Uniform convexity in $\ell_{p[\cdot]}$

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Abstract

In this work, we investigate the variable exponent sequence space $\ell_{p[\cdot]}$. In particular, we prove a geometric property similar to uniform convexity without the assumption $\limsup_{n\to\infty} p(n) < \infty$. This property allows us to prove the analogue to Kirk’s fixed point theorem in the modular vector space $\ell_{p[\cdot]}$ under Nakano’s formulation. ©2017 All rights reserved.

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1. Introduction

The origin of function modulars defined in vector spaces goes back to the 1931 early work of Orlicz [15]. In this work, he introduced the following vector space:

$$X = \left\{ (x_n) \in \mathbb{R}^N; \sum_{n=0}^{\infty} |\lambda x_n|^{p(n)} < \infty \text{ for some } \lambda > 0 \right\},$$

where $\{p(n)\} \subset [1, \infty)$. For interested readers about about the topology and the geometry of $X$, we recommend the references [8, 13, 18, 19]. Note that the vector space $X$ may be seen as a predecessor to the theory of variable exponent spaces [3]. Recently, these spaces have enjoyed a major development. A systematic study of their vector topological properties was initiated in 1991 by Kovačík and Rákosník [9]. But one of the driving forces for the rapid development of the theory of variable exponent spaces has been the model of electrorheological fluids introduced by Rajagopal and Růžička [16, 17]. These fluids are an example of smart materials, whose development is one of the major tools in space engineering.

The general definition of a modular in an abstract vector space was introduced by Nakano [12, 14]. In this work, we focus on establishing a geometric property similar to modular uniform convexity in the vector space $X$ described above. This investigation allows us to discover new unknown properties.

For the readers interested into the metric fixed point theory, we recommend the book by Khamsi and Kirk [4] and the recent book by Khamsi and Kozlowski [5].

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2. Notations and Definitions

First recall the definition of the variable exponent sequence space \( \ell_{p(n)} \).

**Definition 2.1 ([15]).** For a function \( p : \mathbb{N} \to [1, \infty) \), define the vector space

\[
\ell_{p(n)} = \left\{ (x_n) \in \mathbb{R}^N; \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^{p(n)} < \infty \text{ for some } \lambda > 0 \right\}.
\]

Inspired by the vector space \( \ell_{p(n)} \), Nakano [12, 14, 13] came up with the concept of the modular vector structure. The following proposition summarizes Nakano’s main ideas.

**Proposition 2.2 ([8, 12, 18]).** Consider the function \( \rho : \ell_{p(n)} \to [0, \infty] \) defined by

\[
\rho(x) = \rho((x_n)) = \sum_{n=0}^{\infty} \frac{1}{p(n)} |x_n|^{p(n)}.
\]

Then \( \rho \) satisfies the following properties:

1. \( \rho(x) = 0 \) if and only if \( x = 0 \),
2. \( \rho(\alpha x) = \rho(x) \), if \( |\alpha| = 1 \),
3. \( \rho(\alpha x + (1 - \alpha)y) \leq \alpha \rho(x) + (1 - \alpha)\rho(y) \), for any \( \alpha \in [0, 1] \),

for any \( x, y \in X \). The function \( \rho \) is called a convex modular.

Next, we introduce a kind of modular topology that is similar to the classical metric topology.

**Definition 2.3 ([6]).**

(a) We say that a sequence \( \{x_n\} \subset \ell_{p(n)} \) is \( \rho \)-convergent to \( x \in \ell_{p(n)} \) if and only if \( \rho(x_n - x) \to 0 \). Note that the \( \rho \)-limit if it exists.

(b) A sequence \( \{x_n\} \subset \ell_{p(n)} \) is called \( \rho \)-Cauchy if \( \rho(x_n - x_m) \to 0 \) as \( n, m \to \infty \).

(c) A set \( C \subset \ell_{p(n)} \) is called \( \rho \)-closed if for any sequence \( \{x_n\} \subset C \) which \( \rho \)-converges to \( x \) implies that \( x \in C \).

(d) A set \( C \subset \ell_{p(n)} \) is called \( \rho \)-bounded if \( \sup \{\rho(x-y); x, y \in C\} < \infty \).

Note that \( \rho \) satisfies the Fatou property, i.e., \( \rho(x - y) \leq \liminf_{n \to \infty} \rho((x - y_n) \) holds whenever \( \{y_n\} \rho \)-converges to \( y \), for any \( x, y, y_n \in \ell_{p(n)} \). The Fatou property is very useful. For example, Fatou property holds if and only if the \( \rho \)-balls are \( \rho \)-closed. Recall that the subset \( B_{\rho}(x, r) = \{y \in \ell_{p(n)}; \rho(x - y) \leq r\} \), with \( x \in \ell_{p(n)} \) and \( r \geq 0 \), is known as a \( \rho \)-ball.

Recall that \( \rho \) is said to satisfy the \( \Delta_2 \)-condition if there exists \( K \geq 0 \) such that

\[
\rho(2x) \leq K \rho(x)
\]

for any \( x \in \ell_{p(n)} \) [5]. This property is very important in the study of modular functionals. For more on the \( \Delta_2 \)-condition and its variants may be found in [5, 10, 11]. In the case of \( \ell_{p(n)} \), it is easy to see that \( \rho \) satisfies the \( \Delta_2 \)-condition if and only if \( \limsup_{n \to \infty} p(n) < \infty \). Recall that the Minkowski functional associated to the modular unit ball is known as the Luxemburg norm defined by

\[
\|x\|_\rho = \inf \left\{ \lambda > 0; \rho \left( \frac{1}{\lambda} x \right) \leq 1 \right\}.
\]

Recall that \( (\ell_{p(n)}, \|\cdot\|_\rho) \) is a Banach space. Sundaresan [18] proved that \( (\ell_{p(n)}, \|\cdot\|_\rho) \) is reflexive if and only if \( 1 < \inf_{n \to \infty} p(n) \leq \sup_{n \to \infty} p(n) < \infty \). In this case, \( (\ell_{p(n)}, \|\cdot\|_\rho) \) is uniformly convex which implies in fact that \( (\ell_{p(n)}, \|\cdot\|_\rho) \) is superreflexive [1]. In the next section, we will introduce a new modular uniform convexity satisfied by \( \ell_{p(n)} \) even when \( \limsup_{n \to \infty} p(n) < \infty \) is not satisfied.
3. Modular Uniform Convexity

Orlicz function spaces was carried in \[3, 11\]. Its study in Orlicz function spaces was carried in \[3, 11\].

Definition 3.1 ([3, 11]). We define the following uniform convexity type properties of the modular \(\rho\):

(a) [14] Let \(r > 0\) and \(\varepsilon > 0\). Define
\[
\mathcal{D}_1(r, \varepsilon) = \{(x, y); x, y \in \ell_p(\cdot), \rho(x) \leq r, \rho(y) \leq r, \rho(x - y) \geq \varepsilon r\}.
\]
If \(\mathcal{D}_1(r, \varepsilon) \neq \emptyset\), let
\[
\delta_1(r, \varepsilon) = \inf \left\{1 - \frac{1}{r} \rho \left(\frac{x + y}{2}\right); (x, y) \in \mathcal{D}_1(r, \varepsilon)\right\}.
\]
If \(\mathcal{D}_1(r, \varepsilon) = \emptyset\), we set \(\delta_1(r, \varepsilon) = 1\). We say that \(\rho\) satisfies the uniform convexity (UC) if for every \(r > 0\) and \(\varepsilon > 0\), we have \(\delta_1(r, \varepsilon) > 0\). Note that for every \(r > 0\), \(\mathcal{D}_1(r, \varepsilon) \neq \emptyset\), for \(\varepsilon > 0\) small enough.

(b) [5] We say that \(\rho\) satisfies (UUC) if for every \(s \geq 0\) and \(\varepsilon > 0\), there exists \(\eta_1(s, \varepsilon) > 0\) depending on \(s\) and \(\varepsilon\) such that
\[
\delta_1(r, \varepsilon) > \eta_1(s, \varepsilon) > 0 \quad \text{for} \quad r > s.
\]

(c) [5] Let \(r > 0\) and \(\varepsilon > 0\). Define
\[
\mathcal{D}_2(r, \varepsilon) = \{(x, y); x, y \in \ell_p(\cdot), \rho(x) \leq r, \rho(y) \leq r, \rho\left(\frac{x - y}{2}\right) \geq \varepsilon r\}.
\]
If \(\mathcal{D}_2(r, \varepsilon) \neq \emptyset\), let
\[
\delta_2(r, \varepsilon) = \inf \left\{1 - \frac{1}{r} \rho \left(\frac{x + y}{2}\right); (x, y) \in \mathcal{D}_2(r, \varepsilon)\right\}.
\]
If \(\mathcal{D}_2(r, \varepsilon) = \emptyset\), we set \(\delta_2(r, \varepsilon) = 1\). We say that \(\rho\) satisfies (UC2) if for every \(r > 0\) and \(\varepsilon > 0\), we have \(\delta_2(r, \varepsilon) > 0\). Note that for every \(r > 0\), \(\mathcal{D}_2(r, \varepsilon) \neq \emptyset\), for \(\varepsilon > 0\) small enough.

(d) [5] We say that \(\rho\) satisfies (UUC2) if for every \(s \geq 0\) and \(\varepsilon > 0\), there exists \(\eta_2(s, \varepsilon) > 0\) depending on \(s\) and \(\varepsilon\) such that
\[
\delta_2(r, \varepsilon) > \eta_2(s, \varepsilon) > 0 \quad \text{for} \quad r > s.
\]

(e) [14] We say that \(\rho\) is strictly convex, (SC), if for every \(x, y \in \ell_p(\cdot)\) such that \(\rho(x) = \rho(y)\) and
\[
\rho\left(\frac{x + y}{2}\right) = \frac{\rho(x) + \rho(y)}{2},
\]
we have \(x = y\).

The property (UC) was introduced by Nakano [14]. In all the subsequent research done on \(\ell_p(\cdot)\), the authors considered (UC). For example, Sundaresan [18] proved that in \(\ell_p(\cdot)\), \(\rho\) satisfies (UC) if and only if \(1 < \inf_{n\in\mathbb{N}} p(n) \leq \sup_{n\in\mathbb{N}} p(n) < \infty\). Note that (UC) and (UC2) are equivalent if \(\rho\) satisfies the \(\Delta_2\)-condition [5]. In this case, we must have \(\sup_{n\in\mathbb{N}} p(n) < \infty\).

The following technical result is very useful.

Lemma 3.2. The following inequalities are valid:

(i) [2] If \(p \geq 2\), then we have
\[
\left|\frac{a + b}{2}\right|^p + \left|\frac{a - b}{2}\right|^p \leq \frac{1}{2} \left(|a|^p + |b|^p\right)
\]
for any \(a, b \in \mathbb{R}\).
(ii) \[18\] If \( 1 < p \leq 2 \), then we have
\[
\left| \frac{a+b}{2} \right|^p + \frac{p(p-1)}{2} \left| \frac{a-b}{|a|+|b|} \right|^{2-p} \left| \frac{a-b}{2} \right|^p \leq \frac{1}{2} (|a|^p + |b|^p)
\]
for any \( a, b \in \mathbb{R} \) such that \( |a| + |b| \neq 0 \).

Before we state the main result of this work, we will need the following notation:
\[
\rho_K(x) = \rho_K((x_n)) = \sum_{n \in K} \frac{1}{p(n)} |x_n|^p(n)
\]
for any \( K \subseteq \mathbb{N} \) and any \( x \in \ell_p(\cdot) \). If \( K = \emptyset \), we set \( \rho_K(x) = 0 \).

**Theorem 3.3.** Consider the vector space \( \ell_p(\cdot) \). If \( \inf_{n \in \mathbb{N}} p(n) > 1 \), then the modular \( \rho \) is \( (\mathbb{U}, \mathbb{C}) \).

**Proof.** Assume \( A = \inf_{n \in \mathbb{N}} p(n) > 1 \). Let \( r > 0 \) and \( \varepsilon > 0 \). Let \( x, y \in \ell_p(\cdot) \) such that
\[
\rho(x) \leq r, \quad \rho(y) \leq r \quad \text{and} \quad \rho \left( \frac{x-y}{2} \right) \geq \varepsilon.
\]
Since \( \rho \) is convex, then we have
\[
r \varepsilon \leq \rho \left( \frac{x-y}{2} \right) \leq \frac{\rho(x) + \rho(y)}{2} \leq r,
\]
which implies \( \varepsilon \leq 1 \). Next, set \( I = \{ n \in \mathbb{N}; p(n) \geq 2 \} \) and \( J = \{ n \in \mathbb{N}; p(n) < 2 \} = \mathbb{N} \setminus I \). Note that we have \( \rho(z) = \rho_I(z) + \rho_J(z) \), for any \( z \in \ell_p(\cdot) \). From our assumptions, we have either \( \rho_I((x-y)/2) \geq r \varepsilon/2 \) or \( \rho_J((x-y)/2) \geq r \varepsilon/2 \).

Assume first \( \rho_I((x-y)/2) \geq r \varepsilon/2 \). Using Lemma 3.2, we conclude that
\[
\rho_I \left( \frac{x+y}{2} \right) + \rho_I \left( \frac{x-y}{2} \right) \leq \frac{\rho_I(x) + \rho_I(y)}{2},
\]
which implies
\[
\rho_I \left( \frac{x+y}{2} \right) \leq \frac{\rho_I(x) + \rho_I(y)}{2} - \frac{r \varepsilon}{2}.
\]
Since
\[
\rho_J \left( \frac{x+y}{2} \right) \leq \frac{\rho_J(x) + \rho_J(y)}{2},
\]
we get
\[
\rho \left( \frac{x+y}{2} \right) \leq \frac{\rho(x) + \rho(y)}{2} - \frac{r \varepsilon}{2} \leq \rho \left( 1 - \frac{\varepsilon}{2} \right).
\]
For the second case, assume \( \rho_J((x-y)/2) \geq r \varepsilon/2 \). Set \( C = \varepsilon/4 \),
\[
J_1 = \left\{ n \in J_1; \left| x_n - y_n \right| \leq C (|x_n| + |y_n|) \right\} \quad \text{and} \quad J_2 = J \setminus J_1.
\]
We have
\[
\rho_{J_1} \left( \frac{x-y}{2} \right) \leq \sum_{n \in J_1} \frac{C p(n)}{p(n)} \left| \frac{x_n + y_n}{2} \right|^{p(n)} \leq \frac{C}{2} \sum_{n \in J_1} 2 \left| x_n \right|^{p(n)} + \left| y_n \right|^{p(n)} p(n)
\]
because \( C \leq 1 \) and the power function is convex. Hence
\[
\rho_{J_1} \left( \frac{x-y}{2} \right) \leq \frac{C}{2} (\rho_{J_1}(x) + \rho_{J_1}(y)) \leq \frac{C}{2} (\rho(x) + \rho(y)) \leq C r.
\]
Since \( \rho_j((x-y)/2) \geq r \varepsilon/2 \), we get
\[
\rho_j\left(\frac{x-y}{2}\right) = \rho_j\left(\frac{x-y}{2}\right) - \rho_j\left(\frac{x-y}{2}\right) \geq \frac{r \varepsilon}{2} - C r.
\]

For any \( n \in J_2 \), we have
\[
A - 1 \leq p(n)/(p(n) - 1) \quad \text{and} \quad C \leq C^{2-p(n)} \leq \left|\frac{x_n-y_n}{|x_n| + |y_n|}\right|^{2-p(n)},
\]
which implies by Lemma 3.2 that
\[
\left|\frac{x_n+y_n}{2}\right|^{p(n)} + \frac{(A-1)C}{2} \left|\frac{x_n-y_n}{2}\right|^{p(n)} \leq \frac{1}{2} \left(\left|x_n\right|^{p(n)} + \left|y_n\right|^{p(n)}\right).
\]
Hence
\[
\rho_j\left(\frac{x+y}{2}\right) + \frac{(A-1)C}{2} \rho_j\left(\frac{x-y}{2}\right) \leq \rho_j\left(x\right) + \rho_j\left(y\right),
\]
which implies
\[
\rho_j\left(\frac{x+y}{2}\right) \leq \rho_j\left(x\right) + \rho_j\left(y\right) - r \frac{(A-1)\varepsilon^2}{8},
\]
since \( C = \varepsilon/4 \). Therefore, we have
\[
\rho\left(\frac{x+y}{2}\right) \leq r - r \frac{(A-1)\varepsilon^2}{8} = r \left(1 - \frac{(A-1)\varepsilon^2}{8}\right).
\]
Using the definition of \( \delta_2(r, \varepsilon) \), we conclude that
\[
\delta_2(r, \varepsilon) \geq \min \left(\frac{\varepsilon}{2}, (A-1)\frac{\varepsilon^2}{8}\right) > 0.
\]
Therefore, \( \rho \) is (UIC2). Moreover, if we set \( \eta_2(r, \varepsilon) = \min \left(\varepsilon/2, (A-1)\varepsilon^2/8\right) \), we conclude that \( \rho \) is in fact (UUC2).

**Remark 3.4.** Note that in our proof above, we showed that \( \eta_2(r, \varepsilon) \) is in fact a function of \( \varepsilon \) only. We will make use of this fact throughout.

Using this form of uniform convexity, we can prove some interesting modular geometric properties not clear to hold in the absence of the \( \Delta_2 \)-condition. These properties were proved recently in an unpublished work. For the sake of completeness, we include their proofs.

**Proposition 3.5.** Consider the space \( \ell_{p(\cdot)} \). Assume \( \inf_{n \in \mathbb{N}} p(n) > 1 \).

(i) Let \( C \) be a nonempty \( \rho \)-closed convex subset of \( \ell_{p(\cdot)} \). Let \( x \in \ell_{p(\cdot)} \) be such that
\[
d_{\rho}(x, C) = \inf_{y \in C} \rho(x - y); \ y \in C < \infty.
\]
Then there exists a unique \( c \in C \) such that \( d_{\rho}(x, C) = \rho(x - c) \).

(ii) \( \ell_{p(\cdot)} \) satisfies the property (R), i.e., for any decreasing sequence \( \{C_n\}_{n \geq 1} \) of \( \rho \)-closed convex nonempty subsets of \( \ell_{p(\cdot)} \) such that \( \sup_{n \geq 1} d_{\rho}(x, C_n) < \infty \), for some \( x \in \ell_{p(\cdot)} \), then we have \( \bigcap_{n \geq 1} C_n \) is nonempty.

**Proof.** In order to prove (i), we may assume that \( x \notin C \) since \( C \) is \( \rho \)-closed. Therefore, we have \( d_{\rho}(x, C) > 0 \).

Set \( R = d_{\rho}(x, C) \). Hence for any \( n \geq 1 \), there exists \( y_n \in C \) such that \( \rho(x - y_n) < R(1 + 1/n) \). We claim that \( \{y_n/2\} \) is \( \rho \)-Cauchy. Assume otherwise that \( \{y_n/2\} \) is not \( \rho \)-Cauchy. Then there exists a subsequence
\{y_{\varphi(n)}\} and \varepsilon_0 > 0 such that \(\rho\left((y_{\varphi(n)} - y_{\varphi(m)})/2\right) \geq \varepsilon_0\), for any \(n > m \geq 1\). Moreover, we have \(\delta_2(R(1 + 1/n), 2\varepsilon_0/R) \geq \eta_2(\varepsilon_0/2R) > 0\), for any \(n \geq 1\). Since \(\max \left(\rho(x - y_{\varphi(n)}), \rho(x - y_{\varphi(m)})\right) \leq R(1 + 1/\varphi(m))\) and

\[
\rho \left(\frac{y_{\varphi(n)} - y_{\varphi(m)}}{2}\right) \geq \varepsilon_0 \geq R \left(1 + \frac{1}{\varphi(m)}\right) \frac{\varepsilon_0}{2R}
\]

for any \(n > m \geq 1\), we conclude that

\[
\rho \left(x - \frac{y_{\varphi(n)} + y_{\varphi(m)}}{2}\right) \leq R \left(1 + \frac{1}{\varphi(m)}\right) \left(1 - \eta_2 \left(\frac{\varepsilon_0}{2R}\right)\right).
\]

Hence

\[
R = d_\rho(x, C) \leq R \left(1 + \frac{1}{\varphi(m)}\right) \left(1 - \eta_2 \left(\frac{\varepsilon_0}{2R}\right)\right)
\]

for any \(m \geq 1\). If we let \(m \to \infty\), we get

\[
R \leq R \left(1 - \eta_2 \left(\frac{\varepsilon_0}{2R}\right)\right) < R,
\]

which is a contradiction since \(R > 0\). Therefore, \(\{y_{n}/2\}\) is \(\rho\)-Cauchy. Since \(\ell_p(\ast)\) is \(\rho\)-complete, then \(\{y_{n}/2\}\) \(\rho\)-converges to some \(y\). We claim that \(2y\in C\). Indeed, for any \(m \geq 1\), the sequence \(\{(y_n + y_m)/2\}\) \(\rho\)-converges to \(y + y_m/2\). Since \(C\) is \(\rho\)-closed and convex, we get \(y + y_m/2 \in C\). Finally, the sequence \(\{y + y_m/2\}\) \(\rho\)-converges to \(2y\), which implies \(2y \in C\). Set \(c = 2y\). Since \(\rho\) satisfies the Fatou property, we have

\[
d_\rho(x, C) \leq \rho(x - c) \leq \liminf_{m \to \infty} \rho(x - (y + y_m)/2) \leq \liminf_{m \to \infty} \liminf_{n \to \infty} \rho(x - (y_n + y_m)/2) \leq \liminf_{m \to \infty} \liminf_{n \to \infty} \left(\rho(x - y_n) + \rho(x - y_m)/2\right) = d_\rho(x, C).
\]

Hence \(\rho(x - c) = d_\rho(x, C)\). The uniqueness of the point \(c\) follows from the fact that \(\rho\) is (SC) since it is (ULC2).

For the proof of (ii), we assume that \(x \not\in C_{n_0}\) for some \(n_0 \geq 1\). In fact, the sequence \(\{d_\rho(x, C_n)\}\) is increasing and bounded. Set \(\lim_{n \to \infty} d_\rho(x, C_n) = R\). We may assume \(R > 0\). Otherwise \(x \in C_n\), for any \(n \geq 1\). From (i), there exists a unique \(y_n \in C_n\) such that \(d_\rho(x, C_n) = \rho(x - y_n)\), for any \(n \geq 1\). A similar proof will show that \(\{y_{n}/2\}\) \(\rho\)-converges to some \(y \in \ell_p(\ast)\). Since \(\{C_n\}\) are decreasing, convex and \(\rho\)-closed, we conclude that \(2y \in \bigcap_{n \geq 1} C_n\).

\[\square\]

**Remark 3.6.** It is natural to wonder whether the property (R) extends to any family of decreasing subsets. Indeed, assume \(\inf_{n \in N} \rho(n) > 1\). Let \(C\) be a \(\rho\)-closed nonempty convex subset of \(\ell_p(\ast)\) which is \(\rho\)-bounded. Let \(\{C_i\}_{i \in I}\) be a family of \(\rho\)-closed nonempty convex subsets of \(C\) such that \(\bigcap_{i \in F} C_i \neq \emptyset\), for any finite subset \(F\) of \(I\). Then \(\bigcap_{i \in F} C_i \neq \emptyset\). In order to see this, let \(x \in C\). Then \(\sup_{i \in F} \delta_\rho(x, C_i) \leq \delta_\rho(C) < \infty\) holds. For any subset \(F \subseteq I\), set \(d_F = d_\rho(x, \bigcap_{i \in F} C_i)\). Note that if \(F_1 \subseteq F_2 \subseteq I\) are finite subsets, then \(d_{F_1} \leq d_{F_2}\). Set

\[
d = \sup \left\{d_\rho(x, \bigcap_{i \in J} C_i), J \subseteq I \text{ such that } \bigcap_{i \in J} C_i \neq \emptyset\right\}.
\]

For any \(n \geq 1\), there exists a subset \(F_n \subseteq I\) such that \(d_1 - 1/n \leq d_{F_n} \leq d_1\). Set \(F_n = F_1 \cup \cdots \cup F_n\), for \(n \geq 1\). Then \(\bigcap_{i \in F_n} C_i\) is a decreasing sequence of nonempty \(\rho\)-closed convex subsets of \(\ell_p(\ast)\). The property (R) implies \(\bigcap_{i \in J} C_i \neq \emptyset\), where \(J = \bigcup_{n \geq 1} F_n = \bigcup_{n \geq 1} F_n\). Set \(K = \bigcap_{i \in I} C_i\). Note that \(d_\rho(x, K) = d_1\) because
d_1 - 1/n < d_{r_n} \leq d_\rho(x, K) \leq d_1, for any n \geq 1. Proposition 3.5 implies the existence of a unique y \in K such that \rho(x - y) = d_\rho(x, K) = d_1. Let i_0 \in I, then
\[ K \cap C_{i_0} = \bigcap_{i \in I \cup \{i_0\}} C_i \neq \emptyset, \]
because of the same argument using the property (R). Hence \[ d_\rho(x, K) \leq d_\rho(x, K \cap C_{i_0}) \leq d_1. \]
Hence \[ d_\rho(x, K \cap C_{i_0}) = d_\rho(x, K) = d_1 \] which implies y \in K \cap C_{i_0}. Therefore, we have y \in \bigcap_{i \in I} C_i which proves our claim.

If the property (R) is satisfied by the family of convex and closed (for the Luxemburg norm) subsets, we will deduce that \ell_{p(\cdot)} is reflexive. The work of Sundaresan [18] will imply in this case that 1 < \inf_{n \in N} p(n) \leq \sup_{n \in N} p(n) < \infty.

4. Application

In this section, we will show that under the assumption \inf_{n \in N} p(n) > 1, the space \ell_{p(\cdot)} enjoys a nice modular geometric property which will allow us to prove the analogue to Kirk’s fixed point theorem [7].

Definition 4.1. \ell_{p(\cdot)} is said to have the \rho-normal structure property if for any nonempty \rho-closed convex \rho-bounded subset C of \ell_{p(\cdot)} not reduced to one point, there exists x \in C such that
\[ \sup_{y \in C} \rho(x - y) < \delta_{\rho}(C). \]

Theorem 4.2. Assume \inf_{n \in N} p(n) > 1. Then \ell_{p(\cdot)} has the \rho-normal structure property.

Proof. Since \inf_{n \in N} p(n) > 1, Theorem 3.3 implies that \rho is (ULC2). Let C be a \rho-closed convex \rho-bounded subset of \ell_{p(\cdot)} not reduced to one point. Hence \delta_\rho(C) > 0. Set R = \delta_\rho(C). Let x, y \in C such that x \neq y. Hence \rho((x - y)/2) = \frac{\epsilon}{2} > 0. For any c \in C, we have \rho(x - c) \leq R and \rho(y - c) \leq R. Hence
\[ \rho\left(\frac{x + y}{2} - c\right) = \rho\left(\frac{(x - c) + (y - c)}{2}\right) \leq R \left(1 - \delta_2\left(R, \frac{\epsilon}{R}\right)\right) \]
for any c \in C. Hence
\[ \sup_{c \in C} \rho\left(\frac{x + y}{2} - c\right) \leq R \left(1 - \delta_2\left(R, \frac{\epsilon}{R}\right)\right) < R = \delta_\rho(C). \]
This completes the proof of Theorem 4.2 since C is convex.

Before we state the modular analogue to Kirk’s fixed point theorem in \ell_{p(\cdot)}, we will need the following definition.

Definition 4.3. Let C \subset \ell_{p(\cdot)} be nonempty. A mapping T : C \to C is called \rho-Lipschitzian if there exists a constant K \geq 0 such that
\[ \rho(T(x) - T(y)) \leq K \rho(x - y), \quad \text{for any} \ x, y \in C. \]
If K = 1, T is called \rho-nonexpansive. A point x \in C is called a fixed point of T if T(x) = x.

Theorem 4.4. Assume \inf_{n \in N} p(n) > 1. Let C be a nonempty \rho-closed convex \rho-bounded subset of \ell_{p(\cdot)}. Let T : C \to C be a \rho-nonexpansive mapping. Then T has a fixed point.

Proof. Let C be a nonempty \rho-closed convex \rho-bounded subset of \ell_{p(\cdot)}. Let T : C \to C be a \rho-nonexpansive mapping. Without loss of generality, we assume that C is not reduced to one point. Consider the family
\[ \mathcal{F} = \{K \subset C; K \text{ is nonempty \rho-closed convex and } T(K) \subset K\}. \]
The family $\mathcal{F}$ is not empty since $C \in \mathcal{F}$. Since $\inf_{n \in \mathbb{N}} p(\tau_n) > 1$, $\rho$ is (UUC2). Using Remark 3.6 combined with Zorn’s lemma, we conclude that $\mathcal{F}$ has a minimal element $K_0$. We claim that $K_0$ is reduced to one point. Assume not, i.e., $K_0$ has more than one point. Set $\text{co}(T(K_0))$ to be the intersection of all $\rho$-closed convex subset of $C$ containing $T(K_0)$. Hence $\text{co}(T(K_0)) \subset K_0$ since $T(K_0) \subset K_0$. So we have $T(\text{co}(T(K_0))) \subset T(K_0) \subset \text{co}(T(K_0))$. The minimality of $K_0$ implies $K_0 = \text{co}(T(K_0))$. Next, we use Theorem 4.2 to secure the existence of $x_0 \in K_0$ such that

$$r_0 = \sup_{y \in K_0} \rho(x_0 - y) < \delta_\rho(K_0).$$

Define the subset $K = \{x \in K_0; \sup_{y \in K_0} \rho(x - y) \leq r_0\}$. $K$ is not empty since $x_0 \in K$. Note that we have $K = \bigcap_{y \in K_0} B_\rho(y, r_0) \cap K_0$, where $B_\rho(y, r_0) = \{z \in \ell_p(x); \rho(y - z) \leq r_0\}$. Since $\rho$ satisfies the Fatou property and is convex, $B_\rho(y, r_0)$ is $\rho$-closed and convex. Hence $K$ is $\rho$-closed and convex subset of $K_0$. Let us show that $T(K) \subset K$. Let $x \in K$, then $T(x) \in \bigcap_{y \in K_0} B_\rho(T(y), r_0) \cap K_0$ since $T$ is $\rho$-nonexpansive. Hence $T(K_0) \subset B_\rho(T(x), r_0)$ which implies $K_0 = \text{co}(T(K_0)) \subset B_\rho(T(x), r_0)$, i.e., $T(x) \in \bigcap_{y \in K_0} B_\rho(y, r_0) \cap K_0$. Therefore, $T(K) \subset K$ holds. The minimality of $K_0$ implies $K = K_0$, i.e., for any $x \in K_0$, we have $\sup_{y \in K_0} \rho(x - y) \leq r_0$. This clearly will imply $\rho(x - y) \leq r_0$, for any $x, y \in K_0$. Hence $\delta_\rho(K_0) \leq r_0$. This is our sought contradiction. Therefore, $K_0$ is reduced to one point. Since $T(K_0) \subset K_0$, we conclude that $T$ has a fixed point in $C$.

\[ \square \]

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