive mountain-pass solution of (1.1). This completes the proof of Proposition 2.3. \hfill \Box

According to Proposition 2.2 and Proposition 2.3, we can obtain the proof of Theorem 1.1.

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