Hyers-Ulam stability of nonlinear impulsive Volterra integro-delay dynamic system on time scales

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Abstract

This paper proves the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of nonlinear impulsive Volterra integro-delay dynamic system on time scales via a fixed point approach. The uniqueness and existence of the solution of nonlinear impulsive Volterra integro-delay dynamic system is proved with the help of Picard operator. The main tools for proving our results are abstract Grönwall lemma and Banach contraction principle. We also make some assumptions along with Lipschitz condition which make our results appropriate for the approach we are using. ©2017 All rights reserved.

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1. Introduction

Let \((G, \ast)\) and \((H, .)\) be groups, then a function \(\phi: (G, \ast) \rightarrow (H, .)\) is said to be a group homomorphism if it is given by
\[\phi(x \ast y) = \phi(x) \ast \phi(y), \quad \forall x, y \in G.\]

Ulam [23, 24] considered \((H, ., d)\) a metric group with metric \(d(., .)\) and inquired a question, if for any \(\epsilon > 0\) and \(\phi: (G, \ast) \rightarrow (H, .)\) satisfies the inequality
\[d(\phi(x \ast y), \phi(x) \ast \phi(y)) \leq \epsilon, \quad \forall x, y \in G,\]
then for an approximate homomorphism \(\psi: (G, \ast) \rightarrow (H, .)\) can we find a real number \(\delta > 0\) such that
\[d(\phi(x), \psi(x)) \leq \delta, \quad \forall x \in G.\]

To deal this problem, Hyers [10] using direct method, brilliantly gave a partial answer to the case of functional equation by considering \(G\) and \(H\) to be Banach spaces. Afterward, it was called the Hyers-Ulam

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problem and the study of this area has grown-up to be one of the important subjects in mathematical analysis. In 1978, Rassias [21] provided an extension of the Hyers-Ulam stability by introducing new function variables. As a result, another new stability concept, Hyers-Ulam-Rassias stability, was named by mathematicians.

In the literature, many researchers paid attention to the stability properties of different kinds of differential equations. We emphasize that Ulam’s type stability problems have been taken up by a huge amount of mathematicians and the study of this region has grown-up to be one of the vital subjects in mathematical analysis. However, among the functional equations, Obłoza seems to be the first mathematician who has investigated the Hyers-Ulam stability of linear differential equations (see [18, 19]). Thereafter, Alsina and Ger published their paper which handles the Hyers-Ulam stability of the linear differential equation \( y'(t) = y(t) \). They proved that if a differentiable function \( y(t) \) is a solution of the inequality \( |y'(t) - y(t)| < \varepsilon \) for some \( \varepsilon > 0 \) and for all \( t \in (a, \infty) \), then there exists a constant \( c \) such that \( |y(t) - ce^t| < 3\varepsilon \) for all \( t \in (a, \infty) \), where \( a \in \mathbb{R} \) ([2]). Jung in 2004 [11] investigated Hyers-Ulam stability of first order linear differential equations. In 2010, Li and Shen [15] studied the Hyers-Ulam stability of first order linear differential equations of second order. For more details on Hyers-Ulam stability, see [12–14, 16, 18, 25, 26, 28–31].

Many real world phenomena are represented by smooth differential equations. However, the situation becomes quite different in the case when a physical phenomena has sudden changes in its state such as in mechanics with impact, biological systems with heart beats, blood flows, population dynamics, theoretical physics and so on (see [4]). Adequate mathematical models of such processes are systems of differential equations with impulses i.e., impulsive differential equations. An impulsive differential equation is described by three components: a continuous time differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active.

The theory of dynamic equations on time scale has been developing rapidly and has received a lot of attention in recent years. This theory was introduced by Hilger [9] in 1990, with the motivation of providing a unification to continuous and discrete calculus. For more details on time scale, see [5–8, 17, 20, 27]. In 2013, Andráš and Mézháros [3] obtained some results about the Hyers-Ulam stability of some integral equations on time scale via Picard operators. Agarwal et al. [1] in 2014, discussed some results about the stability of linear impulsive Volterra integro-dynamic system on time scales. To the best of our knowledge, only few papers are devoted to the stability of impulsive Volterra integro-dynamic systems. However, as far as we know, the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of nonlinear impulsive Volterra integro-delay dynamic systems have not been studied yet.

In this paper, we obtain Hyers-Ulam stability and Hyers-Ulam-Rassias stability of nonlinear impulsive Volterra integro-delay dynamic system of the form

\[
\begin{align*}
\dot{z}^\lambda(t) &= M(t)z(t) + \int_{t_0}^{t} \mathcal{K}(t, s, z(s), z(h(s))) \Delta s, \quad t \in T_S', \quad t \in T_S' = T_S^0 \setminus \{t_1, t_2, \ldots, t_m\}, \\
\Delta z(t_k) &= z(t_k^+) - z(t_k^-) = \mathcal{Y}_k(z(t_k^-)), \quad k = 1, 2, \ldots, m, \\
z(t) &= \alpha(t), \quad t \in [t_0 - \lambda, t_0], \\
z(t_0) &= \alpha(t_0) = z_0,
\end{align*}
\]  

(1.1)

where \( \lambda > 0 \), \( M(t) \) is piecewise continuous and a regressive square matrix of order \( m \) on \( T_S^0 := [t_0, t_f]_{T_S^0} \), \( t_f > t_0 \neq 0 \) and \( \mathcal{K}(t, s, z(s), z(h(s))) \) is piecewise continuous operator on

\[ \Gamma = \{(t, s, z) : t_0 \leq s < t \leq t_f, \quad z \in \mathbb{R}^m\}. \]

Also \( \mathcal{Y}_k : \mathbb{R} \to \mathbb{R}, \alpha : [t_0 - \lambda, t_0] \to \mathbb{R} \) are continuous functions, \( z(t_k^+) = \lim_{\tau \to 0^+} z(t_k + \tau) \) and \( z(t_k^-) = \lim_{\tau \to 0^+} z(t_k - \tau) \) are respectively the right and left side limits of \( z(t) \) at \( t_k \), where \( t_k \) satisfies

\[ t_0 < t_1 < t_3 < \cdots < t_m < t_{m+1} = t_f < +\infty. \]
Moreover, \( h : \mathcal{T}_S^0 \to \mathcal{T}_S^0 \cap [t_0 - \lambda, t_0] \) is a continuous delay function such that \( h(t) \leq t \).

2. Preliminaries

The time scale is defined to be any nonempty closed subset of real numbers and is denoted by \( \mathcal{T}_S \). The forward jump operator \( \Theta : \mathcal{T}_S \to \mathcal{T}_S \), backward jump operator \( \rho : \mathcal{T}_S \to \mathcal{T}_S \) and graininess function \( \mu : \mathcal{T}_S \to [0, \infty) \) are respectively defined as:

\[
\Theta(s) = \inf\{t \in \mathcal{T}_S : t > s\}, \quad \rho(s) = \sup\{t \in \mathcal{T}_S : t < s\}, \quad \mu(s) = \Theta(s) - s.
\]

For any \( t \in \mathcal{T}_S \), if \( t < \rho(t) \) then point \( t \) is said to be left-scattered and if \( t = \rho(t) \) then \( t \) is called left-dense. If \( t < \Theta(t) \) and \( \Theta(t) = t \), then point \( t \in \mathcal{T}_S \) is called right-scattered and right-dense, respectively. The set \( \mathcal{T}_S^2 \) is known as derived form of time scale \( \mathcal{T}_S \) and is defined as:

\[
\mathcal{T}_S^2 = \begin{cases} 
\mathcal{T}_S \setminus (\rho(\sup \mathcal{T}_S), \sup \mathcal{T}_S), & \text{if } \sup \mathcal{T}_S < \infty, \\
\mathcal{T}_S, & \text{if } \sup \mathcal{T}_S = \infty.
\end{cases}
\]

The real-valued function \( W : \mathcal{T}_S \to \mathbb{R} \) is called right-dense continuous, if it is continuous at every right-dense point on \( \mathcal{T}_S \) and its left-sided limit exists at every left-dense point on \( \mathcal{T}_S \). The real-valued function \( W : \mathcal{T}_S \to \mathbb{R} \) is called regressive, if \( 1 + \mu(t)W(t) \neq 0 \), for all \( t \in \mathcal{T}_S^2 \) and if \( 1 + \mu(t)W(t) > 0 \), then \( W \) is called positively regressive. The set of all right-dense continuous and regressive, right-dense continuous and positively regressive functions, respectively, will be denoted by \( \mathcal{R}_d(\mathcal{T}_S) \) and \( \mathcal{R}_d(\mathcal{T}_S)^+ \). The delta derivative of the function \( W : \mathcal{T}_S \to \mathbb{R} \) at \( t \in \mathcal{T}_S^2 \) is defined by

\[
W^\Delta(t) = \lim_{s \to t, s \neq \Theta(t)} \frac{W(\Theta(t)) - W(s)}{\Theta(t) - s}.
\]

The \( \Delta \)-integral of the rd-continuous function \( W : \mathcal{T}_S \to \mathbb{R} \) is defined by

\[
\int_a^b W(t)\Delta t = w(b) - w(a), \quad \forall a, b \in \mathcal{T}_S,
\]

where the rd-continuous function \( w \) is an anti-derivative of \( W \), i.e., \( w^\Delta = W \) on \( \mathcal{T}_S^2 \).

The generalized exponential function \( e_W(a, b) \) for \( W \in \mathcal{R}_d(\mathcal{T}_S) \) on \( \mathcal{T}_S \) is defined as

\[
e_W(a, b) = \exp \left( \int_a^b \Phi_\mu(s)W(s)\Delta s \right), \quad \forall a, b \in \mathcal{T}_S,
\]

where

\[
\Phi_\mu(t)W(t) = \begin{cases} 
\frac{\log(1 + \mu(t)W(t))}{\mu(t)}, & \text{if } \mu(t) \neq 0, \\
W(t), & \text{if } \mu(t) = 0,
\end{cases}
\]

is the cylindrical transformation.

The fundamental matrix is defined to be the general solution to the matrix dynamic equation \( z^\Delta(t) = M(t)z(t) \), \( z(t_0) = z_0 \), \( t \in \mathcal{T}_S^0 \) and is denoted by \( \Psi_M(t, t_0) \).

Consider the metric space \( \mathcal{T}_{S1} \times \mathcal{T}_{S2} = \{(m, n) : m \in \mathcal{T}_{S1}, n \in \mathcal{T}_{S2}\} \) which is a complete metric space with the metric defined by

\[
d((m_1, n_1), (m_2, n_2)) = \sqrt{(m_1 - m_2)^2 + (n_1 - n_2)^2}, \quad (m_1, n_1), (m_2, n_2) \in \mathcal{T}_{S1} \times \mathcal{T}_{S2},
\]

where \( \mathcal{T}_{S1} \) and \( \mathcal{T}_{S2} \) are the time scales.
The function $W: \mathcal{S}_1 \times \mathcal{S}_2 \rightarrow \mathbb{R}$ is said to be continuous at $(m, n) \in \mathcal{S}_1 \times \mathcal{S}_2$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|g(m, n) - g(m_0, n_0)\| < \varepsilon$ for all $(m_0, n_0) \in \mathcal{S}_1 \times \mathcal{S}_2$ satisfying

$$d((m, n), (m_0, n_0)) < \delta.$$  

Let $C(\mathcal{S}_1^0, \mathbb{R}^m)$ be the Banach space of continuous functions with norm $\|z\| = \sup_{t \in \mathcal{S}_1} \|z(t)\|$, $PC(\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m)$ denotes the Banach space of piecewise continuous functions with norm

$$\|z\| = \sup_{t \in \mathcal{S}_1^0 \cap [t_0 - \lambda, t_0]} \|z(t)\|,$$

and $PC^1(\mathcal{S}_1^0, \mathbb{R}^m) = \{z \in PC(\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) : z^\lambda \in PC(\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m)\}$ is Banach space with norm $\|z\|_{PC^1} = \max(\|z\|_{PC}, \|z^\lambda\|_{PC})$. Consider the following inequalities,

\[
\begin{align*}
\|y^\lambda(t) - M(t)y(t) - \int_{t_0}^t K(t, s, y(s), y(h(s)))\Delta s\| &\leq \varepsilon, \quad t \in \mathcal{S}_1', \\
\|\Delta y(t_k) - \gamma_k(y(t_k))\| &\leq \varepsilon, \quad k = 1, 2, \ldots, m, \\
\|y^\lambda(t) - M(t)y(t) - \int_{t_0}^t K(t, s, y(s), y(h(s)))\Delta s\| &\leq \varphi(t), \quad t \in \mathcal{S}_1', \\
\|\Delta y(t_k) - \gamma_k(y(t_k))\| &\leq \kappa, \quad k = 1, 2, \ldots, m,
\end{align*}
\]

where $\varphi \in C(\mathcal{S}_1^0, \mathbb{R}^m)$ is an increasing function.

**Definition 2.1.** Equation (1.1) is Hyers-Ulam stable on $\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0]$ if for every

$$y \in PC(\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC^1(\mathcal{S}_1^0, \mathbb{R}^m),$$

satisfying (2.1), there exists a solution $y_0 \in PC(\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC^1(\mathcal{S}_1^0, \mathbb{R}^m)$ of (1.1) with

$$\|y_0(t) - y(t)\| \leq K\varepsilon, \quad K > 0, \quad \forall \ t \in \mathcal{S}_1^0 \cap [t_0 - \lambda, t_0].$$

**Definition 2.2.** Equation (1.1) is Hyers-Ulam-Rassias stable on $\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0]$ if for every

$$y \in PC(\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC^1(\mathcal{S}_1^0, \mathbb{R}^m),$$

satisfying (2.2), there exists a solution $y_0 \in PC(\mathcal{S}_1^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC^1(\mathcal{S}_1^0, \mathbb{R}^m)$ of (1.1) with

$$\|y_0(t) - y(t)\| \leq K\varphi(t), \quad K > 0, \quad \forall \ t \in \mathcal{S}_1^0 \cap [t_0 - \lambda, t_0].$$

**Definition 2.3.** Let $(X, d)$ be any metric space. An operator $\Lambda : X \rightarrow X$ is a Picard operator, if it has a unique fixed point $x^* \in X$ such that for all $x \in X$, $(\Lambda^n(x)) \rightarrow x^*$ as $n \rightarrow \infty$.

**Lemma 2.4** ([17]). Let $\tau \in \mathcal{S}_1^+, \ y, b \in \mathbb{R}_G(\mathcal{S}_1^+), \ p \in \mathbb{R}_G(\mathcal{S}_1^+) +$ and $c, b_k \in \mathbb{R}^+$, $k = 1, 2, \ldots, then$

$$y(t) \leq c + \int_{\tau}^{t} p(s)y(s)\Delta s + \sum_{\tau < t_k < t} b_k y(t_k),$$

implies

$$y(t) \leq c \prod_{\tau < t_k < t} (1 + b_k)c p(t, \tau), \quad t \geq \tau.$$  

**Lemma 2.5** (Abstract Grönwall Lemma [22]). Let $(X, d, \leq)$ be an ordered metric space and $\Lambda : X \rightarrow X$ be an increasing Picard operator with fixed point $x^*$. Then for any $x \in X$, $x \leq \Lambda(x)$ implies $x \leq x^*$ and $x \geq \Lambda(x)$ implies $x \geq x^*$.  

Remark 2.6. A function

\[ y \in \text{PC}^1(\mathcal{J}_S^0, \mathbb{R}^m), \]

satisfies (2.1) if and only if there is a function \( f \in \text{PC}(\mathcal{J}_S^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \) and a sequence \( k \) (which depends on \( y \)) such that \( ||f(t)|| \leq \epsilon \) for all \( t \in \mathcal{J}_S^0 \cap [t_0 - \lambda, t_0] \), \( ||f_k|| \leq \epsilon \) for all \( k = 1, 2, \ldots, m \), and

\[
\begin{aligned}
\left\{ 
& y^{\lambda}(t) = M(t)y(t) + \int_{t_0}^{t} \mathcal{K}(t, s, y(s), y(h(s)))DS + f(t), \quad y(t_0) = y_0, \quad t \in \mathcal{J}_S', \\
& \Delta y(t_k) = \gamma_k(y(t_k^+)) + f_k, \quad k = 1, 2, \ldots, m.
\end{aligned}
\]

We have similar remark for (2.2).

Lemma 2.7. Every \( y \in \text{PC}^1(\mathcal{J}_S^0, \mathbb{R}^m) \) that satisfies (1.1) also comes out perfect on the following inequality:

\[
\left| \left| y(t) - \Psi_M(t, t_0)y_0 - \sum_{j=1}^{k} \gamma_j(y(t_j^-)) - \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, y(u), y(h(u)))DU \Delta s \right| \right| \leq (k + t - t_0)\epsilon
\]

for \( t \in (t_k, t_{k+1}] \subset \mathcal{J}_S^0 \).

Proof. If \( y \in \text{PC}^1(\mathcal{J}_S^0, \mathbb{R}^m) \) satisfies (2.1), then by Remark 2.6, we have

\[
\begin{aligned}
& y^{\lambda}(t) = M(t)z(t) + \int_{t_0}^{t} \mathcal{K}(t, s, z(s), z(h(s)))DS + f(t), \quad t \in \mathcal{J}_S, \\
& \Delta y(t_k) = \gamma_k(y(t_k^+)) + f_k, \quad k = 1, 2, \ldots, m.
\end{aligned}
\]

Then

\[ y(t) = y_0 + \Psi_M(t, t_0)y_0 + \sum_{j=1}^{k} \gamma_j(y(t_j^-)) + \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, y(u), y(h(u)))DU \Delta s + \int_{t_0}^{t} f(s)DS. \]

So,

\[
\left| \left| y(t) - y_0 - \Psi_M(t, t_0)y_0 - \sum_{j=1}^{k} \gamma_j(y(t_j^-)) - \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, y(u), y(h(u)))DU \Delta s \right| \right| \leq \int_{t_0}^{t} ||f(s)||DS + \sum_{i=1}^{k} ||f_i|| \leq (k + t - t_0)\epsilon \leq (k + t_f - t_0)\epsilon.
\]

We have similar remarks for (2.2).

3. Main results

Now we are going to give our result on Hyers-Ulam stability.

Theorem 3.1. If

(a) The function \( \mathcal{K} \) is piecewise continuous with the Lipschitz condition

\[ ||\mathcal{K}(t, s, x_1, x_2) - \mathcal{K}(t, s, y_1, y_2)|| \leq \sum_{i=1}^{2} L ||x_i - y_i||, \quad L > 0 \]

for \( t_0 \leq s \leq t \leq t_f \), and for all \( x_i, y_i \in \mathbb{R}^m \), \( i \in \{1, 2\} \);
(b) $\gamma_k : \mathbb{R} \rightarrow \mathbb{R}$ is such that $||\gamma_k(x_1) - \gamma_k(x_2)|| \leq M_k ||x_1 - x_2||$, $M_k > 0$, for all $k \in \{1, 2, \ldots, m\}$ and $x_1, x_2 \in \mathbb{R}$, $i \in \{1, 2\};$

(c) $\left( \sum_{j=1}^{m} M_j + 2 \sup_{t \in T^a \cap [t_0 - \lambda, t_0]} \int_{t_0}^{t} \| \Psi_M(t, \Theta(s)) \| \int_{t_0}^{s} L \Delta u \Delta s \right) < 1;$

(d) for some $C_k \geq 1$, we have $\| \Psi_M(t, \Theta(s)) \| = \sup_{t \in T^a \cap [t_0 - \lambda, t_0]} \| \Psi_M(t, \Theta(s)) \| \leq C_k$;

then equation (1.1) has

(i) a unique solution in $PC(T^a \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC^1(T^a, \mathbb{R}^m);$ 
(ii) Hyers-Ulam stability on $T^a \cap [t_0 - \lambda, t_0].$

Proof.

(i) Define an operator $\Lambda : PC(T^a \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \rightarrow PC(T^a \cap [t_0 - \lambda, t_0], \mathbb{R}^m)$ by

$$
(\Lambda z)(t) = \left\{ \begin{array}{ll}
\alpha(t), & t \in [t_0 - \lambda, t_0], \\
\alpha(t_0) + \psi_M(t, t_0)z_0 + \int_{t_0}^{t} \psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & t \in (t_0, t_1], \\
\alpha(t_0) + \gamma_1(z(t_1^-)) + \psi_M(t, t_0)z_0 + \int_{t_0}^{t} \psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & t \in (t_1, t_2], \\
\vdots & \\
\alpha(t_0) + \sum_{j=1}^{m} \gamma_j(z(t_j^-)) + \psi_M(t, t_0)z_0 + \int_{t_0}^{t} \psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & t \in (t_2, t_3], \\
\vdots & \\
\alpha(t_0) + \sum_{j=1}^{m} \gamma_j(z(t_j^-)) + \psi_M(t, t_0)z_0 + \int_{t_0}^{t} \psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & t \in (t_m, t_{m+1}].
\end{array} \right.
$$

(3.1)

We see that for any $z_1, z_2 \in PC(T^a \cap [t_0 - \lambda, t_0], \mathbb{R}^m)$ and for all $t \in [t_0 - \lambda, t_0]$, we have

$$
|| (\Lambda z_1)(t) - (\Lambda z_2)(t) || = 0.
$$

For $t \in (t_m, t_{m+1}]$ consider,

$$
\left\| (\Lambda z_1)(t) - (\Lambda z_2)(t) \right\| = \sum_{j=1}^{m} \left\| \gamma_j(z_1(t_j^-)) - \gamma_j(z_2(t_j^-)) \right\| \\
+ \left\| \int_{t_0}^{t} \psi_M(t, \Theta(s)) \int_{t_0}^{s} \left( \mathcal{K}(s, u, z_1(u), z_1(h(u))) - \mathcal{K}(s, u, z_2(u), z_2(h(u))) \right) \Delta u \Delta s \right\| \\
\leq \sum_{j=1}^{m} M_j \left\| z_1(t_j^-) - z_2(t_j^-) \right\|.
$$
Following from (c), the operator is strictly contractive and hence a Picard operator on
\[ P \cap \bigcap_{j=1}^{m} M_j \sup_{t \in \mathcal{I}_0 \cap (t_0 - \lambda, t_0]} \| z_1(t_j) - z_2(t_j) \| \]
\[ + \sup_{t \in \mathcal{I}_0 \cap (t_0 - \lambda, t_0]} \| \Psi_M(t, \Theta(s)) \| \int_{t_0}^{s} \| \mathcal{K}(s, u, z_1(u), z_1(h(u))) - \mathcal{K}(s, u, z_2(u), z_2(h(u))) \| \Delta u \Delta s \]
\[ \leq \sum_{j=1}^{m} M_j \|| z_1 - z_2 || + 2 \|| z_1 - z_2 || \sup_{t \in \mathcal{I}_0 \cap (t_0 - \lambda, t_0]} \| \Psi_M(t, \Theta(s)) \| \int_{t_0}^{s} \| \Delta u \Delta s \|
\]
\[ \leq \| z_1 - z_2 \| \left( \sum_{j=1}^{m} M_j + 2 \sup_{t \in \mathcal{I}_0 \cap (t_0 - \lambda, t_0]} \| \Psi_M(t, \Theta(s)) \| \int_{t_0}^{s} \| \Delta u \Delta s \| \right). \]

From (3.1), it follows that the unique fixed point of this operator is in fact the unique solution of (1.1) in
\[ PC(\mathcal{I}_0^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC(\mathcal{I}_0^0, \mathbb{R}^m) \].

(ii) Now let \( y \in PC(\mathcal{I}_0^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC(\mathcal{I}_0^0, \mathbb{R}^m) \) be a solution to (2.1). The unique solution \( z \in PC(\mathcal{I}_0^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap PC(\mathcal{I}_0^0, \mathbb{R}^m) \) of the dynamic equation
\[
\begin{cases}
  z^\Delta(t) = M(t)z(t) + \int_{t_0}^{t} \mathcal{K}(s, t, z(s), z(h(s))) \Delta s, & t \in \mathcal{I}_0',
  \\
  \Delta z(t_k) = z(t_k) - z(t_k-1) = \gamma_k(z(t_k)), & k = 1, 2, \ldots, m,
  \\
  z(t) = y(t), & t \in [t_0 - \lambda, t_0],
  \\
  z(t_0) = y(t_0) = z_0,
\end{cases}
\]
is given by
\[
\begin{align*}
  y(t), & \quad t \in [t_0 - \lambda, t_0], \\
  y(t_0) + \Psi_M(t, t_0) z_0 + \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & \quad t \in (t_0, t_1],
  \\
  y(t_0) + \gamma_1(z(t_0^-)) + \Psi_M(t, t_0) z_0 + \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & \quad t \in (t_1, t_2],
  \\
  y(t_0) + \sum_{j=1}^{2} \gamma_j(z(t_0^-)) + \Psi_M(t, t_0) z_0 + \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & \quad t \in (t_2, t_3],
  \\
  \vdots \\
  y(t_0) + \sum_{j=1}^{m} \gamma_j(z(t_0^-)) + \Psi_M(t, t_0) z_0 + \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, z(u), z(h(u))) \Delta u \Delta s, & \quad t \in (t_m, t_{m+1}].
\end{align*}
\]
We observe that for all \( t \in [t_0 - \lambda, t_0] \), we have \( \|y(t) - z(t)\| = 0 \). For \( t \in (t_m, t_{m+1}) \), using Lemma 2.7, we have

\[
\|y(t) - z(t)\| \leq \|y(t) - \Psi_M(t, t_0) y_0 - \sum_{j=1}^{m} Y_j(y(t_j^-))
\]

\[
- \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \mathcal{K}(s, u, y(u), y(h(u))) \Delta u \Delta s \| + \sum_{j=1}^{m} \|Y_j(y(t_j^-)) - Y_j(z(t_j^-))\|
\]

\[
+ \left| \int_{t_0}^{t} \Psi_M(t, \Theta(s)) \int_{t_0}^{s} \left( \mathcal{K}(s, u, y(u), y(h(u))) - \mathcal{K}(s, z(u), z(h(u))) \right) \Delta u \Delta s \right|
\]

\[
\leq (m + t_f - t_0) e + \sum_{j=1}^{m} M_j \|y(t_j^-) - z(t_j^-)\|
\]

\[
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L \|y(u) - z(u)\| \Delta u \Delta s
\]

\[
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L \|y(h(u)) - z(h(u))\| \Delta u \Delta s.
\]

Next, we show that the operator \( T : PC(J^0_S \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \to PC(J^0_S \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \) given below is an increasing Picard operator on \( PC(J^0_S \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \).

\[
(Tg)(t) = \begin{cases} 
0, & t \in [t_0 - \lambda, t_0], \\
(t_f - t_0) e + \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(u) \Delta u \Delta s \\
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(h(u)) \Delta u \Delta s, & t \in [t_0, t_1], \\
(1 + t_f - t_0) e + M_j g(t_j^-) + \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(u) \Delta u \Delta s \\
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(h(u)) \Delta u \Delta s, & t \in (t_1, t_2], \\
(2 + t_f - t_0) e + \sum_{j=1}^{2} M_j g(t_j^-) + \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(u) \Delta u \Delta s \\
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(h(u)) \Delta u \Delta s, & t \in (t_2, t_3], \\
\vdots \\
(m + t_f - t_0) e + \sum_{j=1}^{m} M_j g(t_j^-) + \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(u) \Delta u \Delta s \\
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g(h(u)) \Delta u \Delta s, & t \in (t_m, t_{m+1}].
\end{cases}
\] (3.2)

For any \( g_1, g_2 \in PC(J^0_S \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \), \( \|(Tg_1)(t) - (Tg_2)(t)\| = 0 \) for all \( t \in [t_0 - \lambda, t_0] \). For \( t \in (t_m, t_{m+1}] \), consider

\[
\|(Tg_1)(t) - (Tg_2)(t)\| \leq \sum_{j=1}^{m} M_j \|g_1(t_j^-) - g_2(t_j^-)\| + \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L \|g_1(u) - g_2(u)\| \Delta u \Delta s.
\]
Since \( \sum_{j=1}^{m} M_j \sup_{t \in \mathcal{T}_s \cap [t_0 - \lambda, t_0]} \|g_1(t_j^-) - g_2(t_j^-)\| \leq \sum_{j=1}^{m} M_j \|g_1 - g_2\| \leq \sum_{j=1}^{m} M_j \|g_1 - g_2\| \leq \sum_{j=1}^{m} M_j \|g_1 - g_2\| \)

\[
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L \sup_{t \in \mathcal{T}_s \cap [t_0 - \lambda, t_0]} \|g_1(t) - g_2(t)\| \Delta u \Delta s
\]

\[
\leq \sum_{j=1}^{m} M_j \|g_1 - g_2\| \left( \sum_{j=1}^{m} M_j + 2 \sup_{t \in \mathcal{T}_s \cap [t_0 - \lambda, t_0]} \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L \Delta u \Delta s \right).
\]

Since \( \left( \sum_{j=1}^{m} M_j + 2 \sup_{t \in \mathcal{T}_s \cap [t_0 - \lambda, t_0]} \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L \Delta u \Delta s \right) < 1 \), the operator is contractive on \( PC(\mathcal{T}_s \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \). Applying Banach contraction principle, \( T \) is Picard operator with unique fixed point \( g^* \in PC(\mathcal{T}_s \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \), i.e.,

\[
g^*(t) = (m + t_f - t_0) e + \sum_{j=1}^{m} M_j g^*(t_j^-) + \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g^*(u) \Delta u \Delta s
\]

\[
+ \int_{t_0}^{t} \|\Psi_M(t, \Theta(s))\| \int_{t_0}^{s} L g^*(h(u)) \Delta u \Delta s.
\]

Since \( g^* \) is increasing, so \( g^*(h(u)) \leq g^*(u) \) and by using (d), we have

\[
g^*(t) \leq (m + t_f - t_0) e + \sum_{j=1}^{m} M_j g^*(t_j^-) + 2 \int_{t_0}^{t} \int_{t_0}^{s} C_k L g^*(u) \Delta u \Delta s.
\]

By Lemma 2.4, we get

\[
g^*(t) \leq (m + t_f - t_0) e \prod_{t_0 < t_j < t} (1 + M_j) e_{P(t, t_0)},
\]

where \( P(s) = 2 \int_{t_0}^{s} C_k L \Delta u \). If we set \( g(t) = \|y(t) - z(t)\| \), then from (3.2), \( g(t) \leq (Tg)(t) \) from which by using abstract Grönwall lemma, it follows that \( g(t) \leq g^*(t) \), thus

\[
\|y(t) - z(t)\| \leq (m + t_f - t_0) e \prod_{t_0 < t_j < t} (1 + M_j) e_{P(t, t_0)}.
\]

Similarly, by following the same process, we can prove that:

**Theorem 3.2.** If

(a) The function \( \mathcal{K} \) is piecewise continuous with the Lipschitz condition \( \|\mathcal{K}(t, s; x_1, x_2) - \mathcal{K}(t, s; y_1, y_2)\| \leq \sum_{i=1}^{2} L|x_i - y_i| \), \( L > 0 \) for \( t_0 \leq s \leq t \leq t_f \) and for all \( x_i, y_i \in \mathbb{R}^m, i \in \{1, 2\} \);

(b) \( \gamma_k : \mathbb{R} \to \mathbb{R} \) is such that \( \|\gamma_k(x_1) - \gamma_k(x_2)\| \leq M_k \|x_1 - x_2\|, M_k > 0, \) for all \( k \in \{1, 2, \cdots , m\} \) and \( x_1, x_2 \in \mathbb{R}, i \in \{1, 2\} \);
\( \left( \sum_{j=1}^{m} M_j + 2 \sup_{t \in \mathcal{T}_S^T \cap [t_0 - \lambda, t_0]} \int_{t_0}^{t} \| \psi_M(t, \Theta(s)) \| \int_{t_0}^{s} \Delta u \Delta s \right) < 1; \)

(d) for some \( C_k \geq 0 \), we have
\[ \| \psi_M(t, \Theta(s)) \| \leq C_k; \]

(e) \( \varphi \in C(\mathcal{T}_S^0, \mathbb{R}^m) \) is increasing such that for some \( \rho > 0 \),
\[ \int_{t_0}^{t} \varphi(r) \Delta r \leq \rho \varphi(t), \]
then (1.1) has

(i) a unique solution in \( \mathcal{PC}(\mathcal{T}_S^0 \cap [t_0 - \lambda, t_0], \mathbb{R}^m) \cap \mathcal{PC}^1(\mathcal{T}_S^0, \mathbb{R}^m); \)

(ii) Hyers-Ulam-Rassias stability on \( \mathcal{T}_S^0 \cap [t_0 - \lambda, t_0]. \)

4. Conclusion

In this paper, we have proved the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of (1.1) using fixed point method. We proved our results by using abstract Grönwall lemma together with Lemma 2.4. Moreover, our results guarantee that there is an exact solution of (1.1) which is close to the approximate solution.

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References


