On subclass of meromorphic multivalent functions associated with Liu-Srivastava operator

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Abstract

In the present paper, we introduce a new subclass related to meromorphically p-valent reciprocal starlike functions associated with the Liu-Srivastava operator. Some sufficient conditions for functions belonging to this class are derived. The results presented here improve and generalize some known results. ©2017 All rights reserved.

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1. Introduction

Let $\Sigma_p$ denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p}, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}),$$

(1.1)

which are analytic and p-valent in the punctured open unit disc $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U}\setminus\{0\}$, where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we set $\Sigma_1 = \Sigma$. Let $f$ and $g$ be two analytic functions in the open unit disk $\mathbb{U}$, we say that the function $f$ is subordinate to $g$ (written as $f \prec g$) if there exists a Schwarz function $\omega$, which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$. Furthermore, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalent relation:

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$
For some details see [2, 12]; see also [15].

A function $f \in \Sigma_p$ is said to be in class $S_p^\ast(\odot)$ of meromorphically $p$-valent starlike of order $\alpha$ if and only if
\[
\Re \left( \frac{zf'(z)}{pf(z)} \right) < -\alpha, \quad (0 \leq \alpha < 1).
\]
It is clear that $S_p^\ast(0) = S_p^\ast$, the class of $p$-valent starlike functions. A function $f \in S_p^\ast$ is said to be in the class $M_p(\odot)$ of meromorphically $p$-valent starlike of reciprocal order $\alpha$ if and only if
\[
\Re \left( \frac{pf(z)}{zf'(z)} \right) < -\alpha, \quad (0 \leq \alpha < 1).
\]
In particular $M_1(\odot) = M(\odot)$.

**Remark 1.1.** In view of the fact that
\[
\Re (p(z)) < 0 \implies \Re \left( \frac{1}{p(z)} \right) = \Re \left( \frac{p(z)}{|p(z)|^2} \right) < 0,
\]
it follows that meromorphically $p$-valent starlike function of reciprocal order 0 is same as a meromorphically $p$-valent starlike function. When $0 < \alpha < 1$, the function $f \in \Sigma_p$ is meromorphically $p$-valent starlike of reciprocal order $\alpha$ if and only if
\[
\left| \frac{zf'(z)}{pf(z)} + \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}.
\]
For $p = 1$, this class was studied by Sun et al. [17]. For arbitrary fixed real numbers $A$ and $B$ ($-1 \leq B < A \leq 1$), we denote by $P(A, B)$ the class of the functions of the form
\[
q(z) = 1 + c_1 z + c_2 z^2 + \cdots,
\]
which is analytic in the unit disk $U$ and satisfies the condition
\[
q(z) \prec \frac{1 + Az}{1 + Bz}, \quad (z \in U).
\]
(1.2)
The class $P(A, B)$ was introduced and studied by Janowski [5]. We also observe from (1.2) (see also [14]) that a function $q(z) \in P(A, B)$ if and only if
\[
\left| q(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2}, \quad (B \neq -1),
\]
(1.3)
and
\[
\Re(q(z)) > \frac{1 - A}{2}, \quad (B = -1).
\]
(1.4)
For function $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by
\[
g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p},
\]
the Hadamard product (convolution) of $f$ and $g$ is given by
\[
(f \ast g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g \ast f)(z).
\]
For complex parameters $\alpha_i$ and $\beta_j$, where $i = 1, 2, \cdots, l$, $j = 1, 2, \cdots, m$ and $\beta_j \notin \mathbb{Z}_0^+ = \{0, -1, -2, \cdots\}$, the generalized hypergeometric function $_{1}F_m$ is defined by
\[
_{1}F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} z^k,
\]
where $l \leq m + 1$, $l, m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $(\lambda)_n$ is pochhammer symbol (or shifted factorial) defined
in terms of Gamma function by
\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 
1, & n = 0, \\
\lambda(\lambda + 1)\cdots(\lambda + n - 1), & n \in \mathbb{N}.
\end{cases}
\]

Now consider the function
\[
h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = z^{-p}F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z),
\]
then the Liu-Srivastava linear operator [8, 9] \(H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \Sigma_p \rightarrow \Sigma_p\) is defined by using the Hadamard product (or convolution) as
\[
H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{(\beta_1)_k \cdots (\beta_m)_k} \alpha_k z^{-k-p}. \tag{1.5}
\]

For convenience, we denote \(H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) \approx H_{p, l, m}[\alpha_1]\).

The Liu-Srivastava operator is studied in [1, 13, 16], is the meromorphic analogue of the Dziok-Srivastava [3] linear operator. Special cases of the Liu-Srivastava linear operator include the meromorphic analogue of the Carlson-Shaffer linear operator \(L\), Liu [6] and Yang [20]. The analogous to the Ruscheweyh derivative operator \(D_n^+ = L_p(n, p, 1)\) was investigated by Yang [19]. The operator
\[
J_{c,p} = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) \, dt = L_p(c, c+1), \quad (c > 0),
\]
was studied by Uraleegaddi and Somanatha [18].

By using operator \(H_{p, l, m}[\alpha_1]\), we introduce the following new class.

**Definition 1.2.** A function \(f \in \Sigma_p\) is said to be in the class \(M(\alpha_1)(p; \beta; \lambda; A_1, B)\), if it satisfies the subordination

\[
\frac{p}{1-p\beta} \left\{ \frac{1-\lambda}{z} H_{p, l, m}[\alpha_1] f(z) + \lambda z \left[H_{p, l, m}[\alpha_1] f(z)\right]' + \beta \right\} < \frac{1+A_1z}{1+Bz},
\]

where \(A_1 = (1-\alpha)A + \alpha B, 0 \leq \alpha < 1, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1, 0 \leq p \beta < 1\) and \(H_{p, l, m}[\alpha_1]\) is defined in (1.5).

**Remark 1.3.** Using (1.3), (1.4) and for \(B \neq -1\), the Definition 1.2 is equivalent to

\[
\left| \frac{p}{1-p\beta} \left\{ \frac{H_{p, l, m}[\alpha_1] F_\lambda(z)}{z[H_{p, l, m}[\alpha_1] F_\lambda(z)]'} + \beta \right\} \right| < \frac{A_1 - B}{1 - B^2}, \tag{1.6}
\]

and for \(B = -1\),

\[
\Re \left\{ \frac{p}{1-p\beta} \left\{ \frac{H_{p, l, m}[\alpha_1] F_\lambda(z)}{z[H_{p, l, m}[\alpha_1] F_\lambda(z)]'} + \beta \right\} \right\} < -\frac{A_1}{2}, \tag{1.7}
\]

also, for \(B = -1, A_1 \neq 1\), (1.7) reduces to

\[
\left| \frac{1-p\beta}{p} z[H_{p, l, m}[\alpha_1] F_\lambda(z)]' + \frac{1}{1-A_1} \right| < \frac{1}{1-A_1}, \tag{1.8}
\]

and for \(B = -1, A_1 = 1\), we obtain

\[
\left| \frac{p}{1-p\beta} \left( \frac{H_{p, l, m}[\alpha_1] F_\lambda(z)}{z[H_{p, l, m}[\alpha_1] F_\lambda(z)]'} + \beta \right) + 1 \right| < 1, \tag{1.9}
\]

where

\[
F_\lambda(z) = (1-\lambda)f(z) + \lambda z f'(z).
\]
By assigning particular values to parameters the class \( M_{[\alpha]} (p; \alpha; \beta; \lambda; A, B) \) generalizes many previously known classes of meromorphic functions.

(i) For \( \lambda = 0, \alpha = 0, 1 = 2, m = 1, \alpha_1 = a, \alpha_2 = 1, \beta_1 = c \), the class \( M_{[\alpha]} (p; \beta; \lambda; A_1, B) \) coincides with the class studied in [10].

(ii) For \( p = 1, A = 1 - 2 \gamma, 0 < \gamma < 1, \beta = 0, B = -1, a = c = 1 \), the class \( M_{[\alpha]} (p; \beta; \lambda; A_1, B) \) coincides with the class studied in [17].

2. Preliminaries

We need the following lemmas for our future investigation.

Lemma 2.1 (Jack’s lemma [4]). Let the (non constant) function \( \omega (z) \) be analytic in \( U \), with \( \omega (0) = 0 \). If \( |\omega (z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \in U \), then \( z_0 \omega' (z_0) = \gamma \omega (z_0) \), where \( \gamma \) is real number and \( \gamma \geq 1 \).

Lemma 2.2 ([11]). Let \( \Omega \) be a set in the complex plane \( C \) and suppose that \( \phi \) is a complex mapping from \( C^2 \times U \) to \( C \) which satisfies \( \phi (ix, y; z) \notin \Omega \) for \( z \in U \), and for all real \( x, y \) such that \( y \leq -\frac{1 + x^2}{2} \). If the function \( p (z) = 1 + c_1 z + c_2 z^2 + \cdots \) is analytic in \( U \) and \( \phi (p(z), z p'(z); z) \in \Omega \) for all \( z \in U \), then \( \Re (p(z)) > 0 \).

Lemma 2.3 ([20]). Let \( p (z) = 1 + b_1 z + b_2 z^2 + \cdots \), be analytic in \( U \) and \( \eta \) be analytic and starlike (with respect to the origin) univalent in \( U \) with \( \eta (0) = 0 \). If \( z p' (z) < \eta (z) \) then

\[
p (z) < 1 + \int_{0}^{z} \frac{\eta (t)}{t} \, dt.
\]

Unless otherwise mentioned, we shall assume that \( 0 \leq \alpha < 1, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1, 0 \leq p \beta < 1 \) and \( p \in \mathbb{N} \).

3. Main results

Theorem 3.1. Let \( f \in \Sigma_p \). Then \( f \in M_{[\alpha]} (p; \alpha; \beta; \lambda; A, B) \) if and only if

\[
\frac{p}{1 - p \beta} \left\{ \frac{\mathcal{H}_{p, l, m} [\alpha] F_\lambda (z)}{z (\mathcal{H}_{p, l, m} [\alpha] F_\lambda (z))'} + \beta \right\} < \frac{-1 + A_1 z}{1 + B z}. \tag{3.1}
\]

Proof. If \( f \in M_{[\alpha]} (p; \alpha; \beta; \lambda; A, B) \), then

\[
\frac{p}{1 - p \beta} \left\{ \frac{(1 - \lambda) \mathcal{H}_{p, l, m} [\alpha] f (z) + \lambda z \mathcal{H}_{p, l, m} [\alpha] f (z)'}{z (\mathcal{H}_{p, l, m} [\alpha] f (z))'} + \beta \right\} < \frac{-1 + A_1 z}{1 + B z}. \tag{3.2}
\]

Let

\[
F_\lambda (z) = (1 - \lambda) f (z) + \lambda z f' (z),
\]

so

\[
\mathcal{H}_{p, l, m} [\alpha] F_\lambda (z) = (1 - \lambda) \mathcal{H}_{p, l, m} [\alpha] f (z) + \lambda z \mathcal{H}_{p, l, m} [\alpha] f' (z). \tag{3.3}
\]

Using (3.2), (3.3) and after some simplifications we have

\[
\frac{p}{1 - p \beta} \left\{ \frac{\mathcal{H}_{p, l, m} [\alpha] F_\lambda (z)}{z (\mathcal{H}_{p, l, m} [\alpha] F_\lambda (z))'} + \beta \right\} < \frac{-1 + A_1 z}{1 + B z},
\]

the converse is straight forward. \( \square \)
Theorem 3.2. If \( f \in \Sigma_p \) satisfies anyone of the following conditions

(i) for \( B \neq -1 \)
\[
\sum_{k=1}^{\infty} \left( \frac{|(k-p)\lambda_1|+\left|\frac{k\lambda_1}{1-p\beta}\right|}{(1-p\beta)(|A_1|-B)} \right) |\Gamma_k(\alpha_1)||a_k| \leq p|1-\lambda-p||1-|B||. \tag{3.4}
\]

(ii) for \( B = -1, \ A_1 \neq 1 \)
\[
\sum_{k=1}^{\infty} \left( \frac{|(1+(k-p)\beta)\lambda_1+(1-A_1)(1-p\beta)(k-p)\lambda_1|}{1-p\beta} \right) |\Gamma_k(\alpha_1)||a_k| \leq (1-p\beta)|1-A_1||1-\lambda+p|; \tag{3.5}
\]

(iii) for \( B = -1, \ A_1 = 1 \)
\[
\sum_{k=1}^{\infty} \left( \frac{|(k-p)\lambda_1|+\left|\frac{k\lambda_1}{1-p\beta}\right|}{(1-p\beta)} \right) |\Gamma_k(\alpha_1)||a_k| < p|\lambda_1|, \tag{3.6}
\]

then \( f \in M(\alpha_1)(p; \alpha; \beta; \lambda; A, B) \), where \( \lambda_1 = 1+\lambda(k-p-1) \) with \( \Gamma_k(\alpha_1) = \frac{\alpha_1 \cdots \alpha_k}{\Gamma(1-p\beta)_k \cdots \Gamma_m(1-p\beta)} \).

Proof. (i): If \( B \neq -1 \), by the condition (1.6) we only need to show that
\[
\left| \frac{p(1-B^2)}{(1-p\beta)(A_1-B)} \left\{ \frac{\mathcal{J}(p, \lambda_m, \alpha_1) F_{\lambda}(z)}{z[\mathcal{J}(p, \lambda_m, \alpha_1) F_{\lambda}(z)]'} + \frac{1-A_1B}{A_1-B} \right\} \right| < 1.
\]

We first observe the
\[
\left| \frac{p(1-B^2)}{(1-p\beta)(A_1-B)} \left\{ \frac{\mathcal{J}(p, \lambda_m, \alpha_1) F_{\lambda}(z)}{z[\mathcal{J}(p, \lambda_m, \alpha_1) F_{\lambda}(z)]'} + \frac{1-A_1B}{A_1-B} \right\} \right| = \left| \frac{pB(1-\lambda-p) + \sum_{k=1}^{\infty} \frac{p(1-B^2)[(1+(k-p)\beta)\lambda_1+(1-A_1)(1-p\beta)(k-p)\lambda_1]}{(1-p\beta)(|A_1|-B)} \Gamma_k(\alpha_1) a_k z^k}{-p(1-\lambda-p) + \sum_{k=1}^{\infty} \frac{p(1-B^2)[(1+(k-p)\beta)\lambda_1+(1-A_1)(1-p\beta)(k-p)\lambda_1]}{(1-p\beta)(|A_1|-B)} \Gamma_k(\alpha_1) a_k z^k} \right|
\]
\[
\leq \frac{p|B(1-\lambda-p)| + \sum_{k=1}^{\infty} \frac{|p(1-B^2)[(1+(k-p)\beta)\lambda_1+(1-A_1)(1-p\beta)(k-p)\lambda_1]|}{(1-p\beta)(|A_1|-B)} |\Gamma_k(\alpha_1)||a_k||z|^k}{p|1-\lambda-p| - \sum_{k=1}^{\infty} \frac{|(k-p)\lambda_1||\Gamma_k(\alpha_1)||a_k|}{|1-\lambda-p| - \sum_{k=1}^{\infty} |(k-p)\lambda_1||\Gamma_k(\alpha_1)||a_k|}}.
\tag{3.7}
\]

Now, by using the inequality (3.4), we have
\[
\frac{p|B(1-\lambda-p)| + \sum_{k=1}^{\infty} \frac{|p(1-B^2)[(1+(k-p)\beta)\lambda_1+(1-A_1)(1-p\beta)(k-p)\lambda_1]|}{(1-p\beta)(|A_1|-B)} |\Gamma_k(\alpha_1)||a_k|}{p|1-\lambda-p| - \sum_{k=1}^{\infty} |(k-p)\lambda_1||\Gamma_k(\alpha_1)||a_k|} < 1,
\]

which, in conjunction with (3.7), completes the proof of (i) for Theorem 3.2.
(ii): If $B = -1, A_1 \neq 1$, by the virtue of the condition (1.8), we only need to show that
\[
\left| \frac{(1 - A_1)(1 - p\beta)}{p} \left( \frac{z(\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z))'}{\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z) + \beta z(\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z))'} + 1 \right) \right| < 1.
\]
We first observe that
\[
\left| \frac{(1 - A_1)(1 - p\beta)}{p} \left( \frac{z(\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z))'}{\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z) + \beta z(\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z))'} + 1 \right) \right| = \left| A_1 (1 - p\beta)(1 - \lambda - \lambda p) + \sum_{k=1}^{\infty} \left( (1 + (k - p) \beta) \lambda_1 + \frac{(1 - A_1)(1 - p\beta)(k-p)\lambda_1}{p} \right) \Gamma_k(\alpha_1) a_k z^k \right| \leq \frac{(1 - p\beta)(1 - \lambda - \lambda p) + \sum_{k=1}^{\infty} |(1 + (k - p) \beta) \lambda_1| |\Gamma_k(\alpha_1)||a_k||z^k|}{(1 - p\beta)(1 - \lambda - \lambda p) - \sum_{k=1}^{\infty} |(1 + (k - p) \beta) \lambda_1| |\Gamma_k(\alpha_1)||a_k||z^k|} |A_1||1 - \lambda - \lambda p| (1 - p\beta) + \sum_{k=1}^{\infty} \left( (1 + (k - p) \beta) \lambda_1 + \frac{(1 - A_1)(1 - p\beta)(k-p)\lambda_1}{p} \right) \Gamma_k(\alpha_1)||a_k| \left| A_1||1 - \lambda - \lambda p| (1 - p\beta) + \sum_{k=1}^{\infty} \left( (1 + (k - p) \beta) \lambda_1 + \frac{(1 - A_1)(1 - p\beta)(k-p)\lambda_1}{p} \right) \Gamma_k(\alpha_1)||a_k| \left| A_1||1 - \lambda - \lambda p| (1 - p\beta) + \sum_{k=1}^{\infty} \left( (1 + (k - p) \beta) \lambda_1 + \frac{(1 - A_1)(1 - p\beta)(k-p)\lambda_1}{p} \right) \Gamma_k(\alpha_1)||a_k| \end{equation}

By using the inequality (3.5), we have
\[
\frac{p}{1 - p\beta} \left( \frac{\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z)}{z(\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z))'} + \beta \right) + 1 \right| < 1,
\]
which, in conjunction with (3.8), completes the proof of (ii) for Theorem 3.2.

(iii): If $B = 1, A_1 = 1$, by virtue of the condition (1.9), we only need to show that
\[
\left| \frac{p}{1 - p\beta} \left( \frac{\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z)}{z(\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z))'} + \beta \right) + 1 \right| < 1, \quad (z \in \mathbb{U}).
\]
We first observe that
\[
\left| \frac{p}{1 - p\beta} \left( \frac{\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z)}{z(\mathcal{H}_{p,1,m} [\alpha_1] F_{\lambda}(z))'} + \beta \right) + 1 \right| = \left| \sum_{k=1}^{\infty} \frac{k\lambda_1}{1 - p\beta} \Gamma_k(\alpha_1) a_k z^k \right| \leq \frac{\sum_{k=1}^{\infty} k\lambda_1 |\Gamma_k(\alpha_1)||a_k||z^k|}{p |1 - \lambda - \lambda p| - \sum_{k=1}^{\infty} |(k - p) \lambda_1| |\Gamma_k(\alpha_1)||a_k||z^k|} \leq \frac{\sum_{k=1}^{\infty} k\lambda_1 |\Gamma_k(\alpha_1)||a_k|}{p |1 - \lambda - \lambda p| - \sum_{k=1}^{\infty} |(k - p) \lambda_1| |\Gamma_k(\alpha_1)||a_k|} \end{equation}

Now, by using the inequality (3.6), we have
\[
\frac{\sum_{k=1}^{\infty} k\lambda_1 |\Gamma_k(\alpha_1)||a_k|}{p |1 - \lambda - \lambda p| - \sum_{k=1}^{\infty} |(k - p) \lambda_1| |\Gamma_k(\alpha_1)||a_k|} < 1,
\]
which, in conjunction with (3.9), completes the proof of (iii), for Theorem 3.2.

\[ \square \]

**Theorem 3.3.** If \( f \in \Sigma_p \) satisfies anyone of the following conditions:

(i) for \( B \neq -1 \),

\[
\left| L_{p,l,m}^{\alpha_i} (F(z)) \right| < \frac{(1-p\beta) (A_1 - B)}{(1-p\beta) (A_1 - B) + (1+|B|)};
\]

(ii) for \( B = -1 \), \(-1 < A_1 \leq 0\)

\[
\left| L_{p,l,m}^{\alpha_i} (F(z)) \right| < \frac{(1-p\beta) (1-A_1) (1+A_1)}{2p\beta (1+A_1) + 2 (1-A_1)};
\]

(iii) for \( B = -1 \), \( A_1 = 1 \)

\[
\left| L_{p,l,m}^{\alpha_i} (F(z)) \right| < \frac{1-p\beta}{2-p\beta},
\]

then \( f \in \mathcal{M}_{[\alpha_i]} p; \alpha; \beta; \lambda; A, B \), where

\[
L_{p,l,m}^{\alpha_i} (F(z)) = 1 + \frac{z F'(z)}{z F''(z)} - \frac{z F'(z)}{z F''(z)} - 1, \quad (z \in \mathbb{U}),
\]

Proof. (i) If \( B \neq -1 \), let

\[
\omega(z) = \frac{1 + \frac{1+|B|}{1+|B|+A_1-B} \cdot \frac{p}{1-p\beta} \left( \frac{H_{p,l,m}^{[\alpha_i]} F(z)}{z H_{p,l,m}^{[\alpha_i]} F(z)} + \beta \right)}{1 - \frac{1+|B|}{1+|B|+A_1-B} - 1}, \quad (z \in \mathbb{U}),
\]

then the function \( \omega(z) \) is analytic in \( \mathbb{U} \) with \( \omega(0) = 0 \). Using (3.11) and after some simplifications, we obtain

\[
\frac{p H_{p,l,m}^{[\alpha_i]} F(z)}{z H_{p,l,m}^{[\alpha_i]} F(z)} = \frac{(1-p\beta) (A_1 - B) \omega(z) - (1+|B|)}{1+|B|}.
\]

Differentiating both sides of (3.12), logarithmically we get

\[
L_{p,l,m}^{\alpha_i} (F(z)) = \frac{(1-p\beta) (A_1 - B) z \omega'(z)}{(1-p\beta) (A_1 - B) \omega(z) - (1+|B|)}.
\]

By virtue of (3.10) and (3.13), we find that

\[
\left| L_{p,l,m}^{\alpha_i} (F(z)) \right| = (1-p\beta) (A_1 - B) \left| \frac{z \omega'(z)}{(1-p\beta) (A_1 - B) \omega(z) - (1+|B|)} \right|,
\]

and

\[
\left| L_{p,l,m}^{\alpha_i} (F(z)) \right| < \frac{(1-p\beta) (A_1 - B)}{(1-p\beta) (A_1 - B) + (1+|B|)}.
\]

Next, we claim that \( |\omega(z)| < 1 \). Indeed, if not, there exists a point \( z_0 \in \mathbb{U} \) such that

\[
\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1, \quad (z_0 \in \mathbb{U}).
\]

Applying Lemma 2.1 to \( \omega(z) \) at the point \( z_0 \), we have

\[
z_0 \omega'(z_0) = \gamma \omega(z_0), \quad (\gamma \geq 1).
\]
By writing
\[ \omega (z_0) = e^{i\theta}, \quad (0 \leq \theta \leq 2\pi), \]
and setting \( z = z_0 \) in (3.13), we get
\[ |L_{p,l,m}^{\alpha_i} (F(z_0))| = (1 - p\beta) (A_1 - B) \left| \frac{\gamma}{(1 - p\beta) (A_1 - B) - (1 + |B|) e^{-i\theta}} \right|, \]
which implies
\[ |L_{p,l,m}^{\alpha_i} (F(z_0))| \geq (1 - p\beta) (A_1 - B) \left| \frac{1}{(1 - p\beta) (A_1 - B) - (1 + |B|) e^{-i\theta}} \right|. \]
This implies that
\[ \left| L_{p,l,m}^{\alpha_i} (F(z)) \right|^2 \geq \frac{[(1 - p\beta) (A_1 - B)]^2}{[(1 - p\beta) (A_1 - B)]^2 + (1 + |B|)^2 - 2 (1 - p\beta) (A_1 - B) (1 + |B|) \cos \theta}. \quad (3.14) \]
Since the right hand side of (3.14) takes its minimum value for \( \cos \theta = -1 \), we have
\[ \left| L_{p,l,m}^{\alpha_i} (F(z_0)) \right|^2 \geq \frac{[(1 - p\beta) (A_1 - B)]^2}{[(1 - p\beta) (A_1 - B) + (1 + |B|)]^2}. \]
This implies that
\[ \left| \frac{p}{1 - p\beta} \left( \frac{\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z)}{z(\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z))^r} + \beta \right) + 1 \right| < \frac{A_1 - B}{1 + |B|}, \]
then, we have
\[ \left| \frac{p}{1 - p\beta} \left( \frac{\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z)}{z(\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z))^r} + \beta \right) + 1 \right| < \frac{A_1 - B}{1 + |B|} + \frac{|B| (A_1 - B)}{1 - B^2} = \frac{A_1 - B}{1 - B^2}, \quad (B \neq -1). \]
Therefore, we conclude that \( f(z) \in M_{(\alpha_1)} (p; \alpha; \beta; \lambda; A, B) \) for \( B \neq -1 \).

Using similar arguments as in proof of (i), (ii) and (iii) can be easily verified. \( \square \)

**Theorem 3.4.** If \( f \in \Sigma_p \) satisfies
\[ \Re \left( L_{p,l,m}^{\alpha_i} (F(z)) \right) < \left\{ \begin{array}{ll} \frac{\beta_2}{2(1 - p\beta)(A_1 - B)^2} & , \quad B + \frac{1 - B}{2(1 - p\beta)} \leq A_1 \leq 1, \\ \frac{1 - B}{2(1 - p\beta)} & , \quad B < A_1 < B + \frac{1 - B}{2(1 - p\beta)}, \end{array} \right. \quad (3.15) \]
then \( f(z) \in M_{(\alpha_1)} (p; \alpha; \beta; \lambda; A, B) \), where \( \beta_2 = (1 - A_1) + p\beta (A_1 - B) \).

**Proof.** Let
\[ g(z) = \frac{-p}{1 - p\beta} \left( \frac{\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z)}{z(\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z))^r} + \beta \right) - \frac{1 - A_1}{1 - B}. \quad (3.16) \]
Then \( g \) is analytic in \( \mathbb{U} \). Using (3.16), we have
\[ \frac{-p\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z)}{z(\mathcal{K}_{p,l,m}^{\alpha_i} \{ \alpha_1 \} F\lambda (z))^r} = \frac{(1 - p\beta) (A_1 - B) g(z) + \beta_2}{1 - B} = \frac{\beta_2}{1 - B}. \quad (3.17) \]
Differentiating (3.17) logarithmically, we obtain

\[-L_{p,l,m}^{\alpha_1} (F(z)) = \frac{(1 - p\beta) (A_1 - B) zg'(z)}{(1 - p\beta) (A_1 - B) g(z) + \beta_2} = \langle g(z), zg'(z); z \rangle,
\]

where

\[\langle r, s; t \rangle = \frac{(1 - p\beta) (A_1 - B) s}{(1 - p\beta) (A_1 - B) r + \beta_2}.
\]

For all real \(x\) and \(y\) satisfying \(y \leq \frac{-1 + x^2}{2}\), we have

\[\Re \left( \langle ix, y; z \rangle \right) = \frac{(1 - p\beta) (A_1 - B) \beta_2 y}{(\beta_2)^2 + [(1 - p\beta) (A_1 - B)]^2 x^2} \leq -\frac{1 + x^2}{2} \frac{(1 - p\beta) (A_1 - B) \beta_2}{(\beta_2)^2 + [(1 - p\beta) (A_1 - B)]^2 x^2} \leq \left\{ \begin{array}{l}
\frac{\beta_2}{2(1 - p\beta)(A_1 - B)}, \quad B + \frac{1 - B}{2(1 - p\beta)} \leq A_1 \leq 1, \\
\frac{\beta_2}{2(1 - p\beta)(A_1 - B)}, \quad B < A_1 \leq B + \frac{1 - B}{2(1 - p\beta)}.
\end{array} \right.
\]

We know put

\[\Omega = \left\{ \xi : \Re(\xi) > \left\{ \begin{array}{l}
\frac{\beta_2}{2(1 - p\beta)(A_1 - B)}, \quad B + \frac{1 - B}{2(1 - p\beta)} \leq A_1 \leq 1, \\
\frac{\beta_2}{2(1 - p\beta)(A_1 - B)}, \quad B < A_1 \leq B + \frac{1 - B}{2(1 - p\beta)}.
\end{array} \right. \}
\]

then \(\langle ix, y; z \rangle \notin \Omega\) for all real \(x, y\) such that \(y \leq \frac{-1 + x^2}{2}\). Moreover, in view of (3.15), we know that

\[\langle g(z), zg'(z); z \rangle \in \Omega. \]

Thus by Lemma 2.2 we deduce that

\[\Re \left( \langle g(z) \rangle \right) > 0, \quad (z \in \mathbb{U}),\]

which shows that the desired assertion of Theorem 3.4 holds.

\[\Box\]

**Theorem 3.5.** If \(f \in \Sigma_p\) satisfies

\[\Re \left\{ \frac{pJ_{p,l,m}(\alpha_1) F_{\lambda}(z)}{z(\mathcal{H}_{p,l,m}(\alpha_1) F_{\lambda}(z))'} \left( 1 + \eta \frac{z(\mathcal{H}_{p,l,m}(\alpha_1) F_{\lambda}(z))''}{z(\mathcal{H}_{p,l,m}(\alpha_1) F_{\lambda}(z))'} \right) \right\} > \frac{1}{2} \delta_1 \eta + p\eta - (1 - \eta) \frac{\beta_2}{1 - B},\]

then \(f \in \mathcal{M}_{\langle \alpha_1 \rangle}(\rho; \alpha; \beta; \lambda; A, B)\) for \(\eta \geq 0\), where \(\delta_1 = (1 - p\beta) \left( \frac{A_1 - B}{1 - B} \right)\).

**Proof.** Let

\[h(z) = \frac{-p}{1 - p\beta} \left\{ \frac{J_{p,l,m}(\alpha_1) F_{\lambda}(z)}{z(\mathcal{H}_{p,l,m}(\alpha_1) F_{\lambda}(z))'} + \frac{1 - A_1}{1 - B} \right\}.
\]

Then \(h\) is analytic in \(\mathbb{U}\). It follows from (3.18) that

\[-pJ_{p,l,m}(\alpha_1) F_{\lambda}(z) \frac{1 - p\beta}{z(\mathcal{H}_{p,l,m}(\alpha_1) F_{\lambda}(z))'} = (1 - p\beta) \left( A_1 - B \right) h(z) + \beta_2,
\]

and

\[1 + \eta \frac{z(\mathcal{H}_{p,l,m}(\alpha_1) F_{\lambda}(z))''}{z(\mathcal{H}_{p,l,m}(\alpha_1) F_{\lambda}(z))'} = \frac{P + Qh(z) + Rzh'(z)}{(1 - p\beta) \left( A_1 - B \right) h(z) + \beta_2},
\]

where

\[P = -p\eta \left( 1 - B \right) + (1 - \eta) \left[ (1 - A_1) + p\beta \left( A_1 - B \right) \right],\]
Then, combining (3.19) and (3.20), we get

$$Q = (1 - p\beta) (A_1 - B) (1 - \eta), \quad R = -(1 - p\beta) (A_1 - B) \eta,$$

where

$$z(\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z))' = -p\eta + (1 - \eta) \frac{\beta_2}{(1 - B)} + \delta_1 (1 - \eta) h(z) - \delta_1 \eta z h'(z) = \phi(h(z), z h'(z); z),$$

Rest of the proof follows by working in similar way as in Theorem 3.4.

**Theorem 3.6.** If $f \in \Sigma_p$ satisfies anyone of the following conditions:

(i) for $B \neq -1$,

$$\left| \left\{ \frac{p}{1 - B^2} \left( \frac{\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z)}{z(\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z))'} + \beta + 1 - A_1 B \right) \right\}' \right| \leq \eta |z|^\tau;$$

(ii) for $B = -1$, $A_1 \neq 1$,

$$\left| \left\{ \frac{p}{1 - \beta} \left( \frac{z(\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z))'}{\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z) + \beta z(\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z))'} \right)' \right\}' \right| \leq \eta |z|^\tau;$$

(iii) for $B = -1$, $A_1 = 1$,

$$\left| \left\{ \frac{p}{1 - \beta} \left( \frac{\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z)}{z(\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z))'} + \beta + 1 \right) \right\}' \right| \leq \eta |z|^\tau,$$

then $f \in M_{\{\alpha_1\}} (p; \alpha; \beta; \lambda; A, B)$, for $0 < \eta \leq \tau + 1$ and $\tau \geq 0$.

**Proof.** (i): If $B \neq -1$, we define the function $\Psi(z)$ by

$$\Psi(z) = z \left\{ \frac{p}{1 - B^2} \left( \frac{\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z)}{z(\mathcal{J}_{p,l,m}[\alpha_1] F_{\lambda}(z))'} + \beta + 1 - A_1 B \right) \right\}' \leq \eta |z|^\tau;$$

then $\Psi(z)$ is regular in $U$ and $\Psi(0) = 0$. The condition of theorem gives us that

$$\left| \left( \frac{\Psi(z)}{z} \right) '' \right| \leq \eta |z|^\tau.$$

It follows that

$$\left| \left( \frac{\Psi(z)}{z} \right) ' \right| \leq \left( \int_0^z \frac{\Psi(t)}{t} \, dt \right) ' \leq \eta |z|^\tau \, dt \leq \frac{\eta}{\tau + 1} |z|^\tau + 1.$$

This implies that

$$\left| \left( \frac{\Psi(z)}{z} \right) ' \right| \leq \frac{\eta}{\tau + 1} |z|^\tau + 1 < 1, \quad (0 < \eta \leq \tau + 1, \quad \tau \geq 0).$$
Therefore, by the definition of \( \Psi(z) \), we conclude that
\[
\left| \frac{p(1 - B^2)}{(1 - p\beta)(A_1 - B)} \left( \frac{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)}{z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]'} + \beta \right) + \frac{1 - A_1 B}{A_1 - B} \right| < 1,
\]
which is equivalent to
\[
\left| \frac{p}{(1 - p\beta)} \left( \frac{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)}{z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]'} + \beta \right) + \frac{1 - A_1 B}{A_1 - B} \right| < \frac{A_1 - B}{1 - B^2}.
\]
Therefore, we conclude that \( f(z) \in \mathcal{M}_{[\alpha]}(p; \alpha; \beta; \lambda; A, B) \).

(ii): If \( B = -1, \ A_1 \neq 1 \), we define the function
\[
\Psi(z) = z \left( 1 + \frac{(1 - A_1)(1 - p\beta)}{p} \left( \frac{z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]'}{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)} + \beta \frac{z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]'}{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)} \right) \right),
\]
then \( \Psi(z) \) is regular in \( \mathcal{U} \) and \( \Psi(0) = 0 \).

Using similar arguments as in proof of (i) and (ii), condition (iii) can be easily verified. \( \square \)

**Theorem 3.7.** If \( f \in \Sigma_p \) satisfies
\[
\left| \frac{1 - p\beta z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]'}{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)} + \beta z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]' \right| \left( 1 + \mathcal{M}_{p,l,m}^\alpha(F(z)) \right) \left| < \frac{A_1 - B}{1 - A_1} \right|
\]
then \( f \in \mathcal{M}_{[\alpha]}(p; \alpha; \beta; \lambda; A, B) \), for \( -1 \leq B < A_1 < \frac{1 + B}{2} \), where
\[
\mathcal{M}_{p,l,m}^\alpha(F(z)) = \frac{z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]''}{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)} - \frac{z \left( \frac{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)}{z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]'} + \beta \right)}{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)}. \tag{3.21}
\]

**Proof.** Let
\[
q(z) = \frac{-p}{1 - p\beta} \left\{ \frac{\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)}{z[\mathcal{H}_{p,l,m}[\alpha_1] F_\lambda(z)]'} + \beta \right\}.
\]
Then \( q(z) \) is analytic in \( \mathcal{U} \). The condition of theorem gives us that
\[
\left| z \left( \frac{1}{q(z)} \right)' \right| < \frac{A_1 - B}{1 - A_1},
\]
that is,
\[
\left| z \left( \frac{1}{q(z)} \right)' \right| < \frac{A_1 - B}{1 - A_1} z. \tag{3.22}
\]
An application of Lemma 2.3 to (3.22) yields
\[
q(z) < \frac{1 - A_1}{1 - A_1 + (A_1 - B)z} = F(z). \tag{3.23}
\]
By noting that
\[
\Re \left( 1 + \frac{zF''(z)}{F'(z)} \right) = \Re \left( \frac{1 - A_1 - (A_1 - B)z}{1 - A_1 + (A_1 - B)z} \right) \geq \frac{1 - A_1 - (A_1 - B)}{1 - A_1 + (A_1 - B)} > 0 \left(-1 \leq B < A_1 < \frac{1 + B}{2}\right),
\]
which implies that the region \( F(\mathcal{U}) \) is symmetric with respect to the real axis and \( F \) is convex univalent in...
which is equivalent to

\[
\Re \left\{ \frac{p}{1-p\beta} \left( \frac{\mathcal{H}_{p,1,m}[\alpha_1] F_\lambda(z)}{z(\mathcal{H}_{p,1,m}[\alpha_1] F_\lambda(z))'} + \beta \right) \right\} < -\frac{1 - A_1}{1 - B_1},
\]

(3.24)

Combining (3.21), (3.23) and (3.24), we deduce that for \((-1 \leq B < A_1 < \frac{1+B}{2})\),

\[
\Re \left\{ \frac{p}{1-p\beta} \left( \frac{\mathcal{H}_{p,1,m}[\alpha_1] F_\lambda(z)}{z(\mathcal{H}_{p,1,m}[\alpha_1] F_\lambda(z))'} + \beta \right) \right\} < -\frac{1 - A_1}{1 - B_1},
\]

which is equivalent to

\[
\frac{p}{1-p\beta} \left( \frac{\mathcal{H}_{p,1,m}[\alpha_1] F_\lambda(z)}{z(\mathcal{H}_{p,1,m}[\alpha_1] F_\lambda(z))'} + \beta \right) < \frac{1 + A_1 z}{1 + B z}.
\]

This evidently completes the proof of Theorem 3.7. \(\square\)

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References