Infinitely many periodic solutions for second-order discrete Hamiltonian systems

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Communicated by K. Q. Lan

Abstract

Infinitely many periodic solutions are obtained for a second-order discrete Hamiltonian systems by using the minimax methods in critical point theory. Our results extend and improve previously known results. ©2017 All rights reserved.

Keywords: Minimax methods, periodic solutions, sublinear, discrete Hamiltonian systems, critical point.

2010 MSC: 34C25, 58E50.

1. Introduction

Consider the following second order discrete Hamiltonian system

\[
\begin{aligned}
\triangle^2 u(t-1) + \nabla F(t, u(t)) &= 0, \quad t \in \mathbb{Z}[1, T], \\
u(0) &= u(T),
\end{aligned}
\] (1.1)

where \( T \in \mathbb{Z}, \mathbb{Z}[1, T] \) denotes the discrete interval \( \{1, 2, \cdots, T\} \), \( \triangle u(t) = u(t+1) - u(t) \), \( \triangle^2 u(t) = \triangle(\triangle u(t)) \) and \( \nabla F(t, x) \) denotes the gradient of \( F \) with respect to the second variable. \( F \) satisfies the following assumption:

(A) \( F(t, x) \in C^1(\mathbb{R}^N, \mathbb{R}) \) for any \( t \in \mathbb{Z}[0, T] \) and \( F \) is \( T \)-periodic in the first variable.

Since Guo and Yu developed a new method to study the existence and multiplicity of periodic solutions of difference equations by using critical point theory (see [4–6, 18], the existence and multiplicity of periodic solutions for problem (1.1) have been extensively studied and lots of interesting results have been worked out, see [1–3, 7, 8, 10–17] and the references therein. In particular, when the nonlinearity \( \nabla F(t, x) \) is bounded, that is, there exists \( M > 0 \) such that \( |\nabla F(t, x)| \leq M \) for all \( (t, x) \in \mathbb{Z}[0, T] \times \mathbb{R}^N \), and that

\[
\sum_{t=0}^{T} F(t, x) \to +\infty \quad \text{as} \quad |x| \to \infty.
\]

Guo and Yu [6] obtained one periodic solution to problem (1.1).

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doi:10.22436/jnsa.010.11.26

Received 2017-07-28
In [12, 13], Xue and Tang generalized these results to the sublinear case:

$$|\nabla F(t, x)| \leq M_1 |x|^{\alpha} + M_2, \quad \forall (t, x) \in \mathbb{Z}[0, T] \times \mathbb{R}^N,$$

and

$$|x|^{-2\alpha} \sum_{t=0}^{T} F(t, x) \to \pm \infty \text{ as } |x| \to \infty,$$

where $M_1 > 0$, $M_2 > 0$ and $\alpha \in [0, 1)$.

In [10], Tang and Zhang considered the nonlinearity $\nabla F(t, x)$ satisfies the following condition:

$$|\nabla F(t, x)| \leq f(t) |x|^{\alpha} + g(t), \quad \forall (t, x) \in \mathbb{Z}[0, T] \times \mathbb{R}^N, \quad (1.2)$$

or

$$|\nabla F(t, x)| \leq f(t) |x| + g(t), \quad \forall (t, x) \in \mathbb{Z}[0, T] \times \mathbb{R}^N, \quad (1.3)$$

where $f, g : \mathbb{Z}[0, T] \to \mathbb{R}^+, \alpha \in (0, 1)$. Under these conditions, periodic solutions of problem (1.1) have been obtained, which completed and extended the results in [12, 13].

Recently, Che and Xue [1] obtained infinitely many periodic solutions for problem (1.1) when (1.2) holds, and

$$\limsup_{r \to +\infty} \inf_{x \in \mathbb{R}^N, |x|=r} \sum_{t=0}^{T} F(t, x) = +\infty, \quad (1.4)$$

and

$$\liminf_{R \to +\infty} \sup_{x \in \mathbb{R}^N, |x|=R} |x|^{-2\alpha} \sum_{t=0}^{T} F(t, x) = -\infty, \quad (1.5)$$

where $\alpha \in (0, 1)$.

In this paper, motivated by the results mentioned above, we will further investigate infinitely many periodic solutions to the problem (1.1) under conditions (1.2) or (1.3).

Let $H_T$ be a Hilbert space defined by

$$H_T = \{u : \mathbb{Z} \to \mathbb{R}^N \mid u(t) = u(t + T), \forall t \in \mathbb{Z}\},$$

with the inner product

$$\langle u, v \rangle = \sum_{t=0}^{T} (u(t), v(t)),$$

and the norm

$$\|u\| = \left(\sum_{t=0}^{T} |u(t)|^2\right)^{\frac{1}{2}}.$$

Let

$$\|u\|_\infty = \max_{t \in \mathbb{Z}[0, T]} |u(t)|.$$

Since $H_T$ is finite dimensional, one has that:

$$\frac{1}{\sqrt{T}} \|u\| \leq \|u\|_\infty \leq \|u\|.$$

Let

$$\Phi(u) = \frac{1}{2} \sum_{t=0}^{T} |\Delta u(t)|^2 - \sum_{t=0}^{T} F(t, u(t)), \quad \forall u \in H_T.$$

It is well-known that the solutions of problem (1.1) correspond to the critical points of $\Phi$ (see [9]).
Lehman 1.2. Suppose that \( \lambda \) where \( \lambda \) is defined by
\[
N_k = \{ u \in H_1 | -\Delta^2 u(t-1) = \lambda_k u(t) \},
\]
where \( \lambda_k = 2 - 2 \cos k\omega, \omega = \frac{2\pi}{T}, k \in \mathbb{Z}[0,\left[ \frac{T}{2} \right]] \) (where \( [c] \) denotes the largest integer less than \( c \)). Then we have

1. \( N_k \perp N_j \) for \( k \neq j \) and \( j, k \in \mathbb{Z}[0,\left[ \frac{T}{2} \right]] \).
2. \( H_T = \oplus_{k=0}^{\left[ \frac{T}{2} \right]} N_k \).

Set \( H_1 = N_0 \) and \( H_2 = \oplus_{k=1}^{\left[ \frac{T}{2} \right]} N_k \). Then \( H_T = H_1 \oplus H_2 \) and
\[
\sum_{t=0}^{T} |\nabla u(t)|^2 \geq \lambda_1 \|u\|, \quad \forall u \in H_2.
\]

The element \( u \) of \( H_T \) is just the eigenvector corresponding to \( \lambda_0 = 0 \) which satisfies \( u(t) \equiv u(0) \) for \( t \in \mathbb{Z}[0,T] \).

Our main results are the following theorems.

**Theorem 1.2.** Suppose that (A), (1.2) and (1.4) hold, and
\[
\liminf_{r \to +\infty} \sup_{x \in \mathbb{R}^N, |x| = r} |x|^{-2\alpha} \sum_{t=0}^{T} F(t, x) < -\frac{\left( \sum_{t=0}^{T} f(t) \right)^2}{2\lambda_1}. \tag{1.6}
\]

Then

(i) the problem (1.1) has infinitely many periodic solutions \( \{u_n\} \) such that \( \Phi(u_n) \to +\infty \) as \( n \to \infty \);

(ii) the problem (1.1) has infinitely many periodic solutions \( \{u_n^*\} \) such that \( \Phi(u_n^*) \to -\infty \) as \( m \to \infty \).

**Theorem 1.3.** Suppose that (A), (1.3) with \( \sum_{t=0}^{T} f(t) < \frac{\lambda_1}{4} \) and (1.4) hold, and
\[
\liminf_{r \to +\infty} \sup_{x \in \mathbb{R}^N, |x| = r} |x|^{-2} \sum_{t=0}^{T} F(t, x) < -\frac{\left( \sum_{t=0}^{T} f(t) \right)^2}{2\left( \lambda_1 - 2\sum_{t=0}^{T} f(t) \right)}. \tag{1.7}
\]

Then

(i) the problem (1.1) has infinitely many periodic solutions \( \{u_n\} \) such that \( \Phi(u_n) \to +\infty \) as \( n \to \infty \);

(ii) the problem (1.1) has infinitely many periodic solutions \( \{u_n^*\} \) such that \( \Phi(u_n^*) \to -\infty \) as \( m \to \infty \).

**Remark 1.4.** Obviously, the condition (1.6) is different from condition (1.5) that of in [1]; Theorem 1.3 is completely new comparing with main result of [1] since we allow \( \alpha = 1 \) although the method using in this paper is same as that of in [1].

**2. Proof of main results**

Since the proof of Theorem 1.2 is similar to that of Theorem 1.3, we only prove Theorem 1.3.

For the sake of convenience, we denote
\[
\gamma = \sum_{t=0}^{T} f(t), \quad \beta = \sum_{t=0}^{T} g(t).
\]
Lemma 2.1. Suppose that (1.3) with $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{4}$ holds, then
\[
\Phi(u) \to +\infty \text{ as } \|u\| \to \infty \text{ in } H_2.
\]

Proof. From (1.3), for all $u \in H_2$ we have
\[
\Phi(u) = \frac{1}{2} \sum_{t=0}^{T} |\triangle u(t)|^2 - \sum_{t=0}^{T} F(t, u(t))
\]
\[
\geq \frac{\lambda_1}{2} |u|^2 - \sum_{t=0}^{T} f(t)|u|^2 - \sum_{t=0}^{T} |g(t)|u(t)|
\]
\[
\geq \frac{\lambda_1}{2} |u|^2 - \|u\|^2 \sum_{t=0}^{T} f(t) - \|u\| \sum_{t=0}^{T} g(t)
\]
\[
\geq \frac{\lambda_1}{2} |u|^2 - \|u\|^2 \sum_{t=0}^{T} f(t) - \|u\| \sum_{t=0}^{T} g(t)
\]
\[
= (\frac{\lambda_1}{2} - \gamma) |u|^2 - \beta |u|.
\]
So, $\Phi(u) \to +\infty$ as $\|u\| \to \infty$ in $H_2$. \qed

Lemma 2.2. Suppose that (1.4) holds. Then there exists positive real sequence $(a_n)$ such that
\[
\lim_{n \to \infty} a_n = +\infty,
\]
\[
\lim_{n \to \infty} \sup_{u \in H_2, \|u\| = a_n} \Phi(u) = -\infty.
\]

Proof. By (1.4), it is easy to obtain this result, so we omit the detail here. \qed

Lemma 2.3. Suppose that (1.3) with $\sum_{t=0}^{T} f(t) < \frac{\lambda_1}{4}$ and (1.7) hold. Then there exists positive real sequence $(b_m)$ such that
\[
\lim_{m \to \infty} b_m = +\infty,
\]
\[
\lim_{m \to \infty} \inf_{u \in H_{b_m}} \Phi(u) = +\infty,
\]
where $H_{b_m} = \{u \in H_1 : \|u\| = b_m\} \bigoplus H_2$.

Proof. By (1.7), let $a > \frac{1}{\lambda_1 - 2\gamma}$ such that
\[
\lim_{r \to +\infty} \sup_{x \in \mathbb{R}^N, |x| = r} |x|^{-2} \sum_{t=0}^{T} F(t, x) < \frac{a}{2} \gamma^2.
\]
Let $u \in H_{b_m}$, $u = \overline{u} + \tilde{u}$, where $\overline{u} \in H_1$, $\tilde{u} \in H_2$. So, we have
\[
\left| \sum_{t=0}^{T} F(t, u(t)) - \sum_{t=0}^{T} F(t, \overline{u}) \right| = \left| \sum_{t=0}^{T} \int_{0}^{1} \nabla F(t, \overline{u}(t) + s\tilde{u}(t), \tilde{u}(t)) ds \right|
\]
\[
\leq \sum_{t=0}^{T} \int_{0}^{1} f(t)|\overline{u}(t)| + s\tilde{u}(t)|\tilde{u}(t)| ds + \sum_{t=0}^{T} \int_{0}^{1} |g(t)|\tilde{u}(t)| ds
\]
\[
\leq \sum_{t=0}^{T} f(t) (|\overline{u}(t)| + |\tilde{u}(t)|) + \sum_{t=0}^{T} g(t)|\tilde{u}(t)|
for all \( u \in H_{b_m} \). Therefore, one has that

\[
\Phi(u) = \frac{1}{2} \sum_{t=0}^{T} |\triangle u(t)|^2 - \sum_{t=0}^{T} F(t, u(t)) \\
= \frac{1}{2} \sum_{t=0}^{T} |\triangle \tilde{u}(t)|^2 - \left( \sum_{t=0}^{T} F(t, u(t)) - \sum_{t=0}^{T} F(t, \tilde{u}(t)) \right) - \sum_{t=0}^{T} F(t, \tilde{u}(t)) \\
\geq \left( \frac{\lambda_1}{2} - \frac{1}{2\alpha} - \gamma \right) ||\tilde{u}||^2 - ||\tilde{u}|| \\
- ||\tilde{u}||^2 \left( ||\tilde{u}||^{-2} \sum_{t=0}^{T} F(t, \tilde{u}(t)) + \frac{a}{2} \gamma^2 \right)
\]

for all \( u \in H_{b_m} \). From condition (1.7) and the above inequality the proof is finished. \( \square \)

Now we give the proof of Theorem 1.3.

The proof of Theorem 1.3. Let \( B_{a_n} \) be a ball in \( H_1 \) with radius \( a_n \). Set

\[
\Gamma_n = \{ \gamma \in C(B_{a_n}, H_1), \gamma|_{\partial B_{a_n}} = \text{Id} |_{\partial B_{a_n}} \},
\]

and

\[
c_n = \inf_{\gamma \in \Gamma_n} \max_{x \in B_{a_n}} \Phi(\gamma(x)).
\]

It is easy to obtain that \( \Phi \) is coercive on \( H_2 \) from Lemma 2.1. So, there is a constant \( M \) such that

\[
\max_{x \in B_{a_n}} \Phi(\gamma(x)) \geq \inf_{u \in H_2} \Phi(u) \geq M.
\]

On the other hand, it is easy to see that \( \gamma(B_{a_n}) \cap H_2 \neq \emptyset \) for any \( \gamma \in \Gamma_n \). Therefore

\[
c_n \geq \inf_{u \in H_2} \Phi(u) \geq M.
\]

By Lemma 2.2, for any large value of \( n \), one has that

\[
c_n > \max_{u \in \partial B_{a_n}} \Phi(u).
\]

For such \( n \), there exists a sequence \( \{\gamma_k\} \) in \( \Gamma_n \) such that

\[
\max_{x \in B_{a_n}} \Phi(\gamma_k(x)) \to c_n, \quad k \to \infty.
\]

Applying [9, Theorem 4.3 and Corollary 4.3], there exists a sequence \( \{v_k\} \) in \( H_1 \) satisfying

\[
\Phi(v_k) \to c_n, \quad \text{dist}(v_k, \gamma_k(B_{a_n})) \to 0, \quad \Phi'(v_k) \to 0,
\]

as \( k \to \infty \). So, for any large enough \( k \), one has that

\[
c_n \leq \max_{x \in B_{a_n}} \Phi(\gamma_k(x)) \leq c_n + 1,
\]
and there exists \( w_k \in \gamma_k(B_{a_n}) \) such that
\[
\|v_k - w_k\| \leq 1.
\]
For fix \( n \), by Lemma 2.3, let \( m \) be large enough such that
\[
b_m > a_n, \quad \text{and} \quad \inf_{u \in H_{b_m}} \Phi(u) > c_n + 1.
\]
This implies that \( \gamma(B_{a_n}) \) cannot intersect the hyperplane \( H_{b_m} \) for each \( k \).

Let \( w_k = \bar{w}_k + \tilde{w}_k \), where \( \bar{w}_k \in H_1 \) and \( \tilde{w}_k \in H_2 \). Then we have \( |\bar{w}_k| < b_m \) for each \( k \).

From (1.3), we have that
\[
\begin{align*}
c_n + 1 & \geq \Phi(w_k) = \frac{1}{2} \sum_{t=0}^{T} |\triangle w_k(t)|^2 - \sum_{t=0}^{T} F(t, w_k(t)) \\geq & \frac{\lambda_1}{2} \|\tilde{w}_k\|^2 - \sum_{t=0}^{T} f(t)|w_k(t)|^2 - \sum_{t=0}^{T} g(t)|w_k(t)| \\geq & \frac{\lambda_1}{2} \|\tilde{w}_k\|^2 - 2 \sum_{t=0}^{T} f(t)|\bar{w}_k(0)|^2 + |\tilde{w}_k(t)|^2 - \sum_{t=0}^{T} g(t)(|\bar{w}_k(0)| + |\tilde{w}_k(t)|)) \\geq & \left( \frac{\lambda_1}{2} - 2\gamma \right) \|\tilde{w}_k\|^2 - 2b_m^2 \gamma - \|\tilde{w}_k\| \beta - b_m \beta.
\end{align*}
\]
Therefore \( \tilde{w}_k(t) \) is bounded. Hence, \( w_k \) is bounded since \( \|w_k\| \leq C(\|\tilde{w}_k\| + \|\bar{w}_k\|) \). Also, \( \{v_k\} \) is bounded in \( H_T \).

From the fact that \( H_T \) is finite dimensional, we know there is a subsequence, which is still be denoted by \( \{v_k\} \) such that \( \{\tilde{w}_k\} \) converges to some point \( u_n \). Therefore, in view of the continuity of \( \Phi \) and \( \Phi' \), it is easy to see that accumulation point \( u_n \) of \( \{v_k\} \) is a critical point and \( c_n \) is a critical value of \( \Phi \).

Let \( n \) large enough such that \( a_n > b_m \), then \( \gamma(B_{a_n}) \) intersects the hyperplane \( H_{b_m} \) for any \( \gamma \in \Gamma_n \). It follows that
\[
\max_{x \in \gamma(B_{a_n})} \Phi(x) \geq \inf_{u \in H_{b_m}} \Phi(u).
\]
In view of above inequality and Lemma 2.3, we get \( \lim_{n \to \infty} c_n = +\infty \). So, the proof of first result of Theorem 1.3 is finished.

Next, we prove (ii) of Theorem 1.3.

For fixed \( m \), let
\[
P_m = \{u \in H_T : u = \tau + \tilde{u}, |\tau| \leq b_m, \tilde{u} \in H_2 \}.
\]
For \( u \in P_m \), one has that
\[
\Phi(u) = \frac{1}{2} \sum_{t=0}^{T} |\triangle u(t)|^2 - \sum_{t=0}^{T} F(t, u(t)) \geq \frac{\lambda_1}{2} \|\tilde{u}\|^2 - \sum_{t=0}^{T} f(t)|u(t)|^2 - \sum_{t=0}^{T} g(t)|u(t)| \geq \frac{\lambda_1}{2} \|\tilde{u}\|^2 - \sum_{t=0}^{T} f(t)|\tau(0)|^2 + |\tilde{u}(t)|^2 - \sum_{t=0}^{T} g(t)(|\tau(0)| + |\tilde{u}(t)|)) \geq \left( \frac{\lambda_1}{2} - 2\gamma \right) \|\tilde{u}\|^2 - 2b_m^2 \gamma - \|\tilde{u}\| \beta - b_m \beta.
\]
So, \( \Phi \) is bounded below on \( P_m \). Let
\[
\mu_m = \inf_{u \in P_m} \Phi(u),
\]
and choose a minimizing sequence \( \{u_k\} \) in \( P_m \), that is
\[
\Phi(u_k) \to \mu_m \text{ as } k \to \infty.
\]

According to (2.1), \( \{u_k\} \) is bounded in \( H_T \). Then there exists a subsequence, which is still be denoted by \( \{u_k\} \) such that
\[
u_k \rightharpoonup \nu^*_m \text{ weakly in } H_T.
\]
Since \( P_m \) is a convex closed subset of \( H_T \) and \( \Phi \) is weakly lower semicontinuous, \( \nu^*_m \in P_m \) and
\[
\mu_m = \lim_{k \to \infty} \Phi(u_k) \geq \Phi(\nu^*_m).
\]

By \( \nu^*_m \in P_m \),
\[
\nu^*_m = v^*_m + \tilde{u}^*_m.
\]
In view of Lemma 2.2 and Lemma 2.3, \( |v^*_m| \neq b_m \) for large \( m \), i.e., \( \nu^*_m \) is in the interior of \( P_m \). Then \( \nu^*_m \) is a local minimum of functional. So, we have
\[
\Phi(\nu^*_m) = \inf_{u \in P_m} \Phi(u) \leq \sup_{|u| = b_m} \Phi(u).
\]

Then from Lemma 2.2 we see that \( \Phi(\nu^*_m) \to -\infty \) as \( m \to \infty \). Therefore, the proof is finished.

**Acknowledgment**

The authors express their sincere thanks to the reviewers and editor for the useful suggestions to improve the paper.

**References**


