Convergence of numerical solutions for a class of stochastic age-dependent capital system with fractional Brownian motion

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Communicated by A. Atangana

Abstract

In this paper, we consider a class of stochastic age-dependent capital system with fractional Brownian motion, and investigate the convergence of numerical approximate solution. It is proved that the numerical approximation solutions converge to the analytic solutions of the equations under given conditions. A numerical example is provided to illustrate the theoretical results. ©2017 All rights reserved.

Keywords: Stochastic age-dependent capital system, numerical solution, Euler approximation, fractional Brownian motion.

1. Introduction

Stochastic partial differential equations have been widely used to model the phenomena arising in many fields of science and industry such as finance, economics, biology and medicine, see e.g., [4, 6, 12, 13, 19, 21] and references therein. Recently, the numerical construction of stochastic age-dependent (vintage) capital system with standard Brownian motion has captured many researchers’ attention. For example, Zhang et al. [21, 22] discussed the exponential stability of numerical solutions for the stochastic age-dependent capital system with Poisson jumps in the case of deterministic magnitude, and further studied the convergence of numerical solutions for a class of stochastic age-dependent capital system with random jump magnitudes. In their subsequent work [20], they constructed a split-step backward Euler method for stochastic age-dependent capital system with Markovian switching and proved that, under the one-sided local Lipschitz condition on the drift and local Lipschitz condition on the diffusion, the split-step backward Euler method converges with strong order of one half to the true solution. Du et al. [2] discussed the convergence of a semi-implicit Euler method for stochastic age-dependent capital system with variable delays and random jump magnitudes. In these studies, uncertainties in the financial market (i.e., randomness of the external environment) were considered in their models in the
form of a standard Brownian motion. However, the randomness of the external environment is not always well modeled by the standard Brownian motion (e.g., because of the long-range dependence or long memory of the processes under consideration). In the last few years, some articles have been published choosing fractional Brownian motion (fBm) as an underlying diffusive process (e.g., Refs. [7–9, 14–17] and references therein). For example, Jiang et al. [9] proposed a class of stochastic heat equations with first order fractional noises and established the existence and uniqueness of the solution of the equation. Wang et al. [15] studied the problem of continuous time option pricing with transaction costs by using the homogeneous subdiffusive fBm as a model of asset prices. Xiao et al. [17] presented a pricing model for equity warrants in a mixed fractional Brownian environment and proposed a hybrid intelligent algorithm to solve the nonlinear optimization problem. Jańczak-Borkowska [8] investigated the existence and uniqueness of generalized backward stochastic differential equation driven by fBm with Hurst parameter H greater than 1/2, and shown the connection between this solution and the solution of the parabolic partial differential equation with Neumann boundary condition. Rostek and Schöbel [14] gave a note on the use of fBm for financial modeling and clarified that fBm is suited for economic modeling. In this paper, motivated by [14], we consider a class of stochastic age-dependent capital system with fBm as following

\[
\begin{align*}
\frac{\partial K(a,t)}{\partial t} + \frac{\partial K(a,t)}{\partial a} &= -\mu(a,t)K(a,t) + f(t, K(a,t)) + g(t, K(a,t)) \frac{dB_H^t}{dt}, \quad \text{in} \mathcal{Q}, \\
N(t) &= \int_0^A K(a, t)da, \\
K(0, t) &= \phi(t) = \gamma(t)A(t)F(L(t), N(t)), \\
K(a, 0) &= K_0(a),
\end{align*}
\]

where \( \mathcal{Q} = (0, A) \times (0, T) \), \( K(a, t) \) is the stock of capital goods of age \( a \) at time \( t \), and \( N(t) \) is the total sum of the capital. The investment \( \phi(t) \) in the new capital and \( f(t, K(a, t)) \) in the capital of age \( a \) are the endogenous (unknown) variables. The maximum physical lifetime of capital \( A \), the planning interval of calendar time \([0, T]\), the depreciation rate \( \mu(a, t) \) of capital, and the capital density \( K_0(a) \) (the initial distribution of capital over age) are given. \( \gamma(t) \in (0, 1) \) and \( A(t) \) denote the accumulative rate and the technical progress at the moment of \( t \), respectively. \( F(L(t), N(t)) \) is the production function and \( L(t) \) is the labor force. \( f(t, K(a, t)) + g(t, K(a, t)) \frac{dB_H^t}{dt} \) denotes effects of the external environment for the system, such as innovations in techniques, introduction of new products, natural disasters and changes in laws and government policies and so on. The effects of external environment include the deterministic and the random parts which depend on \( t \) and \( K(a, t) \). Here \( B_H^t \) is standard fBm with Hurst parameter \( H \in (\frac{1}{2}, 1) \). Fractional Brownian motion has \( H \)-self-similarity, in addition it has long-range dependence when \( H > \frac{1}{2} \) and short-range dependence when \( H < \frac{1}{2} \). When \( H = \frac{1}{4} \), the corresponding fBm is a standard Brownian motion and has independent increments. The standard Brownian motion is a semimartingales if and only if \( H = \frac{1}{2} \), thus Itô stochastic integration theory for semimartingales cannot be applied if \( H \neq \frac{1}{2} \).

The system (1.1) describes the evolution of the composition of the productive capital as a function of purchasing/selling new or used capital. When \( g \equiv 0 \) (i.e., without stochastically perturbed), \( K(a, t) \) denotes the riskless capital and the system becomes to a generalization of the deterministic age-dependent capital system, which has been studied by many authors. For instance, Feichtinger et al. [4, 5] established deterministic (vintage) capital dynamics and the necessary optimality conditions, and developed the vintage capital stock model with technological progress while the model is solved. Goetz et al. [6] studied the capital replacement decision of a firm as a distributed investment and disinvestment optimal control problem. Due to the nonlinear feature of stochastic system with fBm, and its difficulty to bring out explicit solutions, constructing the efficient computational methods becomes extremely important. Thus numerical approximation schemes are invaluable tools for exploring their properties. Numerical solution of stochastic differential equations (SDEs) driven by Brownian motion has been studied by many researchers. For instance, Kloeden et al. [10] have given some useful results in their monograph, which contains an almost complete survey of the theory of numerical solution of such SDEs with regular coefficients. Zhang [18] discussed the existence and uniqueness for stochastic age-dependent population equation with dif-
such that Eq. (1.1) holds.

2. Preliminaries and approximation

Definition 2.1 (fractional Brownian motion (fBm)). Let $B_t^H = B_t^H(t), t \in \mathbb{R}$ be a stochastic process, and $0 < H < 1$. $B_t^H$ is called a fBm with Hurst parameter $H$, if it is a centered Gaussian process, which starts at zero ($B_0^H = 0$), has expectation zero for all $t$, i.e., $E[|B_t^H|] = 0$, and has the covariance function

$$E[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad t, s \in \mathbb{R}.$$ 

Let

$$V = H^1([0, A]) \equiv \{ \varphi | \varphi \in \mathbb{L}^2([0, A]), \frac{\partial \varphi}{\partial x_i} \in \mathbb{L}^2([0, A]) \}, \quad \frac{\partial \varphi}{\partial x_i} \text{ are generalized partial derivatives}.$$ 

$V$ is a Sobolev space. $H = \mathbb{L}^2([0, A])$ such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

Then $V' = H^{-1}([0, A])$ is the dual space of $V$. We denote by $| \cdot |$ and $\| \cdot \|$ the norms in $V$ and $V'$, respectively, by $\langle \cdot, \cdot \rangle$ the duality product between $V$, $V'$, and by $(\cdot, \cdot)$ the scalar product in $H$. $m$ is a constant such that

$$\|x\| \leq m \|x\|, \quad \forall x \in V.$$ 

Throughout this paper, $f(t, \cdot) : V \rightarrow V'$ is a family of nonlinear operators a.e. $t$ for all $t$. $g(t, \cdot) : V \rightarrow \mathcal{L}(K, H)$ is the family of all bounded linear operators from $K$ into $H$. Let $C = C([0, T]; H)$ be the space of all continuous functions from $[0, T]$ into $H$ with sup-norm $\|\varphi\|_C = \sup_{0 \leq s \leq T} \|\varphi(s)\|$, $L^p_V = L^p([0, T]; V)$ and $L^p_H = L^p([0, T]; H)$. We assume that there exists a unique process $K(a, t) \in L^p([0, T]; V) \cap \mathbb{L}^2 = L^p(\omega, C([0, T]; H))$ such that Eq. (1.1) holds.

The integral version of Eq. (1.1) is given as follows

$$K_t = K_0 - \int_0^t \frac{\partial K_s}{\partial a} ds - \int_0^t \mu(a, s) ds + \int_0^t f(s, K_s) ds + \int_0^t g(s, K_s) dB^H_s, \quad t \in [0, T],$$

$$K(0, t) = \phi(t) = \gamma(t)A(t)F(L(t), \int_0^a K(a, t) da), \quad t \in [0, T] \tag{2.1}$$

where $K_t = K(a, t)$. For system (1.1) the discrete Euler approximate solution on $t = 0, h, 2h, \cdots, Nh$ is defined by the following iterative scheme

$$Q_t^{k+1} = Q_t^k - \frac{\partial Q_t^k}{\partial a} h - \mu(a, t_k)Q_t^k h + f(t_k, Q_t^k) h + g(t_k, Q_t^k) \Delta B^H_t, \quad t \in [0, T],$$

with initial value $Q_0^k \equiv K(a, 0), Q^k(0, t) \equiv (t)A(t)F(L(t), \int_0^a Q_t^k da)$. Where $Q_t^k$ is an approximation to $K(a, t_k)$ for $t_k = kh$, $h = \Delta t = T/N \ll 1$ is the time increment, and $\Delta B^H_t = B^H_{t+\Delta t} - B^H_t$ is fBm increment.
For convenience, we shall extend the discrete numerical solution (2.2) to continuous time,

\[ Q_t = K_0 - \int_0^t \frac{\partial z_s}{\partial a} ds - \int_0^t \mu(a,s)Z_s ds + \int_0^t f(t,Z_s) ds + \int_0^t g(s,Z_s) dB_s^H, \quad (2.3) \]

where we have defined the piecewise constant function

\[ Z_t = Z(a,t) \equiv \sum_{k=0}^{N-1} Q^k_t I_{[kh,(k+\gamma)h]}(t) \]

and \( I_G \) is the indicator function for the set \( G \). It is straightforward to check that \( Z(a,t_k) = Q^k_t = Q(a,t_k) \). For convenience, the expression (2.3) may also be expressed in the stochastic differential form

\[ dQ_t = -\frac{\partial Z_t}{\partial a} dt - \mu(a,t)Z_t dt + f(t,Z_t) dt + g(t,Z_t) dB_t^H. \quad (2.4) \]

Let \( G \) be an open subset of \( V \), and \( D \subseteq G \) be any compact set. We denote the unique solution of Eq. (1.1) for all \( t \in [0,T] \) by \( K[a,t] \in G \) with initial condition \( K_0 \in G \), and the Euler approximation solution to Eq. (1.1) by \( Q^k_t \).

As the standing hypotheses we always assume that the following conditions are satisfied:

(a) \( \mu(a,t) \) is non-negative measurable in \( Q \), and \( \gamma(t) \) and \( \Lambda(t) \) are nonnegative continuous in \([0,T]\) such that

\[ \left\{ \begin{array}{l}
0 \leq \mu_0 \leq \mu(a,t) \leq \bar{\mu} < \infty, \\
\gamma(t)\Lambda(t) \leq \eta, \quad \eta \text{ is a non-negative constant, in } [0,T],
\end{array} \right. \]

where \( \int_0^1 \mu(a,t) da = +\infty \). \( F(L,N) \) is satisfied with

\[ \left\{ \begin{array}{l}
F(L,N) \geq 0 (F(L,0) = 0), \quad \frac{\partial F}{\partial N} > 0, \\
0 < \frac{\partial F}{\partial N} < F_1, \text{where } F_1 \text{ is a positive constant.}
\end{array} \right. \]

(b) (Lipschitz condition). There exists a positive constant \( m_1(D) \) for all \( K_1, K_2 \in D \) such that

\[ |f(t,K_2) - f(t,K_1)|^2 + \|g(t,K_2) - g(t,K_1)\|_2 \leq m_1(D)|K_2 - K_1|^2. \]

(c) There exists a \( C^{1,2} \)-positive function \( V(t,\cdot) : G \to \mathbb{R}_+ \), such that \( K \in G : V(t,K) \leq r \) is compact for any \( r > 0 \).

(d) Let \( \psi_1(t), \psi_2(t) \) be two continuous non-negative functions, and there exists a positive constant \( m_2(D) \). Assume that for all \( K \in G \),

\[ LV(t,K) \leq m_2(D) + \psi_1(t) + \psi_2(t) V(t,K), \]

where

\[ \int_0^T LV(t,K) dt = \int_0^T \left( V(t,K) + \left( V_K(t,K) - \frac{\partial K}{\partial a} - \mu(a,t)K + f(t,K) \right) \right) dt \\
+ \frac{1}{2} \int_0^T t^{2H-1} V_{KK}(t,K) \| g(t,K) \|_2^2 dt. \]

(e) There exists a positive constant \( m_3(D) \) such that for all \( K_1, K_2 \in D \)

\[ |V(t,K_1) - V(t,K_2)| \vee |V(t,K_1) - V(t,K_2)| \vee |V(t,K_1) - V(t,K_2)| \vee |V(t,K_1) - V(t,K_2)| \leq m_3(D)|K_1 - K_2|. \]

**Remark 2.2.** When \( H = \frac{1}{2} \), the fBm is a standard Brownian motion.
If condition (b) holds, there exists a positive constant $m_4(\mathcal{D})$ such that
\[
|f(t, K)|^2 \vee \|g(t, K)\|^2 \leq m_4(\mathcal{D}) \tag{2.5}
\]
for all $K \in \mathcal{D}$. Since $K(a)$ is bounded, there exists a positive constant $m_5(\mathcal{D})$ such that
\[
|K(a, t)|^2 \leq m_5(\mathcal{D}). \tag{2.6}
\]
Since $K(a, t) \in V = H^1([0, \Lambda]), \frac{\partial}{\partial a} : V \to V'$ is a family of linear operators, then there also exists a positive constant $m_6(\mathcal{D})$ such that
\[
\left|\frac{\partial K}{\partial a}\right|^2 \leq m_6(\mathcal{D}) \tag{2.7}
\]
for $K \in \mathcal{D}$.

3. Main results

In this section, we will give and prove the following main results that the numerical solutions converge to the true ones of the stochastic age-dependent capital equations with respect to fBm under the given conditions.

**Theorem 3.1.** If $\tau$ is the first exit time of the solution $K(a, t)$ or the Euler approximate solution $Q_t$ in a bounded region $\mathcal{D}$, and for all $a \in (0, \Lambda)$, $\sup_{t \in \mathcal{D}} |\frac{Q_t - K[a, t]}{Z_t - K[a, t]}|$ exists, and $f(t, K)$ and $g(t, K)$ satisfy condition (b), then for $\Delta t T < 1$
\[
E \left[ \sup_{0 \leq t \leq \tau \wedge T} |Q_t - K(a, t)|^2 \right] \leq C_1(\mathcal{D}) e^{C_2(\mathcal{D})^T} \Delta t = C(\mathcal{D}) \Delta t,
\]
where
\[
C_1(\mathcal{D}) = 8 \left\{ [\alpha F^2_1 \eta^2 M(\mathcal{D}) + \mu_0 + m_1(\mathcal{D})(1 + 2HT^2_t)] \ast \left[ m_6(\mathcal{D}) + \mu_0^2 m_5(\mathcal{D}) + m_4(\mathcal{D}) + m_4(\mathcal{D}) T^2_1 \right] 
+ m_4^2(\mathcal{D}) m_5^2(\mathcal{D}) L^2_1 \right\}
\]
\[
C_2(\mathcal{D}) = 2\alpha F^2_1 \eta^2 M(\mathcal{D}) + 3\mu_0 + 1 + 2m_1(\mathcal{D})(1 + 2HT^2_t).
\]

Thus, the Euler approximation solution $Q_t$ converges to the unique exact solution $K(a, t)$ when $\Delta t \to 0$ as long as $Q_t$ and $K(a, t)$ remain within the domain $\mathcal{D}$.

**Proof.** We consider one and only trajectories $K(a, t)$ and $Q_t$ which remain within a bounded domain $\mathcal{D}$. Firstly, we introduce the stopping time $\tau = \rho \wedge \theta$ where
\[
\rho = \inf\{t \geq 0 : Q_t \not\in \mathcal{D}\} \text{ and } \theta = \inf\{t \geq 0 : K(a, t) \not\in \mathcal{D}\}
\]
are the first exit time that $Q_t$ and $K(a, t)$ leave from the domain $\mathcal{D}$, respectively.

The bounded domain $\mathcal{D}$ is defined as
\[
\mathcal{D} = \mathcal{D}(r) \equiv \{ K \in \mathcal{G} \text{ such that } V(t, K) \leq r \}.
\]

Applying Itô formula to $|Q_t - K(a, t)|^2$ yields
\[
|Q_t - K(a, t)|^2 = 2 \int_0^t \left\langle -\frac{\partial Z_s}{\partial a} + \frac{\partial K(a, s)}{\partial a} - \mu(a, s)(Z_s - K(a, s)), Q_s - K(a, s) \right\rangle ds
+ 2 \int_0^t (f(s, Z_s) - f(s, K(a, s)), Q_s - K(a, s)) ds + 2H \int_0^t s^{2H-1} \|g(s, Z_s) - g(s, K(a, s))\|^2 ds.
\]
\[\begin{align*}
+2 \int_0^t (Q_s - K(a, s), \mathcal{G}(s, Z_s) - g(s, K(a, s))) dB^H_s \\
\leq -2 \int_0^t \left( \frac{\partial(Z_s - K(a, s))}{\partial a}, Q_s - K(a, s) \right) ds - 2\mu_0 \int_0^t (Z_s - K(a, s), Q_s - K(a, s)) ds \\
+2 \int_0^t (f(s, Z_s) - f(s, K(a, t)), Q_s - K(a, s)) ds + 2H \int_0^t s^{2H-1} \|g(s, Z_s) - g(s, K(a, s))\|_2^2 ds \\
+2 \int_0^t (Q_s - K(a, s), g(s, Z_s) - g(s, K(a, s))) dB^H_s.
\end{align*}\]

Since
\[\begin{align*}
&- \left( \frac{\partial(Z_t - K(a, t))}{\partial a}, Q_t - K(a, t) \right) \\
&= - \int_0^A \frac{\partial(Z_t - K(a, t))}{\partial a} (Q_t - K(a, t)) da \\
&= - \int_0^A \frac{\partial(Z_t - K(a, t))}{\partial a} (Z_t - K(a, t)) \frac{Q_t - K(a, t)}{Z_t - K(a, t)} \frac{Q_t - K(a, t)}{Z_t - K(a, t)} da \\
&= - \int_0^A \frac{Q_t - K(a, t)}{Z_t - K(a, t)} \frac{\partial(Z_t - K(a, t))}{\partial a} da (Z_t - K(a, t)) \\
&\leq \frac{1}{2} \gamma^2(s) A^2(s) \left( F(L(s), \int_0^A Z_t - K(a, t) da) - F(L(s), 0) \right)^2 \sup_{t \in \mathcal{D}} \left| \frac{Q_t - K(a, t)}{Z_t - K(a, t)} \right| da \\
&\leq \frac{1}{2} M(\mathcal{D}) \eta^2 \left( \frac{\partial F(L, N)}{\partial N} \right)^2 \left( \int_0^A Z_t - K(a, t) da \right) \\
&\leq \frac{1}{2} A F_L^2 \eta^2 M(\mathcal{D}) |Z_t - K(a, t)|^2,
\end{align*}\]

where \( M(\mathcal{D}) = \sup_{t \in \mathcal{D}} \left| \frac{Q_t - K(a, t)}{Z_t - K(a, t)} \right| \) for all \( a \), therefore, we get that
\[|Q_t - K(a, t)|^2 \leq A F_L^2 \eta^2 M(\mathcal{D}) |Z_t - K(a, t)|^2 \]
\[+ 2\mu_0 \int_0^t |Z_s - K(a, s)||Q_s - K(a, s)||f(s, Z_s) - f(s, K(a, t))||ds \\
+ 2H \int_0^t s^{2H-1} \|g(s, Z_s) - g(s, K(a, s))\|_2^2 ds + 2 \int_0^t (Q_s - K(a, s), g(s, Z_s) - g(s, K(a, s))) dB^H_s \\
\leq A F_L^2 \eta^2 M(\mathcal{D}) |Z_t - K(a, t)|^2 + \mu_0 \int_0^t |Z_s - K(a, s)|^2 + \mu_0 \int_0^t |Q_s - K(a, s)|^2 ds \\
+ \int_0^t |Q_s - K(a, s)|^2 + \int_0^t |f(s, Z_s) - f(s, K(a, t))|^2 ds + 2H \int_0^t s^{2H-1} \|g(s, Z_s) \\
- g(s, K(a, s))\|_2^2 ds + 2 \int_0^t (Q_s - K(a, s), g(s, Z_s) - g(s, K(a, s))) dB^H_s.
\]

Since the coefficients of Eq. (1.1) are Lipschitz continuous and \( K(a, s) \) and \( Q_s \) are bounded, we get that
\[|f(s, Z_s) - f(s, K(a, s))|^2 \vee \|g(s, Z_s) - g(s, K(a, s))\|_2^2 \leq m_1(\mathcal{D}) |Z_s - K(a, s)|^2 \]
for \( s \in [0, \tau \wedge T_1] \). Let \( T_1 \in [0, T] \) be an arbitrary time. For any \( t \in [0, \tau \wedge T_1] \), we have

\[\begin{align*}
\end{align*}\]
\[
\int_0^{\tau \wedge T_1} s^{2H-1} \| g(s, Z_s) - g(s, K(a, s)) \|^2 ds \leq m_1(\mathcal{D}) \int_0^{\tau \wedge T_1} s^{2H-1} |Z_s - K(a, s)|^2 ds
\]

\[
\leq m_1(\mathcal{D}) \sup_{0 \leq s \leq \tau \wedge T_1} s^{2H-1} \int_0^{\tau \wedge T_1} |Z_s - K(a, s)|^2 ds
\]

\[
\leq m_1(\mathcal{D}) T_1^{2H-1} \int_0^{\tau \wedge T_1} |Z_s - K(a, s)|^2 ds.
\]

Therefore,

\[
E \left( \sup_{0 \leq t \leq \tau \wedge T_1} |Q_t - K(a, t)|^2 \right) \leq \left[ AF_1^2 \eta^2 \mathbb{E}(\mathcal{D}) + \mu_0 + m_1(1 + 2HT_1^{2H-1}) \right] \int_0^{\tau \wedge T_1} E|Z_s - K(a, s)|^2 ds + (\mu_0 + 1) \int_0^{\tau \wedge T_1} E|Q_s - K(a, s)|^2 ds + 2E \sup_{0 \leq t \leq \tau \wedge T_1} \int_0^t (Q_s - K(a, s), g(s, Z_s) - g(s, K(a, s))) dB_s^H.
\]

(3.1)

By the basic inequality \(a^2 + b^2 \geq 2ab\) and an Itô formula for a finite dimensional fractional Brownian motion applied to \((K(t, a), t \in [0, T])\) (e.g., [3]), we can get

\[
E \sup_{0 \leq t \leq \tau \wedge T_1} \int_0^t (Q_s - K(a, s), g(s, Z_s) - g(s, K(a, s))) dB_s^H
\]

\[
\leq E \int_0^{\tau \wedge T_1} |Q_s - K(a, s)| \| g(s, Z_s) - g(s, K(a, s)) \|_2 dB_s^H
\]

\[
\leq 2m_2^H(\mathcal{D}) L_2 \left[ E \left( \int_0^{\tau \wedge T_1} \| g(s, Z_s) - g(s, K(a, s)) \|_2^2 dB_s^H \right)^{1/2} \right]
\]

\[
\leq 4m_4^H(\mathcal{D}) m_2^H(\mathcal{D}) L_1 L_2^H \Delta t,
\]

(3.2)

where \(L_1\) and \(L_2\) are two positive constants. Thus, it follows from (3.1) and (3.2)

\[
E \left( \sup_{0 \leq t \leq \tau \wedge T_1} |Q_t - K(a, t)|^2 \right)
\]

\[
\leq \left[ AF_1^2 \eta^2 \mathbb{E}(\mathcal{D}) + \mu_0 + m_1(1 + 2HT_1^{2H-1}) \right] \int_0^{\tau \wedge T_1} E|Z_s - Q_s + Q_s - K(a, s)|^2 ds + (\mu_0 + 1) \int_0^{\tau \wedge T_1} E|Q_s - K(a, s)|^2 ds + 8m_4^H(\mathcal{D}) m_2^H(\mathcal{D}) L_1^2 L_2^H \Delta t
\]

\[
\leq \left[ 2AF_1^2 \eta^2 \mathbb{E}(\mathcal{D}) + 3\mu_0 + 1 + 2m_1(1 + 2HT_1^{2H-1}) \right] \int_0^{T_1} \left( E \left( \sup_{0 \leq t \leq \tau \wedge T_1} |Q_t - K(a, r)|^2 \right) \right) dr
\]

\[
+ 2 \left[ AF_1^2 \eta^2 \mathbb{E}(\mathcal{D}) + \mu_0 + m_1(1 + 2HT_1^{2H-1}) \right] \int_0^{\tau \wedge T_1} E|Z_s - Q_s|^2 ds + 8m_4^H(\mathcal{D}) m_2^H(\mathcal{D}) L_1 L_2^H \Delta t
\]

(3.3)

for all \(t \in [0, T]\).

Applying Gronwall’s inequality, Eq. (2.3) and expression \(Z_s = Z(a, s) = Q(a, \left[ \frac{s}{\Delta t} \right] \Delta t)\) (where \(\left[ \frac{s}{\Delta t} \right] \) is
the integer part of \( \frac{s}{\Delta t} \), we obtain a bound of the form

\[
|z_s - q_s|^2 = \left| Q\left( a, \left[ \frac{s}{\Delta t} \right] \Delta t \right) - q_s \right|^2
\]

Applying the definition of fBm on the last term of (3.4) again, we have

\[
E \left( \int_{\left[ \frac{s}{\Delta t} \right] \Delta t}^{s} g \left( u, Q \left( a, \left[ \frac{s}{\Delta t} \right] \Delta t \right) \right) du \right)^2 \leq m_4(\mathcal{D})T_1^{2H}\Delta t.
\]

Thus, by conditions (2.6) and (2.7) and inequality (3.4), for \( T\Delta t < 1 \),

\[
E \left( \int_{0}^{T\wedge T_1} |z_s - q_s|^2 ds \right) \leq 4T_1 \left[ m_6(\mathcal{D}) + \mu_0^2m_5(\mathcal{D}) + m_4(\mathcal{D}) \right] \Delta t^2 + 4m_4(\mathcal{D})T_1^{2H}\Delta t
\]

Substituting (3.5) into (3.3), one can get

\[
E \left( \sup_{0 \leq t \leq \tau \wedge T_1} |Q_t - K(a, t)|^2 \right) \leq C_1(\mathcal{D})\Delta t + C_2(\mathcal{D}) \int_{0}^{T_1} E \left( \sup_{0 \leq t \leq \tau \wedge T_1} |Q_t - K(a, t)|^2 \right) dt,
\]

where

\[
C_1(\mathcal{D}) = 8 \left[ AF_1^2\eta^2M(\mathcal{D}) + \mu_0 + m_1(\mathcal{D})(1 + 2HT_1^{2H-1}) \right] * \left[ m_6(\mathcal{D}) + \mu_0^2m_5(\mathcal{D}) + m_4(\mathcal{D}) + m_4(\mathcal{D})T_1^{2H} \right]
\]

\[
C_2(\mathcal{D}) = 2AF_1^2\eta^2M(\mathcal{D}) + 3\mu_0 + 1 + 2m_1(\mathcal{D})(1 + 2HT_1^{2H-1}).
\]

On applying the Gronwall inequality we then have the following inequality

\[
E \left( \sup_{0 \leq t \leq \tau \wedge T} |Q_t - K(a, t)|^2 \right) \leq C_1(\mathcal{D})e^{C_2(\mathcal{D})T}\Delta t = C(\mathcal{D})\Delta t.
\]

So, the proof is complete. \( \square \)

**Theorem 3.2.** If \( \theta \) is the first exit time of the solution \( K(a, t) \) to (2.1) from the domain \( \mathcal{D}(r) \), and a function \( V(t, K) \) exists which satisfies conditions (c) and (d), then the probability

\[
P(\theta \geq T) \geq 1 - \varepsilon.
\]
Proof. Assuming that there exists a non-negative function $V(t, K)$ satisfying condition (c), and applying Itô’s formula to $V(t, K)$, we have

$$dV(t, K(a, t)) = LV(t, K(a, t)) + V_K(t, K(a, t))g(t, K(a, t))dB^H_t.$$ 

Taking integral over the interval $[0, t \wedge \theta]$ and taking expectations, yields

$$EV(t \wedge \theta, K(a, t)) = V(t_0, K_0) + E \int_0^{t \wedge \theta} LV(s, K(a, s))ds.$$ 

By condition (d), it follows that

$$EV(t \wedge \theta, K(a, t \wedge \theta)) \leq V(t_0, K_0) + E \int_0^T (m_2(D) + \psi_1(s) + \psi_2(s)V(s, K(a, s)))ds.$$ 

Applying Gronwall’s lemma, one can easily obtain that almost surely

$$EV(t, K(a, t)) \leq \left( V(t_0, K_0) + m_2(D)T + \int_0^T \psi_1(s)ds \right) \times \exp \left( \int_0^T \psi_2(s)ds \right).$$ 

Let

$$N_1 = \left( V(t_0, K_0) + m_2(D)T + \int_0^T \psi_1(s)ds \right) \times \exp \left( \int_0^T \psi_2(s)ds \right).$$ 

Note that $V(\theta, K(a, \theta)) = r$, and $K(a, \theta)$ is valued on the boundary of $D(r)$, the probability $p(\theta < T)$ can be bounded as

$$N_1 \geq E[V(t \wedge \theta, K(a, t \wedge \theta))] \geq E[V(t \wedge \theta, K(a, t \wedge \theta))]I_{[\theta < T]}(\omega) \geq rE[I_{[\theta < T]}(\omega)] \geq rP(\theta < T).$$ (3.6)

Rearranging (3.6), we have

$$P(\theta < T) \leq \frac{N_1}{r} = \epsilon.$$ (3.7)

Theorem 3.2 is proved. □

We can obtain the following results deduced from Theorem 3.2 directly.

**Lemma 3.3.** Let $\theta$ be the first exit time of the solution $K(a, t)$ to Eq. (1.1) from the domain $D(r)$, and the coefficients of Eq. (1.1) satisfy condition (a). If there exists a function $V(t, K)$ satisfying conditions (c) and (d), then $\lim_{r \to \infty} D(r) = G$ and, for any $t \in [0, T]$ and $K_0 \in G$, $K(a, t)$ remains in $G$. Thus, for all finite $T$, $K(a, t)$ is the unique solution to Eq. (1.1) on $t \in [0, T]$.

**Proof.** The proof of this lemma is similar to that in [19]. □

For the Euler approximate solution $Q_t$, one can get the following Theorem 3.4 similarly:

**Theorem 3.4.** If $\rho$ is the first exit time of the Euler approximate solution $Q_t$ to Eq. (2.3) from the domain $D(r)$, and if $f(t, K)$ and $g(t, K)$ satisfy condition (b) and there exists a function $V(t, K(t, a))$ satisfying conditions (c)-(e), then

$$P(\rho \geq T) \geq 1 - \epsilon(1 + \tilde{N}(\omega)\Delta t)$$

for sufficiently small $\Delta t$, where $\tilde{N}(\omega) = N(\omega)e^{-\int_0^T \psi_2(s)ds} \left( V(0, K_0) + \int_0^T \psi_1(s)ds \right)$. 
Proof. Note that $Q_t$ is also the solution to Eq. (2.4), and applying Itô’s formula to $V(t, Q_t)$ yields,

$$
\begin{align*}
dV(t, Q_t) &= V_t(t, Q_t)dt + \left\langle V_k(t, Q_t), -\frac{\partial Z_t}{\partial a} - \mu(t, a)Z_t + f(t, Z_t) \right\rangle dt \\
& \quad + \frac{1}{2}h^{2H-1}V_{kk}(t, Q_t)||g(t, Z_t)||^2 dt + \left\langle V_k(t, Q_t), g(t, Z_t)dB^H_t \right\rangle \\
& \quad = LV(Z_t) + \left\langle V_t(t, Q_t) - V_t(t, Z_t)dt + \left\langle V_k(t, Q_t) - V_k(t, Z_t), -\frac{\partial Z_t}{\partial a} - \mu(t, a)Z_t + f(t, Z_t) \right\rangle dt \\
& \quad + \frac{1}{2}h^{2H-1}(V_{kk}(t, Q_t) - V_{kk}(t, Z_t))||g(t, Z_t)||^2 dt + \left\langle V_k(t, Q_t), g(t, Z_t)dB^H_t \right\rangle.
\end{align*}
$$

Applying condition (d), we have

$$
\begin{align*}
dV(t, Q_t) &\leq m_2(D) + \psi_1(t) + \psi_2(t)V(t, Z_t) + ||V_t(t, Q_t) - V_t(t, Z_t)||dt \\
& \quad + \left\langle V_k(t, Q_t) - V_k(t, Z_t), -\frac{\partial Z_t}{\partial a} - \mu(t, a)Z_t + f(t, Z_t) \right\rangle dt \\
& \quad + \frac{1}{2}h^{2H-1}(V_{kk}(t, Q_t) - V_{kk}(t, Z_t))||g(t, Z_t)||^2 dt + \left\langle V_k(t, Q_t), g(t, Z_t)dB^H_t \right\rangle \\
& \leq m_2(D) + \psi_1(t) + \psi_2(t)V(t, Q_t) + \psi_2(t) ||V(t, Z_t) - V(t, Q_t)|| dt \\
& \quad + \left\langle V_k(t, Q_t) - V_k(t, Z_t), -\frac{\partial Z_t}{\partial a} - \mu(t, a)Z_t + f(t, Z_t) \right\rangle dt \\
& \quad + \frac{1}{2}h^{2H-1}(V_{kk}(t, Q_t) - V_{kk}(t, Z_t))||g(t, Z_t)||^2 dt + \left\langle V_k(t, Q_t), g(t, Z_t)dB^H_t \right\rangle.
\end{align*}
$$

Taking integral over the interval $[0, \rho \wedge t]$ and taking expectations, yields

$$
\begin{align*}
\mathbb{E}\left[V(\rho \wedge t, Q_{\rho \wedge t}) \right] &\leq V(0, K_0) + m_2(D)T + \int_0^{\rho \wedge t} \psi_1(s)ds + \mathbb{E}\int_0^{\rho \wedge t} \psi_2(s)V(s, Q_s)ds \\
& \quad + \mathbb{E}\int_0^{\rho \wedge t} \left\langle V_k(s, Q_s) - V_k(s, Z_s), -\frac{\partial Z_s}{\partial a} - \mu(s, a)Z_s + f(s, Z_s) \right\rangle ds \\
& \quad + \frac{1}{2}h^{2H-1}(V_{kk}(s, Q_s) - V_{kk}(s, Z_s))||g(s, Z_s)||^2 ds \\
& \leq V(0, K_0) + m_2(D)T + \int_0^{\rho \wedge t} \psi_1(s)ds + \mathbb{E}\int_0^{\rho \wedge t} \psi_2(s)V(s, Q_s)ds \\
& \quad + \mathbb{E}\int_0^{\rho \wedge t} \left| V_k(s, Q_s) - V_k(s, Z_s) \right| \left| -\frac{\partial Z_s}{\partial a} - \mu(s, a)Z_s + f(s, Z_s) \right| ds \\
& \quad + \frac{1}{2}h^{2H-1}(V_{kk}(s, Q_s) - V_{kk}(s, Z_s))||g(s, Z_s)||^2 ds.
\end{align*}
$$

Therefore, applying inequalities (2.5)-(2.7) and condition (e), we get that

$$
\begin{align*}
\mathbb{E}\left[V(\rho \wedge t, Q_{\rho \wedge t}) \right] &\leq V(0, K_0) + m_2(D)T + \int_0^{t} \psi_1(s)ds \\
& \quad + N_2(D)m_3(D)\mathbb{E}\int_0^{\rho \wedge t} |Z_s - Q_s| ds + \mathbb{E}\int_0^{t} \psi_2(s)V(\rho \wedge s, Q_{\rho \wedge s})ds,
\end{align*}
$$
where
\[ N_2(D) = m_4^{1/2}(D) + \mu_0 m_5^{1/2}(D) + m_6^{1/2}(D) + m_4(D)T^{2H-1/2} + \sup_{0 \leq t \leq T} \psi_2(t) + 1. \]

Applying Hölder inequality and (3.5), for all \( t \in [0, T] \), we have the bound
\[ \int_0^t \mathbb{E} |Z_s - Q_s| ds \leq (4T(m_6(D) + \mu_0 m_5(D) + m_4(D) + m_4(D)T^{2H}))^{1/2} \Delta t^{1/2}. \]

Thus
\[ \mathbb{E}[V(\rho \wedge t, Q_{\rho \wedge t})] \leq V(0, K_0) + m_2(D)T + \int_0^T \psi_1(s) ds \]
\[ + N_2(D)m_3(D) \left( 4T(m_6(D) + \mu_0 m_5(D) + m_4(D) + m_4(D)T^{2H}) \right)^{1/2} \Delta t^{1/2} \]
\[ + \int_0^T \psi_2(s)EV(\rho \wedge s, Q_{\rho \wedge s}) ds. \]

Applying the Gronwall inequality, yields
\[ \mathbb{E} \left[ V(\rho \wedge t, Q_{\rho \wedge t}) \right] \leq \left[ V(0, K_0) + m_2(D)T + \int_0^T \psi_1(s) ds \right] e^{\int_0^T \psi_2(s) ds} + N(D)\Delta t^{1/2}, \]

where \( N(D) = e^{\int_0^T \psi_2(s) ds} N_2(D)m_3(D) \left( 4T \left( m_6(D) + \mu_0 m_5(D) + m_4(D) + m_4(D)T^{2H} \right) \right)^{1/2}. \)

A similar argument, being used to prove Theorem 3.2, can be used to bound \( P(\rho < T) \). Since \( Q_\rho \) is valued on the boundary of \( D(\tau) \) then \( V(\rho, Q_\rho) = \tau \), one can get
\[ \mathbb{E} \left[ V(\rho \wedge t, Q_{\rho \wedge t}) \right] \leq \left[ V(0, K_0) + m_2(D)T + \int_0^T \psi_1(s) ds \right] e^{\int_0^T \psi_2(s) ds} + N(D)\Delta t^{1/2}, \]

where \( N(D) = e^{\int_0^T \psi_2(s) ds} N_2(D)m_3(D) \left( 4T \left( m_6(D) + \mu_0 m_5(D) + m_4(D) + m_4(D)T^{2H} \right) \right)^{1/2}. \)

Rearranging the inequality and defining \( \bar{N}(D) = N(D)e^{-\int_0^T \psi_2(s) ds} / \left( V(0, K_0) + \int_0^T \psi_1(s) ds \right) \), and defining \( \epsilon = \frac{\bar{N}(D)}{T} \in (0, 1) \) as in (3.7), we have
\[ P(\rho < T) \leq \left( 1 + \bar{N}(D)\Delta t^{1/2} \right) \epsilon. \]

Theorems 3.2 and 3.4 show that both \( K(a, t) \) and \( Q_t \) remain in the domain \( D(\tau) \). And therefore, by Theorem 3.1, the Euler approximate solution \( Q_t \) will converge to the \( K(a, t) \) with probability
\[ P(\tau < T) \leq P(\rho < T) + P(s < T) \leq \left( 2 + \bar{N}(D)\Delta t^{1/2} \right) \epsilon. \]

\[ \square \]

**Theorem 3.5.** Let \( G \) be an open subset of \( V \), and denote the unique solution to Eq. (1.1) for \( t \in [0, T] \) with initial value \( K_0 \in G \) by \( K(a, t) \in G \). Let \( Q_t \), be the Euler approximate solution to (2.2) and also let \( D \subseteq G \) be any compact set. Suppose conditions (a)-(e) are satisfied, and for all \( a \in (0, A), \sup_{t \in D} \left| \frac{Q_t - K(a, t)}{\sqrt{T}} \right| \) exists. Then for any \( \epsilon, \delta > 0 \) there exists \( \Delta t^* > 0 \) such that (when \( \Delta t \leq \Delta t^* \))
\[ P \left( \sup_{0 \leq t \leq T} |Q_t - K(a, t)|^2 \geq \delta \right) \leq \epsilon. \]
Proof. Introducing the event sub-space \( \bar{\Omega} = \{ \omega : \sup_{0 \leq t \leq T} |Q_t - K(a, t)|^2 \geq \delta \} \) and using Theorem 3.1, we find that
\[
C(\bar{\Omega}) \Delta t \geq E \left[ \sup_{0 \leq t \leq T} |Q_t - K(a, t)|^2 \right] \\
\geq E \left[ I_{[\tau \geq T]}(\omega) \sup_{0 \leq t \leq T} |Q_t - K(a, t)|^2 \right] \\
\geq E \left[ I_{[\tau \geq T]}(\omega) I_{[\Omega]} \sup_{0 \leq t \leq T} |Q_t - K(a, t)|^2 \right] \\
\geq \delta E \left[ I_{[\tau \geq T]}(\omega) I_{[\Omega]}(\omega) \right] = \delta P \left( (\tau \geq T) \cap \bar{\Omega} \right) \geq \delta \left[ P(\bar{\Omega}) - P(\tau < T) \right].
\]

Whence on using (3.8) we conclude that probability
\[
K(a, \bar{\Omega}) = P \left( \sup_{0 \leq t \leq T} |Q_t - K(a, t)|^2 \geq \delta \right) \leq 2\epsilon + \epsilon N((|D|) \Delta t^{1/2} + \frac{C(|D|)}{\delta} \Delta t,
\]
which, for appropriate choice of \( \Delta t, \) proves Theorem 3.5. \( \square \)

4. An example

We consider the following stochastic age-dependent capital system with respect to fBm
\[
\begin{align*}
\frac{\partial K(a, t)}{\partial t} + \frac{\partial K(a, t)}{\partial a} &= -\frac{1}{(1-a)^2} K(a, t) - tK(a, t) + K(a, t) dB_t^H, \quad (a, t) \in Q, \\
K(a, 0) &= \exp\left(-\frac{1}{1-a}\right), \quad a \in [0, 1], \\
K(0, t) &= \gamma(t) A(t) = 2t, \quad F(L(t), N(t)) = \frac{1}{(2-t)^2} \int_0^1 K(a, t) da, \quad L(t) = \frac{1}{(2-t)^2} f(t, K) = -tK, \quad g(t, K) = K, \quad \text{and} \quad K(a, 0) = \exp\left(-\frac{1}{1-a}\right).
\end{align*}
\]
(4.1)

where \( Q = (0, 1) \times (0, T). \) \( B_t^H \) is a fBm with Hurst number \( H = \frac{3}{4} \). For the simulation of \( B_t^H \) with \( H = 3/4 \), the generators of the fBm described in [1] are used. We can set this problem in our formulation by taking \( H = L^2([0,1]) \) and \( V = B_t^H([0,1]) \) (a Sobolev space with elements satisfying the boundary conditions above), and set \( \mu(t, a) = \frac{1}{(1-a)^2}, \gamma(t) A(t) = 2t, \quad F(L(t), N(t)) = \frac{1}{(2-t)^2} \int_0^1 K(a, t) da, \quad L(t) = \frac{1}{(2-t)^2} f(t, K) = -tK, \quad g(t, K) = K, \quad \text{and} \quad K(a, 0) = \exp\left(-\frac{1}{1-a}\right). \)

Obviously, \( F(L, N) \) and \( \mu(t, a) \) satisfy condition (a) and the operators \( f \) and \( g \) satisfy condition (b). Then, the approximate solution will converge to the true solution of Eq. (4.1) for any \( (a, t) \in Q \) in the sense of Theorem 3.5. Here we take \( T = 1 \) in Eq. (4.1). The results are obtained with fixed step sizes \( \Delta t = 0.005, \Delta a = 0.05 \) in our calculation.

Figure 1 (a) shows the explicit solution to Eq. (4.1) without perturbation, that is \( EK(a, t) = \exp\left(-\frac{1}{1-a} - \frac{t^2}{2}\right) \). Figures 1 (b)-(d) show Euler approximate solution of Eq. (4.1) with 100, 1000, and 10000 experiments, respectively, where \( EQ(a, t) = \frac{1}{n} \sum_{k=1}^n K_k(a, t) \). These figures reveal the fact that the numerical approximate solution will tend to the true one in the mean sense.

For Eq. (4.1), it is difficult to obtain the analytic (explicit) solution, so the explicit solution \( K(a, t) \) to Eq. (4.1) can be replaced by \( \exp\left(-\frac{1}{1-a} - \frac{t^2}{2}\right)(1 + \Delta B_t^H) \). Figures 2 (a) and (b) give the simulation of \( \exp\left(-\frac{1}{1-a} - \frac{t^2}{2}\right)(1 + \Delta B_t^H) \) and the Euler approximate solution \( Q(a, t) \), respectively. The absolutely error and square error are also respectively shown in Figures 2 (c) and (d). The maximum of square error is not greater than 0.02. It is obvious that the numerical approximation will tend to the true solution in the mean square sense.
Figure 1: Numerical simulation of stochastic age-dependent capital system: (a) analysis solution and Euler approximate solution for several tests, (b) \( n = 100 \), (c) \( n = 1000 \), (d) \( n = 10000 \).

Figure 2: Error simulation of stochastic age-dependent capital system with fBm.
Acknowledgment

This work was supported by the National Natural Science Foundation of China under Grants 11662016 and 11661064, by the Natural Science Foundation of Ningxia Hui Autonomous Region (China) under Grants NZ16005 and NZ14048, and supported in part by the Scientific Research Foundation of the Higher Education Institutions of Ningxia Hui Autonomous Region (China) under Grant NGY2016061.

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