



Well-posedness for systems of time-dependent hemivariational inequalities in Banach spaces

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Abstract

In this paper, we generalize the concept of α -well-posedness to a system of time-dependent hemivariational inequalities without Volterra integral terms in Banach spaces. We establish some metric characterizations of α -well-posedness and prove some equivalence results of strong α -well-posedness (resp., in the generalized sense) between a system of time-dependent hemivariational inequalities and its derived system of inclusion problems. ©2017 All rights reserved.

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1. Introduction

In 1983, Panagiotopoulos [27] introduced and studied the hemivariational inequality concerning non-smooth and nonconvex energy functions, which is an important generalization of variational inequality. As a useful tool, this type of inequalities and their systems have been used to describe many important problems arising in mechanics and engineering, such as unilateral contact problems in nonlinear elasticity, thermoviscoelastic frictional contact problems, and obstacles problems, see, e.g., [25, 26, 28]. By using the generalized directional derivative (in the sense of Clarke) and the Clarke's generalized gradient (see definitions in Section 2), many kinds of hemivariational inequalities and systems of hemivariational inequalities, including stationary hemivariational inequalities, evolutionary hemivariational inequalities and their systems, etc., have been studied by many researchers in recent years. We can refer to, e.g., [1, 2, 6, 12, 24–26, 28, 34, 37–39] and the references therein for more details.

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There are two important approaches to study the existence and uniqueness of solution to various types of hemivariational inequalities in the recent literature on hemivariational inequalities. One is closely related to Knaster-Kuratowski-Mazurkiewicz theorems and fixed point theorems, which are used by Panagiotopoulos et al. [29], Repovš and Varga [30], Costea and Radulescu [10], and Zhang and He [41] to study stationary hemivariational inequalities and systems of stationary hemivariational inequalities. The other is closely related to surjectivity theorems concerning pseudomonotone and coercive operators, which are captured by Migorski et al. [24], Carl et al. [1], Naniewicz and Panagiotopoulos [26], Xiao and Huang [33], and Liu [22] for various types of stationary hemivariational inequalities and evolutionary hemivariational inequalities. Before that, Panagiotopoulos formulated two systems of hemivariational inequalities for the behavior of von Karman plates and linear thermoelastic materials in Chapter 7.3 of monograph [28], respectively. However, the regularity hypotheses on the multi-valued terms were quite unnatural for the existence of solutions to the two systems of hemivariational inequalities. Very recently, Xiao et al. [35] studied a system of time-dependent hemivariational inequalities with Volterra integral terms by using a surjectivity theorem for pseudomonotone and coercive operators, and the Banach fixed point theorem for contraction mappings. Under some appropriate conditions, the existence and uniqueness result of solution to the problem considered was established by proving that a derived vector inclusion problem with Volterra integral term is solvable.

On the other hand, Tykhonov [31] first introduced and considered the well-posedness for a minimization problem in 1966, which not only is one of the classical concepts in the optimization theory but also has a profound impact on the development of optimization problems and their related problems. After that, various kinds of results concerned with well-posedness for many optimization problems were established and various kinds of well-posedness for optimization problems and their related ones were studied extensively in recent years by a large number of researchers in many fields; see e.g., [8, 15, 21, 42] and the references therein. In 1981, Lucchetti and Patrone [23] extended the notion of well-posedness for optimization problems to a variational inequality for the first time. By using Ekeland's theorem, they gave a characterization of Tykhonov's well-posedness for a minimizing problem with a convex lower semi-continuous function on a closed convex set. Since then, various kinds of well-posedness for optimization problems, such as LP well-posedness and generalized well-posedness etc., are extended to develop the well-posedness for variational inequalities and their related problems, such as equilibrium problems, fixed point problems, inclusion problems and others; see e.g., [3, 11, 14, 17, 20].

It seems to be natural and easy to generalize the concept of well-posedness to hemivariational inequalities and to many people that most results on well-posedness for variational inequalities should hold for hemivariational inequalities under some similar conditions. However, it is not the truth. The Clarke's generalized directional derivative of a nonconvex and nonsmooth Lipschitz functional in hemivariational inequalities makes it much difficult. Thus, the literature on well-posedness for hemivariational inequalities is limit. In 1995, Goeleven and Motreanu [13] first introduced the well-posedness for a hemivariational inequality and presented some basic results concerning the well-posed hemivariational inequality. Later, using the concept of approximating sequence, Xiao et al. [34, 36] defined a concept of well-posedness for a hemivariational inequality and a variational-hemivariational inequality. They gave some metric characterizations for the well-posed hemivariational inequality and the well-posed variational-hemivariational inequality, and proved the equivalence of well-posedness between the hemivariational inequality and the corresponding inclusion problem. Till now, there are a few papers discussing the solvability of systems of hemivariational inequalities since, due to the complex structure of systems of hemivariational inequalities, it is much more difficult than the study of hemivariational inequalities. Very recently, Wang et al. [32] introduced and considered the well-posedness for systems of hemivariational inequalities, gave some metric characterizations of well-posedness and established the equivalence between well-posedness of a system of hemivariational inequalities and its derived system of inclusion problems.

Inspired by the above research work of [32], in this paper we generalize the notion of α -well-posedness to a system of time-dependent hemivariational inequalities without Volterra integral terms, establish some metric characterizations of α -well-posedness and prove the equivalence between α -well-posedness of the

system of time-dependent hemivariational inequalities and its derived system of inclusion problems. The paper is structured as follows. In Section 2, we briefly recall some preliminaries. In Section 3, we define several notions of α -well-posedness for the system of time-dependent hemivariational inequalities and, under some mild conditions on the operators involved, give some metric characterizations of α -well-posedness for the system of time-dependent hemivariational inequalities. In Section 4, we establish two equivalence results of α -well-posedness between the system of time-dependent hemivariational inequalities and its derived system of inclusion problems. Finally, in Section 5, we give some concluding remarks on our main results.

2. Preliminaries

In this section, we present the problem considered in this paper and recall some important notions and useful results on nonlinear analysis, nonsmooth analysis and operators of monotone type, which can be found in [9, 24, 26, 40].

Let V_1 and V_2 be real, separable and reflexive Banach spaces with dual spaces V_1^* and V_2^* , respectively. Assume that X_1 and X_2 are real reflexive Banach spaces with dual spaces X_1^* and X_2^* , respectively. For some $T \in (0, +\infty)$, we suppose that $\mathcal{V}_1 = L^2(0, T; V_1)$ and $\mathcal{V}_2 = L^2(0, T; V_2)$, and thus $\mathcal{V}_1^* = L^2(0, T; V_1^*)$ and $\mathcal{V}_2^* = L^2(0, T; V_2^*)$. Unless otherwise specifically stated in the paper, we always denote by $\langle \cdot, \cdot \rangle_{E^* \times E}$ the duality pairing between Banach space E and its dual E^* , and by $\|\cdot\|_E$, $\|\cdot\|_{E^*}$ the norms on the space E and its dual E^* , respectively, where $E \in \{V_i, X_i, \mathcal{V}_i, i = 1, 2\}$. In this paper, we consider the following system of time-dependent hemivariational inequalities without Volterra integral terms, which is specified as follows: Find $(u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ such that

$$(\text{STDHVI}) \quad \begin{cases} \langle A_1(t, u_1(t)) + B_1(t, u_2(t)), v_1 \rangle_{V_1^* \times V_1} + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1 v_1) \\ \quad \geq \langle f_1(t), v_1 \rangle_{V_1^* \times V_1}, \quad \forall v_1 \in V_1, \quad \text{a.e. } t \in (0, T), \\ \langle A_2(t, u_2(t)) + B_2(t, u_1(t)), v_2 \rangle_{V_2^* \times V_2} + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2 v_2) \\ \quad \geq \langle f_2(t), v_2 \rangle_{V_2^* \times V_2}, \quad \forall v_2 \in V_2, \quad \text{a.e. } t \in (0, T), \end{cases}$$

where, for $i, j = 1, 2$ and $j \neq i$, $A_i : (0, T) \times V_i \rightarrow V_i^*$, $B_i : (0, T) \times V_j \rightarrow V_i^*$, and $f_i : (0, T) \rightarrow V_i^*$ are operators with images in V_i^* . For $i \neq j = 1, 2$, $M_i : V_i \rightarrow X_i$ is a linear continuous and compact operator, and $J_i^\circ(t, x_1, x_2; v)$ is the partial generalized directional derivative (in the sense of Clarke) of the locally Lipschitz functional $J : (0, T) \times X_1 \times X_2 \rightarrow \mathbf{R}$ with respect to the i -th argument at the point $x_i \in X_i$ in the direction $v \in X_i$ for the given $x_j \in X_j$.

Suppose Z is a real Banach space with its dual Z^* and $\langle \cdot, \cdot \rangle_{Z^* \times Z}$ is the duality pairing between Z^* and Z . Let u and $\{u_n\}$ be a point and a sequence in Z , and let u^* and $\{u_n^*\}$ be a point and a sequence in Z^* , respectively. We use the notations $u_n \rightarrow u$, $u_n \rightharpoonup u$ and $u_n^* \xrightarrow{*} u^*$ to indicate the strong convergence of $\{u_n\}$ to u , the weak convergence of $\{u_n\}$ to u and the weak* convergence of $\{u_n^*\}$ to u^* , respectively. Recall that if Z is not reflexive, then the weak* topology of Z^* is weaker than its weak topology and that if Z is reflexive, then the weak* topology of Z^* coincides with its weak topology. It is easy to see that if $\{u_n\} \subset Z$, $\{u_n^*\} \subset Z^*$, $u_n \rightarrow u$ in Z and $u_n^* \xrightarrow{*} u^*$ in Z^* , then $\langle u_n^*, u_n \rangle_{Z^* \times Z} \rightarrow \langle u^*, u \rangle_{Z^* \times Z}$ as $n \rightarrow \infty$.

Definition 2.1. Let $A : Z \rightarrow Z^*$ be an operator. A is said to be

- (i) demicontinuous, if for any sequence $\{u_n\} \subset Z$ with $u_n \rightarrow u \in Z$, $A(u_n) \rightharpoonup A(u)$ in Z^* ;
- (ii) hemicontinuous, if for all $u, v \in Z$, the function $t \mapsto \langle A(u + t(v - u)), v - u \rangle_{Z^* \times Z}$ from $[0, 1]$ into \mathbf{R} is continuous at 0^+ ;
- (iii) continuous, if for any sequence $\{u_n\} \subset Z$ with $u_n \rightarrow u \in Z$, $A(u_n) \rightarrow A(u)$ in Z^* .

It is clear that, if $A : Z \rightarrow Z^*$ is continuous, then it is demicontinuous which, in turn, implies that A is hemicontinuous. If $A : Z \rightarrow Z^*$ is linear and demicontinuous, then it is continuous. It can be shown that for monotone operator $A : Z \rightarrow Z^*$ with $D(A) = Z$, the notions of demicontinuity and hemicontinuity coincide.

Definition 2.2. An operator $A : Z \rightarrow Z^*$ is said to be monotone if

$$\langle Au - Av, u - v \rangle_{Z^* \times Z} \geq 0, \quad \forall u, v \in Z.$$

Definition 2.3. Let $h : Z \rightarrow \mathbf{R}$ be a functional. h is said to be

- (i) Lipschitz continuous, if there exists a constant $L > 0$ such that

$$|h(u_1) - h(u_2)| \leq L \|u_1 - u_2\|_Z, \quad \forall u_1, u_2 \in Z;$$

- (ii) locally Lipschitz continuous, if for every $u \in Z$, there exist a neighborhood $N(u)$ and a constant $L_u > 0$ such that

$$|h(u_1) - h(u_2)| \leq L_u \|u_1 - u_2\|_Z, \quad \forall u_1, u_2 \in N(u).$$

Definition 2.4. Let Z_1, Z_2 be two real Banach spaces and $h : Z_1 \times Z_2 \rightarrow \mathbf{R}$ be a functional. The functional h is said to be

- (i) Lipschitz continuous in the first variable if the functional $h(\cdot, u_2) : Z_1 \rightarrow Z_1^*$ is Lipschitz continuous on Z_1 for any given $u_2 \in Z_2$;
(ii) locally Lipschitz continuous in the first variable, if the functional $h(\cdot, u_2) : Z_1 \rightarrow Z_1^*$ is locally Lipschitz continuous on Z_1 for any given $u_2 \in Z_2$.

By the similar way, we can define the Lipschitz continuity and locally Lipschitz continuity of the functional $h : Z_1 \times Z_2 \rightarrow \mathbf{R}$ in the second variable.

Definition 2.5. Let $h : Z \rightarrow \mathbf{R}$ be a locally Lipschitz functional on Z and let $u, v \in Z$ be two given elements. The Clarke's generalized directional derivative of h at the point u in the direction v , denoted by $h^\circ(u; v)$, is defined by

$$h^\circ(u; v) = \limsup_{w \rightarrow u, \lambda \downarrow 0} \frac{h(w + \lambda v) - h(w)}{\lambda}.$$

The Clarke's generalized gradient of h at u , denoted by $\partial h(u)$, the subset of the dual Z^* , which is defined by

$$\partial h(u) = \{\rho \in Z^* : h^\circ(u; v) \geq \langle \rho, v \rangle_{Z^* \times Z}, \forall v \in Z\}.$$

Remark 2.6. As pointed out in [1], if $h : Z \rightarrow \mathbf{R}$ is convex and continuous, then the Clarke's generalized gradient $\partial h(x)$ coincides with the subdifferential of h at x in the sense of convex analysis, and if h is continuously differentiable, then, for all $x \in Z$, $\partial h(x) = \{h'(x)\}$, where $h'(x)$ denotes the Fréchet differential of h at x .

Proposition 2.7. Let $h : Z \rightarrow \mathbf{R}$ be a locally Lipschitz functional on Z and let $u, v \in Z$ be two given elements. Then

- (i) the function $v \mapsto h^\circ(u; v)$ is finite, positively homogeneous, subadditive and thus convex on Z ;
(ii) $h^\circ(u; v)$ is upper semicontinuous on $Z \times Z$ as a function of (u, v) , i.e., for all $u, v \in Z$, $\{u_n\} \subset Z$, $\{v_n\} \subset Z$ such that $u_n \rightarrow u$ and $v_n \rightarrow v$ in Z , we have

$$\limsup_{n \rightarrow \infty} h^\circ(u_n; v_n) \leq h^\circ(u; v);$$

- (iii) $h^\circ(u; -v) = (-h)^\circ(u; v)$;
(iv) for all $u \in Z$, $\partial h(u)$ is a nonempty, convex, bounded and weak*-compact subset of Z^* ;
(v) for all $v \in Z$, one has

$$h^\circ(u; v) = \max\{\langle \xi, v \rangle_{Z^* \times Z} : \xi \in \partial h(u)\};$$

- (vi) the graph of the Clarke's generalized gradient $\partial h(u)$ is closed in $Z \times (w^*-Z^*)$ topology, where (w^*-Z^*) denotes the space Z^* equipped with weak* topology, i.e., if $\{u_n\} \subset Z$ and $\{u_n^*\} \subset Z^*$ are sequences such that $u_n^* \in \partial h(u_n)$, $u_n \rightarrow u$ in Z and $u_n^* \rightarrow u^*$ weakly* in Z^* , then $u^* \in \partial h(u)$.

Definition 2.8. Let $h : Z \rightarrow \mathbf{R}$ be a locally Lipschitz functional on Banach space Z . h is said to be regular (in the sense of Clarke) at $u \in Z$ if

- (i) for all $v \in Z$, the directional derivative $h'(u, v)$ exists;
- (ii) for all $v \in Z$, $h'(u, v) = h^\circ(u; v)$, where $h'(u, v)$ is the directional derivative of h at $u \in Z$ in the direction $v \in Z$.

Proposition 2.9. Let Z_1 and Z_2 be two Banach spaces. If $H : Z_1 \times Z_2 \rightarrow \mathbf{R}$ is locally Lipschitz on $Z_1 \times Z_2$ and either H or $-H$ is regular at $(u_1, u_2) \in Z_1 \times Z_2$, then $\partial H(u_1, u_2) \subseteq \partial_1 H(u_1, u_2) \times \partial_2 H(u_1, u_2)$, or for any point $(v_1, v_2) \in Z_1 \times Z_2$, $H^\circ(u_1, u_2; v_1, v_2) \leq H_1^\circ(u_1, u_2; v_1) + H_2^\circ(u_1, u_2; v_2)$. However, in general, the converses of the above inclusion and inequality are not true.

Definition 2.10 (see [16]). Let S be a nonempty subset of Z . The measure of noncompactness μ of the set S is defined by

$$\mu(S) = \inf\{\epsilon > 0 : S \subseteq \bigcup_{i=1}^n S_i, \text{diam}(S_i) < \epsilon, i = 1, 2, \dots, n\},$$

where $\text{diam}(S_i)$ denotes the diameter of set S_i .

Definition 2.11. Let A_1, A_2 be nonempty subsets of Z . The Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ between A_1 and A_2 is defined by

$$\mathcal{H}(A_1, A_2) = \max\{e(A_1, A_2), e(A_2, A_1)\},$$

where $e(A_1, A_2) = \sup_{a \in A_1} d(a, A_2)$ with $d(a, A_2) = \inf_{b \in A_2} \|a - b\|_Z$.

Note that, in [16], we can find some more properties of the Hausdorff metric between two sets. At the end of this section, we give a lemma, which is important to our main results.

Theorem 2.12 (see [24]). Let $C \subseteq Z$ be nonempty, closed and convex, $C^* \subseteq Z^*$ be nonempty, closed, convex and bounded, $\varphi : Z \rightarrow \mathbf{R}$ be proper, convex and lower semicontinuous and $y \in C$ be arbitrary. Assume that, for each $x \in C$, there exists $x^*(x) \in C^*$ such that

$$\langle x^*(x), x - y \rangle_{Z^* \times Z} \geq \varphi(y) - \varphi(x).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, x - y \rangle_{Z^* \times Z} \geq \varphi(y) - \varphi(x), \quad \forall x \in C.$$

3. Well-posedness of STDHVI with metric characterizations

In this section, based on the concepts of well-posedness for a system of hemivariational inequalities SHVI in [32], and α -well-posedness for a system of mixed quasivariational-like inequalities in [4], we introduce the concept of α -well-posedness for a system of time-dependent hemivariational inequalities STDHVI, and establish some metric characterizations of α -well-posedness for STDHVI under some appropriate conditions.

Let $V = V_1 \times V_2$. Endowed with the norm defined by $\|u\|_V := \|u_1\|_{V_1} + \|u_2\|_{V_2}$ for all $u = (u_1, u_2) \in V$, V is a reflexive Banach space with its dual V^* and the duality pairing between V and V^* is given by

$$\langle u^*, u \rangle_{V^* \times V} = \langle u_1^*, u_1 \rangle_{V_1^* \times V_1} + \langle u_2^*, u_2 \rangle_{V_2^* \times V_2}, \quad \forall u^* = (u_1^*, u_2^*) \in V^*, \quad u = (u_1, u_2) \in V.$$

Similarly, we can also construct the product spaces $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ and $X = X_1 \times X_2$ and their dual \mathcal{V}^* and X^* .

For $i = 1, 2$, let $\alpha_i : (0, T) \times V_i \rightarrow \mathbf{R}^+ = [0, \infty)$ be a functional such that for a.e. $t \in (0, T)$, $\alpha_i(t, \cdot)$ is convex, continuous and positively homogeneous, i.e., $\alpha_i(t, \lambda_i v_i) = \lambda_i \alpha_i(t, v_i)$ for all $v_i \in V_i$ and $\lambda_i \geq 0$.

Definition 3.1. A sequence $\{\mathbf{u}^n\} \subset \mathcal{V}$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ is said to be an α -approximating sequence with $\alpha = (\alpha_1, \alpha_2)$ for STDHVI if there exists $\epsilon_n > 0$ with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\begin{cases} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} + J_1^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_1(v_1 - u_1^n(t))) \\ \quad \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \quad \forall v_1 \in V_1, \quad \text{a.e. } t \in (0, T), \\ \langle A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} + J_2^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_2(v_2 - u_2^n(t))) \\ \quad \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)), \quad \forall v_2 \in V_2, \quad \text{a.e. } t \in (0, T). \end{cases}$$

In particular, if for $i = 1, 2$, $\alpha_i(t, x_i - y_i) = \|x_i - y_i\|_{V_i}$ for all $x_i, y_i \in V_i$, then $\{\mathbf{u}^n\}$ is said to be an approximating sequence for STDHVI.

Definition 3.2. The system STDHVI of time-dependent hemivariational inequalities is said to be strongly (resp., weakly) α -well-posed with $\alpha = (\alpha_1, \alpha_2)$ if it has a unique solution \mathbf{u} and every α -approximating sequence $\{\mathbf{u}^n\}$ for STDHVI has the property that for a.e. $t \in (0, T)$, $\mathbf{u}^n(t) \rightarrow \mathbf{u}(t)$ (resp., $\mathbf{u}^n(t) \rightharpoonup \mathbf{u}(t)$) in V . In particular, if for $i = 1, 2$, $\alpha_i(t, x_i - y_i) = \|x_i - y_i\|_{V_i}$ for all $x_i, y_i \in V_i$, then the system STDHVI of time-dependent hemivariational inequalities is said to be strongly (resp., weakly) well-posed.

Remark 3.3. It is clear that, the strong α -well-posedness of STDHVI implies the weak α -well-posedness of STDHVI, but the converse is not true in general.

Definition 3.4. The system STDHVI of time-dependent hemivariational inequalities is said to be strongly (resp., weakly) α -well-posed in the generalized sense if the solution set S of STDHVI is nonempty and, for every α -approximating sequence $\{\mathbf{u}^n\}$, there always exists a subsequence $\{\mathbf{u}^{n_k}\}$ such that for a.e. $t \in (0, T)$, $\mathbf{u}^{n_k}(t) \rightarrow \mathbf{u}(t)$ (resp., $\mathbf{u}^{n_k}(t) \rightharpoonup \mathbf{u}(t)$) in V for some $\mathbf{u} \in S$. In particular, if for $i = 1, 2$,

$$\alpha_i(t, x_i - y_i) = \|x_i - y_i\|_{V_i}, \quad \forall x_i, y_i \in V_i,$$

then the system STDHVI of time-dependent hemivariational inequalities is said to be strongly (resp., weakly) well-posed in the generalized sense.

Remark 3.5. Similarly, the strong α -well-posedness in the generalized sense for STDHVI implies the weak α -well-posedness in the generalized sense for STDHVI, but the converse is not true in general.

Remark 3.6. The notions of strong and weak α -well-posedness of STDHVI introduced in this paper are quite different from Definitions 3.1-3.2 and 3.4 in Wang et al. [32].

In order to establish the metric characterizations for the α -well-posedness of STDHVI, we first define two sets in $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ for any $\epsilon > 0$ as follows:

$$\begin{aligned} \Omega_{\alpha, \epsilon} = \{ \mathbf{u} = (u_1, u_2) \in \mathcal{V} : \text{the following hold for all } \mathbf{v} = (v_1, v_2) \in V \text{ and a.e. } t \in (0, T) \\ \langle A_1(t, u_1(t)) + B_1(t, u_2(t)) - f_1(t), v_1 - u_1(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(v_1 - u_1(t))) \geq -\epsilon \alpha_1(t, v_1 - u_1(t)), \\ \langle A_2(t, u_2(t)) + B_2(t, u_1(t)) - f_2(t), v_2 - u_2(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(v_2 - u_2(t))) \geq -\epsilon \alpha_2(t, v_2 - u_2(t)) \}, \end{aligned}$$

and

$$\begin{aligned} \Delta_{\alpha, \epsilon} = \{ \mathbf{u} = (u_1, u_2) \in \mathcal{V} : \text{the following hold for all } \mathbf{v} = (v_1, v_2) \in V \text{ and a.e. } t \in (0, T) \\ \langle A_1(t, v_1) + B_1(t, u_2(t)) - f_1(t), v_1 - u_1(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(v_1 - u_1(t))) \geq -\epsilon \alpha_1(t, v_1 - u_1(t)), \\ \langle A_2(t, v_2) + B_2(t, u_1(t)) - f_2(t), v_2 - u_2(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(v_2 - u_2(t))) \geq -\epsilon \alpha_2(t, v_2 - u_2(t)) \}. \end{aligned}$$

Moreover, we also define two multi-valued mappings $\Omega_{\alpha,\epsilon}(\cdot) : (0, T) \rightarrow 2^V$ and $\Delta_{\alpha,\epsilon}(\cdot) : (0, T) \rightarrow 2^V$ as follows:

$$\Omega_{\alpha,\epsilon}(t) := \{\mathbf{u}(t) : \mathbf{u} \in \Omega_{\alpha,\epsilon}\} \quad \text{and} \quad \Delta_{\alpha,\epsilon}(t) := \{\mathbf{u}(t) : \mathbf{u} \in \Delta_{\alpha,\epsilon}\}, \quad \forall t \in (0, T).$$

In order to prove some properties of the sets $\Omega_{\alpha,\epsilon}$ and $\Delta_{\alpha,\epsilon}$, we first give some hypotheses on the operators A_1, A_2, B_1, B_2 and J in the STDHVI.

(HAB) For $i, j = 1, 2$ and $j \neq i$, $A_i(\cdot, v_i) : (0, T) \rightarrow V_i^*$ and $B_i(\cdot, v_j) : (0, T) \rightarrow V_i^*$ both are measurable on $(0, T)$ for all $\mathbf{v} = (v_1, v_2) \in V$, and the following hold:

- (1) for a.e. $t \in (0, T)$, $A_1(t, \cdot) : V_1 \rightarrow V_1^*$ is monotone;
- (2) for a.e. $t \in (0, T)$, $A_2(t, \cdot) : V_2 \rightarrow V_2^*$ is monotone;
- (3) for a.e. $t \in (0, T)$, $A_1(t, \cdot) : V_1 \rightarrow V_1^*$ is demicontinuous on V_1 ;
- (4) for a.e. $t \in (0, T)$, $A_2(t, \cdot) : V_2 \rightarrow V_2^*$ is demicontinuous on V_2 ;
- (5) for a.e. $t \in (0, T)$, $B_1(t, \cdot) : V_2 \rightarrow V_1^*$ is demicontinuous on V_2 ;
- (6) for a.e. $t \in (0, T)$, $B_2(t, \cdot) : V_1 \rightarrow V_2^*$ is demicontinuous on V_1 .

(HJ) $J(\cdot, x_1, x_2) : (0, T) \rightarrow \mathbf{R}$ is measurable on $(0, T)$ for all $(x_1, x_2) \in X = X_1 \times X_2$, and the following hold:

- (1) for a.e. $t \in (0, T)$, $J(t, \cdot, \cdot) : X_1 \times X_2 \rightarrow \mathbf{R}$ is locally Lipschitz w.r.t. the first variable and second variable on $X_1 \times X_2$;
- (2) for a.e. $t \in (0, T)$, $J(t, x_1, x_2) + J(t, y_1, y_2) = J(t, x_1, y_2) + J(t, y_1, x_2)$, $\forall (x_1, x_2), (y_1, y_2) \in X_1 \times X_2$.

Lemma 3.7 ([32, Lemma 3.6]). Suppose that the functional $J : (0, T) \times X_1 \times X_2 \rightarrow \mathbf{R}$ satisfies the hypothesis **(HJ)**. Then, for any sequence $\mathbf{x}^n = (x_1^n, x_2^n) \in X_1 \times X_2$ converging strongly to $\mathbf{x} = (x_1, x_2) \in X_1 \times X_2$ and $y_i^n \in X_i$ converging strongly to $y_i \in X_i$, one has

$$\limsup_{n \rightarrow \infty} J_i^\circ(t, x_1^n, x_2^n; y_i^n) \leq J_i^\circ(t, x_1, x_2; y_i), \quad \text{for a.e. } t \in (0, T),$$

where $i = 1, 2$.

Lemma 3.8. Suppose that for $i, j = 1, 2$ and $j \neq i$, $A_i(\cdot, v_i) : (0, T) \rightarrow V_i^*$ and $B_i(\cdot, v_j) : (0, T) \rightarrow V_i^*$ satisfy the hypotheses (1)-(4) in **(HAB)** and $J : (0, T) \times V_1 \times V_2 \rightarrow \mathbf{R}$ satisfies the hypothesis **(HJ)**. Then, $\Omega_{\alpha,\epsilon} = \Delta_{\alpha,\epsilon}$ for all $\epsilon > 0$.

Proof. For a.e. $t \in (0, T)$, by the monotonicity of operators $A_1(t, \cdot) : V_1 \rightarrow V_1^*$ and $A_2(t, \cdot) : V_2 \rightarrow V_2^*$, we can readily see that $\Omega_{\alpha,\epsilon} \subseteq \Delta_{\alpha,\epsilon}$ for any $\epsilon > 0$. Thus we only need to show $\Delta_{\alpha,\epsilon} \subseteq \Omega_{\alpha,\epsilon}$. Indeed, take an arbitrary $\mathbf{u} = (u_1, u_2) \in \Delta_{\alpha,\epsilon}$. Then the following hold for all $\mathbf{v} = (v_1, v_2) \in V$ and a.e. $t \in (0, T)$,

$$\begin{cases} \langle A_1(t, v_1) + B_1(t, u_2(t)) - f_1(t), v_1 - u_1(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(v_1 - u_1(t))) \geq -\epsilon \alpha_1(t, v_1 - u_1(t)), \\ \langle A_2(t, v_2) + B_2(t, u_1(t)) - f_2(t), v_2 - u_2(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(v_2 - u_2(t))) \geq -\epsilon \alpha_2(t, v_2 - u_2(t)). \end{cases} \quad (3.1)$$

For any $\mathbf{w} = (w_1, w_2) \in V_1 \times V_2$, $\lambda \in (0, 1)$, and a.e. $t \in (0, T)$, by letting $v_1 = u_1(t) + \lambda(w_1 - u_1(t))$ and $v_2 = u_2(t) + \lambda(w_2 - u_2(t))$ in (3.1), we deduce from (3.1) and the positive homogeneousness of $\alpha_1(t, \cdot)$ and $\alpha_2(t, \cdot)$ that for a.e. $t \in (0, T)$

$$\begin{cases} \langle A_1(t, u_1(t) + \lambda(w_1 - u_1(t))) + B_1(t, u_2(t)) - f_1(t), \lambda(w_1 - u_1(t)) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); \lambda M_1(w_1 - u_1(t))) \geq -\epsilon \lambda \alpha_1(t, w_1 - u_1(t)), \\ \langle A_2(t, u_2(t) + \lambda(w_2 - u_2(t))) + B_2(t, u_1(t)) - f_2(t), \lambda(w_2 - u_2(t)) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); \lambda M_2(w_2 - u_2(t))) \geq -\epsilon \lambda \alpha_2(t, w_2 - u_2(t)). \end{cases}$$

From property (i) of Proposition 2.7, we know that Clarke's generalized directional derivative is positively homogeneous w.r.t. its direction. So, it follows that

$$\begin{cases} \langle A_1(t, u_1(t) + \lambda(w_1 - u_1(t))) + B_1(t, u_2(t)) - f_1(t), w_1 - u_1(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(w_1 - u_1(t))) \geq -\epsilon \alpha_1(t, w_1 - u_1(t)), \\ \langle A_2(t, u_2(t) + \lambda(w_2 - u_2(t))) + B_2(t, u_1(t)) - f_2(t), w_2 - u_2(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(w_2 - u_2(t))) \geq -\epsilon \alpha_2(t, w_2 - u_2(t)). \end{cases} \quad (3.2)$$

It is easy to see from the hypotheses (3), (4) in **(HAB)** that for a.e. $t \in (0, T)$, operators $A_1(t, \cdot) : V_1 \rightarrow V_1^*$ and $A_2(t, \cdot) : V_2 \rightarrow V_2^*$ both are hemicontinuous. Thus, taking the limit as $\lambda \rightarrow 0^+$ at both sides of two inequalities in (3.2) implies that for a.e. $t \in (0, T)$,

$$\begin{cases} \langle A_1(t, u_1(t)) + B_1(t, u_2(t)) - f_1(t), w_1 - u_1(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(w_1 - u_1(t))) \geq -\epsilon \alpha_1(t, w_1 - u_1(t)), \\ \langle A_2(t, u_2(t)) + B_2(t, u_1(t)) - f_2(t), w_2 - u_2(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(w_2 - u_2(t))) \geq -\epsilon \alpha_2(t, w_2 - u_2(t)), \end{cases}$$

which together with the arbitrariness of $w = (w_1, w_2) \in V_1 \times V_2$, implies that $u \in \Omega_{\alpha, \epsilon}$, and hence $\Delta_{\alpha, \epsilon} \subseteq \Omega_{\alpha, \epsilon}$. This completes the proof. \square

Lemma 3.9. Let $B_1 : (0, T) \times V_2 \rightarrow V_1^*$ satisfy the hypothesis (5) in **(HAB)**, $B_2 : (0, T) \times V_1 \rightarrow V_2^*$ satisfy the hypothesis (6) in **(HAB)**, and $J : (0, T) \times X_1 \times X_2 \rightarrow \mathbf{R}$ satisfy the hypothesis **(HJ)**. Then, for any $\epsilon > 0$ and a.e. $t \in (0, T)$, $\Delta_{\alpha, \epsilon}(t)$ is closed in $V_1 \times V_2$.

Proof. We let $t \in (0, T)$ be fixed. Let $u^n(t) = (u_1^n(t), u_2^n(t)) \in \Delta_{\alpha, \epsilon}(t)$ be a sequence converging strongly to $u(t) = (u_1(t), u_2(t))$ in $V_1 \times V_2$. Then

$$\begin{cases} \langle A_1(t, v_1) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_1(v_1 - u_1^n(t))) \geq -\epsilon \alpha_1(t, v_1 - u_1^n(t)), \quad \forall v_1 \in V_1, \\ \langle A_2(t, v_2) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_2(v_2 - u_2^n(t))) \geq -\epsilon \alpha_2(t, v_2 - u_2^n(t)), \quad \forall v_2 \in V_2. \end{cases} \quad (3.3)$$

Note that by the hypotheses (5), (6) in **(HAB)**, the mappings $B_1(t, \cdot) : V_2 \rightarrow V_1^*$ and $B_2(t, \cdot) : V_1 \rightarrow V_2^*$ both are demicontinuous. So, it follows that $B_1(t, u_2^n(t)) \rightarrow B_1(t, u_2(t))$ and $B_2(t, u_1^n(t)) \rightarrow B_2(t, u_1(t))$. Thus, we get

$$\begin{cases} \lim_{n \rightarrow \infty} \langle A_1(t, v_1) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ \quad = \langle A_1(t, v_1) + B_1(t, u_2(t)) - f_1(t), v_1 - u_1(t) \rangle_{V_1^* \times V_1}, \\ \lim_{n \rightarrow \infty} \langle A_2(t, v_2) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \\ \quad = \langle A_2(t, v_2) + B_2(t, u_1(t)) - f_2(t), v_2 - u_2(t) \rangle_{V_2^* \times V_2}. \end{cases} \quad (3.4)$$

Moreover, by the hypothesis **(HJ)** on the functional J , Lemma 3.7 implies that

$$\begin{cases} \limsup_{n \rightarrow \infty} J_1^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_1(v_1 - u_1^n(t))) \\ \quad \leq J_1^\circ(t, M_1 u_1(t), M_2 u_2(t); M_1(v_1 - u_1(t))), \\ \limsup_{n \rightarrow \infty} J_2^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_2(v_2 - u_2^n(t))) \\ \quad \leq J_2^\circ(t, M_1 u_1(t), M_2 u_2(t); M_2(v_2 - u_2(t))). \end{cases} \quad (3.5)$$

Also, from the continuity of $\alpha_1(t, \cdot)$ and $\alpha_2(t, \cdot)$ we get

$$\lim_{n \rightarrow \infty} \alpha_1(t, v_1 - u_1^n(t)) = \alpha_1(t, v_1 - u_1(t)) \quad \text{and} \quad \lim_{n \rightarrow \infty} \alpha_2(t, v_2 - u_2^n(t)) = \alpha_2(t, v_2 - u_2(t)). \quad (3.6)$$

Therefore, taking the limsup as $n \rightarrow \infty$ at both sides of (3.3), we conclude from (3.4), (3.5) and (3.6) that

$$\begin{cases} \langle A_1(t, v_1) + B_1(t, u_2(t)) - f_1(t), v_1 - u_1(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(v_1 - u_1(t))) \geq -\epsilon \alpha_1(t, v_1 - u_1(t)), \quad \forall v_1 \in V_1, \\ \langle A_2(t, v_2) + B_2(t, u_1(t)) - f_2(t), v_2 - u_2(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(v_2 - u_2(t))) \geq -\epsilon \alpha_2(t, v_2 - u_2(t)), \quad \forall v_2 \in V_2, \end{cases}$$

which implies that $\mathbf{u}(t) = (u_1(t), u_2(t)) \in \Delta_{\alpha, \epsilon}(t)$. Thus $\Delta_{\alpha, \epsilon}(t)$ is closed in $V_1 \times V_2$. This completes the proof. \square

Remark 3.10. Let $A_1 : (0, T) \times V_1 \rightarrow V_1^*$, $A_2 : (0, T) \times V_2 \rightarrow V_2^*$, $B_1 : (0, T) \times V_2 \rightarrow V_1^*$ and $B_2 : (0, T) \times V_1 \rightarrow V_2^*$ satisfy the hypothesis **(HAB)** and $J : (0, T) \times X_1 \times X_2 \rightarrow \mathbf{R}$ satisfy the hypothesis **(HJ)**. Then it can be readily seen from Lemmas 3.8 and 3.9 that $\Omega_{\alpha, \epsilon}(t) = \Delta_{\alpha, \epsilon}(t)$ is closed in $V_1 \times V_2$ for any $\epsilon > 0$ and a.e. $t \in (0, T)$.

Theorem 3.11. Assume that there hold the hypothesis (3) on $A_1 : (0, T) \times V_1 \rightarrow V_1^*$, hypothesis (4) on $A_2 : (0, T) \times V_2 \rightarrow V_2^*$, hypothesis (5) on $B_1 : (0, T) \times V_2 \rightarrow V_1^*$ and hypothesis (6) on $A_2 : (0, T) \times V_1 \rightarrow V_2^*$ in **(HAB)**. Let $J : (0, T) \times X_1 \times X_2 \rightarrow \mathbf{R}$ satisfy the hypothesis **(HJ)**. Then the system STDHVI is strongly α -well-posed if and only if for a.e. $t \in (0, T)$,

$$\Omega_{\alpha, \epsilon}(t) \neq \emptyset, \quad \forall \epsilon > 0 \quad \text{and} \quad \text{diam}(\Omega_{\alpha, \epsilon}(t)) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0.$$

Proof. Necessity. Let STDHVI be strongly α -well-posed. Then STDHVI admits a unique solution $\mathbf{u} = (u_1, u_2) \in \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, i.e., for a.e. $t \in (0, T)$,

$$\begin{cases} \langle A_1(t, u_1(t)) + B_1(t, u_2(t)) - f_1(t), v_1 - u_1(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(v_1 - u_1(t))) \geq 0, \quad \forall v_1 \in V_1, \\ \langle A_2(t, u_2(t)) + B_2(t, u_1(t)) - f_2(t), v_2 - u_2(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(v_2 - u_2(t))) \geq 0, \quad \forall v_2 \in V_2. \end{cases}$$

This implies that $\mathbf{u}(t) \in \Omega_{\alpha, \epsilon}(t)$ for any $\epsilon > 0$ and a.e. $t \in (0, T)$, i.e., $\Omega_{\alpha, \epsilon}(t) \neq \emptyset$ for all $\epsilon > 0$ and a.e. $t \in (0, T)$. If for some $t_0 \in (0, T)$, $\text{diam}(\Omega_{\alpha, \epsilon}(t_0)) \not\rightarrow 0$ as $\epsilon \rightarrow 0$, then there exist

$$\mathbf{u}^n(t_0) = (u_1^n(t_0), u_2^n(t_0)) \in \Omega_{\alpha, \epsilon_n}(t_0), \quad \mathbf{p}^n(t_0) = (p_1^n(t_0), p_2^n(t_0)) \in \Omega_{\alpha, \epsilon_n}(t_0), \quad d > 0,$$

and $0 < \epsilon_n \rightarrow 0$ such that

$$\|\mathbf{u}^n(t_0) - \mathbf{p}^n(t_0)\|_{V_1 \times V_2} = \|u_1^n(t_0) - p_1^n(t_0)\|_{V_1} + \|u_2^n(t_0) - p_2^n(t_0)\|_{V_2} > d. \quad (3.7)$$

Since the definition of $\Omega_{\alpha, \epsilon_n}(t_0)$ implies that $\mathbf{u}^n \in \Omega_{\alpha, \epsilon_n}$ and $\mathbf{p}^n \in \Omega_{\alpha, \epsilon_n}$, by the definition of the α -approximating sequence for STDHVI, $\{\mathbf{u}^n\}$ and $\{\mathbf{p}^n\}$ are two α -approximating sequences for STDHVI. Thus it follows from the strong α -well-posedness of STDHVI that $\mathbf{u}^n(t_0) \rightarrow \mathbf{u}(t_0)$ and $\mathbf{p}^n(t_0) \rightarrow \mathbf{u}(t_0)$ in V , which contradicts (3.7).

Sufficiency. Assume that for a.e. $t \in (0, T)$, $\Omega_{\alpha, \epsilon}(t) \neq \emptyset$, for all $\epsilon > 0$ and $\text{diam}(\Omega_{\alpha, \epsilon}(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$. Then we show that the system of time-dependent hemivariational inequalities STDHVI is strongly α -well-posed. Indeed, suppose that $\{\mathbf{u}^n\}$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ is an α -approximating sequence for STDHVI. Then there exists $0 < \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$\begin{cases} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_1(v_1 - u_1^n(t))) \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \quad \forall v_1 \in V_1, \quad \text{a.e. } t \in (0, T), \\ \langle A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_2(v_2 - u_2^n(t))) \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)), \quad \forall v_2 \in V_2, \quad \text{a.e. } t \in (0, T), \end{cases}$$

which implies $\mathbf{u}^n \in \Omega_{\alpha, \epsilon_n}$ for all $n \geq 1$. Since $\text{diam}(\Omega_{\alpha, \epsilon_n}(t)) \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $t \in (0, T)$, $\{\mathbf{u}^n(t)\}$ is a Cauchy sequence in V for a.e. $t \in (0, T)$. Without loss of generality, we may assume that for a.e. $t \in (0, T)$, $\{\mathbf{u}^n(t)\}$ converges strongly to $\mathbf{u}(t) = (u_1(t), u_2(t))$ in $V = V_1 \times V_2$.

Now, we claim that \mathbf{u} is a unique solution to the system STDHVI of time-dependent hemivariational inequalities. Indeed, since operators $A_1(t, \cdot) + B_1(t, \cdot)$ and $A_2(t, \cdot) + B_2(t, \cdot)$ are demicontinuous on $V_1 \times V_2$ (due to hypotheses (3)-(6) in **(HAB)**) and the functional J satisfies the hypothesis **(HJ)**, we can obtain by the similar arguments to those in (3.4) and (3.5) that for a.e. $t \in (0, T)$,

$$\begin{aligned} & \langle A_1(t, u_1(t)) + B_1(t, u_2(t)) - f_1(t), v_1 - u_1(t) \rangle_{V_1^* \times V_1} + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1(v_1 - u_1(t))) \\ & \geq \lim_{n \rightarrow \infty} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ & \quad + \limsup_{n \rightarrow \infty} J_1^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_1(v_1 - u_1^n(t))) \\ & = \limsup_{n \rightarrow \infty} \{ \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ & \quad + J_1^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_1(v_1 - u_1^n(t))) \} \\ & \geq \lim_{n \rightarrow \infty} -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)) \\ & = 0. \end{aligned}$$

By the similar way, one has for a.e. $t \in (0, T)$,

$$\langle A_2(t, u_2(t)) + B_2(t, u_1(t)) - f_2(t), v_2 - u_2(t) \rangle_{V_2^* \times V_2} + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2(v_2 - u_2(t))) \geq 0.$$

Therefore, \mathbf{u} is a solution to the system STDHVI.

Finally, we show the uniqueness of solutions of the system STDHVI. Suppose that \mathbf{u}' is another solution to the system STDHVI of time-dependent hemivariational inequalities. Since, for any $\epsilon > 0$, $\mathbf{u}, \mathbf{u}' \in \Omega_{\alpha, \epsilon}$,

$$\|\mathbf{u}(t) - \mathbf{u}'(t)\|_{V_1 \times V_2} \leq \text{diam}(\Omega_{\alpha, \epsilon}(t)),$$

which together with the condition $\text{diam}(\Omega_{\alpha, \epsilon}(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$ for a.e. $t \in (0, T)$, implies that $\mathbf{u} = \mathbf{u}'$. This completes the proof. \square

Theorem 3.12. Let $A_1 : (0, T) \times V_1 \rightarrow V_1^*$, $A_2 : (0, T) \times V_2 \rightarrow V_2^*$, $B_1 : (0, T) \times V_2 \rightarrow V_1^*$ and $B_2 : (0, T) \times V_1 \rightarrow V_2^*$ satisfy the hypothesis **(HAB)** and $J : (0, T) \times X_1 \times X_2 \rightarrow \mathbf{R}$ satisfy the hypothesis **(HJ)**. Then the system STDHVI is strongly α -well-posed in the generalized sense if and only if for a.e. $t \in (0, T)$,

$$\Omega_{\alpha, \epsilon}(t) \neq \emptyset, \quad \forall \epsilon > 0 \quad \text{and} \quad \mu(\Omega_{\alpha, \epsilon}(t)) \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0.$$

Proof. Necessity. Suppose that the system STDHVI of time-dependent hemivariational inequalities is strongly α -well-posed in the generalized sense. Then the solution set of the system STDHVI, $S \neq \emptyset$. We define the multi-valued mapping $S(t) : (0, T) \rightarrow 2^V$ as follows:

$$S(t) := \{\mathbf{u}(t) : \mathbf{u} \in S\}, \quad \forall t \in (0, T).$$

This means that, for any $\epsilon > 0$ and a.e. $t \in (0, T)$, $\Omega_{\alpha, \epsilon}(t) \neq \emptyset$ because $S(t) \subseteq \Omega_{\alpha, \epsilon}(t)$. Moreover, we claim here that the set $S(t)$ is compact in V for a.e. $t \in (0, T)$. Indeed, for any sequence $\{\mathbf{u}^n(t)\} \subset S(t)$ with $\mathbf{u}^n(t) = (u_1^n(t), u_2^n(t))$, the corresponding $\{\mathbf{u}^n\} \subset S$ is an α -approximating sequence for STDHVI and thus there exists a subsequence $\{\mathbf{u}^{n_k}\}$ such that for a.e. $t \in (0, T)$, $\mathbf{u}^{n_k}(t) \rightarrow \mathbf{u}(t)$ in V for some $\mathbf{u} \in S$, which implies that $S(t)$ is compact in V for a.e. $t \in (0, T)$. To complete the proof of necessity, we show that $\mu(\Omega_{\alpha, \epsilon}(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$ for a.e. $t \in (0, T)$. It follows from $S(t) \subseteq \Omega_{\alpha, \epsilon}(t)$ for a.e. $t \in (0, T)$ that

$$\mathcal{H}(\Omega_{\alpha, \epsilon}(t), S(t)) = \max\{e(\Omega_{\alpha, \epsilon}(t), S(t)), e(S(t), \Omega_{\alpha, \epsilon}(t))\} = e(\Omega_{\alpha, \epsilon}(t), S(t)).$$

Since the set $S(t)$ is compact for a.e. $t \in (0, T)$, one has for a.e. $t \in (0, T)$,

$$\mu(\Omega_{\alpha,\epsilon}(t)) \leq 2\mathcal{H}(\Omega_{\alpha,\epsilon}(t), S(t)) = 2e(\Omega_{\alpha,\epsilon}(t), S(t)).$$

Now, to show $\mu(\Omega_{\alpha,\epsilon}(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$ for a.e. $t \in (0, T)$, it suffices to show that $e(\Omega_{\alpha,\epsilon}(t), S(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$ for a.e. $t \in (0, T)$. Assume by contradiction that for some $t_0 \in (0, T)$, $e(\Omega_{\alpha,\epsilon}(t_0), S(t_0)) \not\rightarrow 0$ as $\epsilon \rightarrow 0$. Then there exist a constant $l > 0$, a sequence $\{\epsilon_n\} \subset [0, \infty)$ with $\epsilon_n \rightarrow 0$ and $\mathbf{u}^n(t_0) \in \Omega_{\alpha,\epsilon_n}(t_0)$ such that

$$\mathbf{u}^n(t_0) \notin S(t_0) + B(0, l), \quad (3.8)$$

where $B(0, l)$ is the closed ball centered at 0 with radius l . Since $\mathbf{u}^n(t_0) \in \Omega_{\alpha,\epsilon_n}(t_0)$ with $\epsilon_n \rightarrow 0$, the corresponding $\{\mathbf{u}^n\}$ is an α -approximating sequence for STDHVI. So, there exists a subsequence $\{\mathbf{u}^{n_k}\}$ such that for a.e. $t \in (0, T)$, $\mathbf{u}^{n_k}(t) \rightarrow \mathbf{u}(t)$ in V for some $\mathbf{u} \in S$ (due to the strong α -well-posedness in the generalized sense of STDHVI). This is a contradiction to (3.8). Consequently, $\mu(\Omega_{\alpha,\epsilon}(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$ for a.e. $t \in (0, T)$.

Sufficiency. Assume that for a.e. $t \in (0, T)$, $\Omega_{\alpha,\epsilon}(t) \neq \emptyset$, for all $\epsilon > 0$ and $\mu(\Omega_{\alpha,\epsilon}(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$. We show that the system STDHVI of time-dependent hemivariational inequalities is strongly α -well-posed in the generalized sense. Indeed, we observe that for a.e. $t \in (0, T)$,

$$S(t) = \bigcap_{\epsilon > 0} \Omega_{\alpha,\epsilon}(t).$$

Furthermore, since for a.e. $t \in (0, T)$, $\mu(\Omega_{\alpha,\epsilon}(t)) \rightarrow 0$ as $\epsilon \rightarrow 0$ and by Remark 3.10, $\Omega_{\alpha,\epsilon}(t)$ is nonempty and closed for any $\epsilon > 0$, it follows from the theorem in [10] that for a.e. $t \in (0, T)$, $S(t)$ is nonempty compact and

$$e(\Omega_{\alpha,\epsilon}(t), S(t)) = \mathcal{H}(\Omega_{\alpha,\epsilon}(t), S(t)) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.9)$$

Next, to show the strong α -well-posedness in the generalized sense of STDHVI, let $\{\mathbf{u}^n\} \subset \mathcal{V}$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ be an α -approximating sequence for STDHVI. Then there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \rightarrow 0$ such that

$$\begin{cases} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} + J_1^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_1(v_1 - u_1^n(t))) \\ \quad \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \quad \forall v_1 \in V_1, \quad \text{a.e. } t \in (0, T), \\ \langle A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} + J_2^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_2(v_2 - u_2^n(t))) \\ \quad \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)), \quad \forall v_2 \in V_2, \quad \text{a.e. } t \in (0, T), \end{cases}$$

which implies $\mathbf{u}^n(t) \in \Omega_{\alpha,\epsilon_n}(t)$. This together with (3.9), implies that for a.e. $t \in (0, T)$,

$$d(\mathbf{u}^n(t), S(t)) \leq e(\Omega_{\alpha,\epsilon_n}(t), S(t)) \rightarrow 0. \quad (3.10)$$

We claim that there exists a subsequence $\{\mathbf{u}^{n_k}\}$ of $\{\mathbf{u}^n\}$ such that for a.e. $t \in (0, T)$, $\mathbf{u}^{n_k}(t) \rightarrow \hat{\mathbf{u}}(t)$ in V for some $\hat{\mathbf{u}} \in S$. Indeed, suppose that for every subsequence $\{\mathbf{u}^{n_j}\}$ of $\{\mathbf{u}^n\}$, there exists $t_0 \in (0, T)$ such that $\mathbf{u}^{n_j}(t_0) \not\rightarrow \mathbf{u}(t_0)$ in V for all $\mathbf{u} \in S$. Then, from (3.10) and the compactness of $S(t_0)$, it follows that there exists $\bar{\mathbf{u}}^{n_j}(t_0) \in S(t_0)$ such that as $j \rightarrow \infty$

$$\|\mathbf{u}^{n_j}(t_0) - \bar{\mathbf{u}}^{n_j}(t_0)\|_{V_1 \times V_2} = d(\mathbf{u}^{n_j}(t_0), S(t_0)) \leq e(\Omega_{\alpha,\epsilon_{n_j}}(t_0), S(t_0)) \rightarrow 0.$$

Again from the compactness of $S(t_0)$, we may, without loss of generality, that $\bar{\mathbf{u}}^{n_j}(t_0) \rightarrow \bar{\mathbf{u}}(t_0)$ in V for some $\bar{\mathbf{u}} \in S$. So, it follows that

$$\|\mathbf{u}^{n_j}(t_0) - \bar{\mathbf{u}}(t_0)\|_{V_1 \times V_2} \leq \|\mathbf{u}^{n_j}(t_0) - \bar{\mathbf{u}}^{n_j}(t_0)\|_{V_1 \times V_2} + \|\bar{\mathbf{u}}^{n_j}(t_0) - \bar{\mathbf{u}}(t_0)\|_{V_1 \times V_2} \rightarrow 0,$$

that is, $\mathbf{u}^{n_j}(t_0) \rightarrow \bar{\mathbf{u}}(t_0)$ in V for some $\bar{\mathbf{u}} \in S$, which reaches a contradiction. Therefore, the system STDHVI is strongly α -well-posed in the generalized sense. This completes the proof. \square

Remark 3.13. The system SHVI of hemivariational inequalities considered in [32] is extended to develop the system STDHVI of time-dependent hemivariational inequalities without Volterra integral term. Our Lemmas 3.8-3.9 and Theorems 3.11-3.12 improve, extend and develop Lemmas 3.7-3.8 and Theorems 3.10-3.11 in [32] to a great extent because the system STDHVI of time-dependent hemivariational inequalities without Volterra integral term is very different from the system SHVI of hemivariational inequalities considered in [32, Lemmas 3.7-3.8 and Theorems 3.10-3.11].

4. Relations with well-posedness of SIP

In this section, we first introduce systems of inclusion problems in the product space $V_1 \times V_2$ and then define the concept of α -well-posedness for the system of inclusion problems. Moreover, we show the equivalence results between the α -well-posedness of the system of time-dependent hemivariational inequalities and the α -well-posedness of the corresponding system of inclusion problems.

Suppose that, for $i = 1, 2$, $\Gamma_i : (0, T) \times V_1 \times V_2 \rightarrow 2^{V_i^*}$ is a nonempty set-valued mapping. A system of inclusion problems associated with mappings Γ_1 and Γ_2 is defined as follows:

Find $(u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ such that for a.e. $t \in (0, T)$,

$$(\text{SIP}) \quad \begin{cases} 0_1 \in \Gamma_1(t, u_1(t), u_2(t)), \\ 0_2 \in \Gamma_2(t, u_1(t), u_2(t)), \end{cases}$$

where for $i = 1, 2$, $0_i \in V_i^*$ represent the zero element in V_i^* . For simplicity, we use the symbols as follows:

$$\mathbf{u} = (u_1, u_2) \in \mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2, \quad \mathbf{0} = (0_1, 0_2) \in V^* = V_1^* \times V_2^*, \quad \Gamma(t, \mathbf{u}(t)) = (\Gamma_1(t, \mathbf{u}(t)), \Gamma_2(t, \mathbf{u}(t))).$$

This allows us to simplify the system of inclusion problems as follows:

Find $\mathbf{u} \in \mathcal{V}$ such that for a.e. $t \in (0, T)$,

$$\mathbf{0} \in \Gamma(t, \mathbf{u}(t)).$$

Definition 4.1. A sequence $\{\mathbf{u}^n\} \subset \mathcal{V}_1 \times \mathcal{V}_2$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ is called an α -approximating sequence for the system SIP of inclusion problems if there exist $\mathbf{p}^n(t) = (p_1^n(t), p_2^n(t)) \in \Gamma(t, \mathbf{u}^n(t))$, $n \in \mathbf{N}$ for a.e. $t \in (0, T)$ and a nonnegative sequence $\{\epsilon_n\}$ with $\|\mathbf{p}^n(t)\|_{V_1^* \times V_2^*} + \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $t \in (0, T)$, such that for a.e. $t \in (0, T)$,

$$\begin{cases} \langle p_1^n(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), & \forall v_1 \in V_1, \quad n \in \mathbf{N}, \\ \langle p_2^n(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)), & \forall v_2 \in V_2, \quad n \in \mathbf{N}. \end{cases}$$

Definition 4.2. The system SIP is said to be strongly (resp., weakly) α -well-posed if it has a unique solution \mathbf{u} and every α -approximating sequence $\{\mathbf{u}^n\}$ has the property that for a.e. $t \in (0, T)$, $\mathbf{u}^n(t) \rightarrow \mathbf{u}(t)$ (resp., $\mathbf{u}^n(t) \rightharpoonup \mathbf{u}(t)$) in V .

Definition 4.3. The system SIP is said to be strongly (resp., weakly) α -well-posed in the generalized sense if the solution set S of SIP is nonempty and, for every α -approximating sequence $\{\mathbf{u}^n\}$, there always exists a subsequence $\{\mathbf{u}^{n_k}\}$ such that for a.e. $t \in (0, T)$, $\mathbf{u}^{n_k}(t) \rightarrow \mathbf{u}(t)$ (resp., $\mathbf{u}^{n_k}(t) \rightharpoonup \mathbf{u}(t)$) in V for some $\mathbf{u} \in S$.

For $t \in (0, T)$ and $\mathbf{v} = (v_1, v_2) \in V$, we define some operators as follows:

$$\begin{cases} A : (0, T) \times V \rightarrow V^*, & A(t, \mathbf{v}) := (A_1(t, v_1) + B_1(t, v_2), A_2(t, v_2) + B_2(t, v_1)), \\ M : V \rightarrow X = X_1 \times X_2, & M\mathbf{v} := (M_1 v_1, M_2 v_2), \\ f : (0, T) \rightarrow V^*, & f(t) := (f_1(t), f_2(t)). \end{cases}$$

In order to show that the α -well-posedness of the system of time-dependent hemivariational inequalities is equivalent to the α -well-posedness of its corresponding system of inclusion problems, we first give a lemma which establishes the equivalence between the system STDHVI and its derived system of inclusion problems.

Lemma 4.4. Let locally Lipschitz functional $J(t, \cdot, \cdot)$ be regular on X for a.e. $t \in (0, T)$. Then, $\mathbf{u} = (u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ is a solution to the system STDHVI of time-dependent hemivariational inequalities if and only if it solves the following derived system of inclusion problems: Find $\mathbf{u} = (u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ such that for a.e. $t \in (0, T)$,

$$(\mathbf{DSIP}) \quad \begin{cases} f_1(t) \in A_1(t, u_1(t)) + B_1(t, u_2(t)) + M_1^* \circ \partial_1 J(t, M_1 u_1(t), M_2 u_2(t)), \\ f_2(t) \in A_2(t, u_2(t)) + B_2(t, u_1(t)) + M_2^* \circ \partial_2 J(t, M_1 u_1(t), M_2 u_2(t)), \end{cases}$$

where for $i \neq j = 1, 2$, M_i^* is the adjoint operator of M_i , $\partial_i J(t, x_1, x_2)$ denotes the Clarke's generalized gradient of the functional $J(t, \cdot, x_j)$ at x_i .

Proof. First of all, we show the necessity. Indeed, assume that $\mathbf{u} = (u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ is a solution to the system STDHVI of time-dependent hemivariational inequalities, i.e., for all $\mathbf{v} = (v_1, v_2) \in V$ and a.e. $t \in (0, T)$,

$$\begin{cases} \langle A_1(t, u_1(t)) + B_1(t, u_2(t)) - f_1(t), v_1 \rangle_{V_1^* \times V_1} + J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1 v_1) \geq 0, \\ \langle A_2(t, u_2(t)) + B_2(t, u_1(t)) - f_2(t), v_2 \rangle_{V_2^* \times V_2} + J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2 v_2) \geq 0, \end{cases}$$

which is hence equivalent to the following

$$\begin{cases} J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1 v_1) \geq \langle f_1(t) - A_1(t, u_1(t)) - B_1(t, u_2(t)), v_1 \rangle_{V_1^* \times V_1}, \\ J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2 v_2) \geq \langle f_2(t) - A_2(t, u_2(t)) - B_2(t, u_1(t)), v_2 \rangle_{V_2^* \times V_2}. \end{cases} \quad (4.1)$$

Moreover, since $J(t, \cdot, \cdot)$ is regular and M_i is linear bounded for $i = 1, 2$, it follows from Proposition 3.37 in monograph [24] that $J_1^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_1 v_1) = (J(t, \cdot, M_2 u_2(t)) \circ M_1)^\circ(u_1(t); v_1)$ and $\partial(J(t, \cdot, M_2 u_2(t)) \circ M_1)(u_1(t)) = M_1^* \circ \partial_1 J(t, M_1 u_1(t), M_2 u_2(t))$ for all $v_1 \in V_1$ and a.e. $t \in (0, T)$ and that $J_2^\circ(t, M_1(u_1(t)), M_2(u_2(t)); M_2 v_2) = (J(t, M_1 u_1(t), \cdot) \circ M_2)^\circ(u_2(t); v_2)$ and

$$\partial(J(t, M_1 u_1(t), \cdot) \circ M_2)(u_2(t)) = M_2^* \circ \partial_2 J(t, M_1 u_1(t), M_2 u_2(t))$$

for all $v_2 \in V_2$ and a.e. $t \in (0, T)$. So, it follows from (4.1) that for a.e. $t \in (0, T)$,

$$\begin{cases} f_1(t) \in A_1(t, u_1(t)) + B_1(t, u_2(t)) + M_1^* \circ \partial_1 J(t, M_1 u_1(t), M_2 u_2(t)), \\ f_2(t) \in A_2(t, u_2(t)) + B_2(t, u_1(t)) + M_2^* \circ \partial_2 J(t, M_1 u_1(t), M_2 u_2(t)), \end{cases}$$

which implies that $\mathbf{u} = (u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ is the solution to the system DSIP.

Sufficiency. Suppose that $\mathbf{u} = (u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ is a solution to the derived system DSIP of inclusion problems, i.e., for a.e. $t \in (0, T)$,

$$(\mathbf{DSIP}) \quad \begin{cases} f_1(t) \in A_1(t, u_1(t)) + B_1(t, u_2(t)) + M_1^* \circ \partial_1 J(t, M_1 u_1(t), M_2 u_2(t)), \\ f_2(t) \in A_2(t, u_2(t)) + B_2(t, u_1(t)) + M_2^* \circ \partial_2 J(t, M_1 u_1(t), M_2 u_2(t)). \end{cases}$$

Then, there exist $\eta_1(t) \in \partial_1 J(t, M_1 u_1(t), M_2 u_2(t))$ and $\eta_2(t) \in \partial_2 J(t, M_1 u_1(t), M_2 u_2(t))$ such that for a.e. $t \in (0, T)$,

$$f_1(t) = A_1(t, u_1(t)) + B_1(t, u_2(t)) + M_1^* \eta_1(t) \quad \text{and} \quad f_2(t) = A_2(t, u_2(t)) + B_2(t, u_1(t)) + M_2^* \eta_2(t). \quad (4.2)$$

By multiplying the above two equalities in (4.2) with v_1 and v_2 , respectively, we can obtain by the definition of the Clarke's generalized gradient, that for all $\mathbf{v} = (v_1, v_2) \in V_1 \times V_2$ and a.e. $t \in (0, T)$,

$$\begin{aligned} \langle f_1(t), v_1 \rangle_{V_1^* \times V_1} &= \langle A_1(t, u_1(t)) + B_1(t, u_2(t)) + M_1^* \eta_1(t), v_1 \rangle_{V_1^* \times V_1} \\ &= \langle A_1(t, u_1(t)) + B_1(t, u_2(t)), v_1 \rangle_{V_1^* \times V_1} + \langle \eta_1(t), M_1 v_1 \rangle_{V_1^* \times V_1} \\ &\leq \langle A_1(t, u_1(t)) + B_1(t, u_2(t)), v_1 \rangle_{V_1^* \times V_1} + J_1^\circ(t, M_1 u_1(t), M_2 u_2(t); M_1 v_1), \end{aligned}$$

and

$$\begin{aligned}\langle f_2(t), v_2 \rangle_{V_2^* \times V_2} &= \langle A_2(t, u_2(t)) + B_2(t, u_1(t)) + M_2^* \eta_2(t), v_2 \rangle_{V_2^* \times V_2} \\ &= \langle A_2(t, u_2(t)) + B_2(t, u_1(t)), v_2 \rangle_{V_2^* \times V_2} + \langle \eta_2(t), Mv_2 \rangle_{V_2^* \times V_2} \\ &\leq \langle A_2(t, u_2(t)) + B_2(t, u_1(t)), v_2 \rangle_{V_2^* \times V_2} + J_2^\circ(t, M_1 u_1(t), M_2 u_2(t); M_2 v_2).\end{aligned}$$

Therefore, \mathbf{u} is a solution of the system STDHVI. This completes the proof. \square

Remark 4.5. It is easy to simplify the system DSIP as follows: Find $\mathbf{u} \in \mathcal{V}$ such that for a.e. $t \in (0, T)$,

$$0 \in A(t, \mathbf{u}) - f(t) + \Theta(t, \mathbf{u}),$$

where $\Theta(t, \mathbf{v}) := (M_1^* \circ \partial_1 J(t, M_1 v_1, M_2 v_2), M_2^* \circ \partial_2 J(t, M_1 v_1, M_2 v_2))$ for all $\mathbf{v} = (v_1, v_2) \in V$ and $t \in (0, T)$.

Theorem 4.6. Let locally Lipschitz functional $J(t, \cdot, \cdot)$ be regular on X for a.e. $t \in (0, T)$. Then, the system STDHVI of time-dependent hemivariational inequalities is strongly α -well-posed if and only if the derived system DSIP of inclusion problems is strongly α -well-posed.

Proof. Necessity. Suppose that the system STDHVI of time-dependent hemivariational inequalities is strongly α -well-posed. Then there exists a unique $\mathbf{u} = (u_1, u_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ solving STDHVI. It follows from Lemma 4.4 that \mathbf{u} is the unique solution to DSIP. To show the strong α -well-posedness of DSIP, we let $\mathbf{u}^n = (u_1^n, u_2^n)$ be an α -approximating sequence for DSIP. We claim that for a.e. $t \in (0, T)$, $\mathbf{u}^n(t) \rightarrow \mathbf{u}(t)$ in V . Indeed, one knows that there exist $\mathbf{p}^n(t) = (p_1^n(t), p_2^n(t)) \in A(t, \mathbf{u}(t)) - f(t) + \Theta(t, \mathbf{u}^n(t))$, $n \in \mathbf{N}$ for a.e. $t \in (0, T)$ and a nonnegative sequence $\{\epsilon_n\}$ with $\|\mathbf{p}^n(t)\|_{V_1^* \times V_2^*} + \epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $t \in (0, T)$, such that for a.e. $t \in (0, T)$,

$$\begin{cases} \langle p_1^n(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), & \forall v_1 \in V_1, \quad n \in \mathbf{N}, \\ \langle p_2^n(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)), & \forall v_2 \in V_2, \quad n \in \mathbf{N}. \end{cases} \quad (4.3)$$

It is clear that for a.e. $t \in (0, T)$,

$$\begin{cases} p_1^n(t) - A_1(t, u_1^n(t)) - B_1(t, u_2^n(t)) + f_1(t) \in M_1^* \circ \partial_1 J(t, M_1 u_1^n(t), M_2 u_2^n(t)), \\ p_2^n(t) - A_2(t, u_2^n(t)) - B_2(t, u_1^n(t)) + f_2(t) \in M_2^* \circ \partial_2 J(t, M_1 u_1^n(t), M_2 u_2^n(t)). \end{cases}$$

By the similar arguments to those in the proof of Lemma 4.4, one easily obtains that for all $\mathbf{v} = (v_1, v_2) \in V$ and a.e. $t \in (0, T)$,

$$\begin{cases} J_1^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_1(v_1 - u_1^n(t))) \\ \quad \geq \langle -A_1(t, u_1^n(t)) - B_1(t, u_2^n(t)) + f_1(t) + p_1^n(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1}, \\ J_2^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_2(v_2 - u_2^n(t))) \\ \quad \geq \langle -A_2(t, u_2^n(t)) - B_2(t, u_1^n(t)) + f_2(t) + p_2^n(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2}, \end{cases}$$

which together with (4.3), implies that for all $\mathbf{v} = (v_1, v_2) \in V$ and a.e. $t \in (0, T)$,

$$\begin{cases} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} + J_1^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_1(v_1 - u_1^n(t))) \\ \quad \geq \langle p_1^n(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \\ \langle A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} + J_2^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_2(v_2 - u_2^n(t))) \\ \quad \geq \langle p_2^n(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)). \end{cases}$$

This means that $\{\mathbf{u}^n\}$ is an α -approximating sequence for STDHVI. Therefore, it follows from the strong α -well-posedness of STDHVI that for a.e. $t \in (0, T)$, $\mathbf{u}^n(t) \rightarrow \mathbf{u}(t)$ in V . Consequently, DSIP is strongly α -well-posed.

Sufficiency. Let the system DSIP of inclusion problems be strongly α -well-posed. Then there exists a unique solution \mathbf{u} to DSIP, which together with Lemma 4.4, implies that \mathbf{u} is the unique solution to the system STDHVI of time-dependent hemivariational inequalities as well. Let $\{\mathbf{u}^n\}$ be an α -approximating sequence for STDHVI. Then there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\begin{cases} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ \quad + J_1^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_1(v_1 - u_1^n(t))) \\ \quad \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \quad \forall v_1 \in V_1, \quad \text{a.e. } t \in (0, T), \\ \langle A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \\ \quad + J_2^\circ(t, M_1(u_1^n(t)), M_2(u_2^n(t)); M_2(v_2 - u_2^n(t))) \\ \quad \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)), \quad \forall v_2 \in V_2, \quad \text{a.e. } t \in (0, T). \end{cases}$$

By virtue of Proposition 2.7 (v), one observes that for all $\mathbf{v} = (v_1, v_2) \in V$ and a.e. $t \in (0, T)$,

$$\begin{cases} J_1^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_1(v_1 - u_1^n(t))) \\ \quad = \max\{\langle h_1, M_1(v_1 - u_1^n(t)) \rangle_{V_1^* \times V_1} : h_1 \in \partial_1 J(t, M_1 u_1^n(t), M_2 u_2^n(t))\}, \\ J_2^\circ(t, M_1 u_1^n(t), M_2 u_2^n(t); M_2(v_2 - u_2^n(t))) \\ \quad = \max\{\langle h_2, M_2(v_2 - u_2^n(t)) \rangle_{V_2^* \times V_2} : h_2 \in \partial_2 J(t, M_1 u_1^n(t), M_2 u_2^n(t))\}. \end{cases}$$

Thus, for any $\mathbf{v} = (v_1, v_2) \in V$ and a.e. $t \in (0, T)$, there exist

$$h_1(t, u_1^n(t), u_2^n(t), v_1) \in \partial_1 J(t, M_1 u_1^n(t), M_2 u_2^n(t)) \text{ and } h_2(t, u_1^n(t), u_2^n(t), v_2) \in \partial_2 J(t, M_1 u_1^n(t), M_2 u_2^n(t))$$

such that

$$\begin{cases} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ \quad + \langle h_1(t, u_1^n(t), u_2^n(t), v_1), M_1(v_1 - u_1^n(t)) \rangle_{V_1^* \times V_1} \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \\ \langle A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \\ \quad + \langle h_2(t, u_1^n(t), u_2^n(t), v_2), M_2(v_2 - u_2^n(t)) \rangle_{V_2^* \times V_2} \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)). \end{cases} \quad (4.4)$$

According to Proposition 2.7 (iv), one knows that for a.e. $t \in (0, T)$, both $\partial_1 J(t, M_1 u_1^n(t), M_2 u_2^n(t))$ and $\partial_2 J(t, M_1 u_1^n(t), M_2 u_2^n(t))$ are nonempty, convex, bounded and closed in V_1^* and V_2^* , respectively. So, it follows that for a.e. $t \in (0, T)$, the sets $\{A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) + p_1 - f_1(t) : p_1 \in \partial_1 J(t, M_1 u_1^n(t), M_2 u_2^n(t))\}$ and $\{A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) + p_2 - f_2(t) : p_2 \in \partial_2 J(t, M_1 u_1^n(t), M_2 u_2^n(t))\}$ are also nonempty, convex, bounded and closed in V_1^* and V_2^* , respectively. Therefore, for $i = 1, 2$, it follows from (4.4) and Theorem 2.12 with $\varphi_i(t, \cdot) = \epsilon_n \alpha_i(t, \cdot - u_i^n(t))$ that for a.e. $t \in (0, T)$, there exists

$$\mathbf{h}^n(t) = (h_1^n(t), h_2^n(t)) \in \partial_1 J(t, M_1 u_1^n(t), M_2 u_2^n(t)) \times \partial_2 J(t, M_1 u_1^n(t), M_2 u_2^n(t)),$$

which is independent from $\mathbf{v} = (v_1, v_2)$, such that for all $\mathbf{v} = (v_1, v_2) \in V$ and a.e. $t \in (0, T)$,

$$\begin{cases} \langle A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \\ \quad + \langle h_1^n(t), M_1(v_1 - u_1^n(t)) \rangle_{V_1^* \times V_1} \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \\ \langle A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \\ \quad + \langle h_2^n(t), M_2(v_2 - u_2^n(t)) \rangle_{V_2^* \times V_2} \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)). \end{cases} \quad (4.5)$$

So, it follows from (4.5) that for a.e. $t \in (0, T)$,

$$\begin{cases} \langle p_1^n(t), v_1 - u_1^n(t) \rangle_{V_1^* \times V_1} \geq -\epsilon_n \alpha_1(t, v_1 - u_1^n(t)), \quad \forall v_1 \in V_1, \quad n \in \mathbf{N}, \\ \langle p_2^n(t), v_2 - u_2^n(t) \rangle_{V_2^* \times V_2} \geq -\epsilon_n \alpha_2(t, v_2 - u_2^n(t)), \quad \forall v_2 \in V_2, \quad n \in \mathbf{N}, \end{cases} \quad (4.6)$$

where

$$p_1^n(t) = A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t) + M_1^* h_1^n(t),$$

and

$$p_2^n(t) = A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t) + M_2^* h_2^n(t).$$

It is clear that for a.e. $t \in (0, T)$,

$$\begin{cases} p_1^n(t) \in A_1(t, u_1^n(t)) + B_1(t, u_2^n(t)) - f_1(t) + M_1^* \circ \partial_1 J(t, M_1 u_1^n(t), M_2 u_2^n(t)), & \forall n \in \mathbf{N}, \\ p_2^n(t) \in A_2(t, u_2^n(t)) + B_2(t, u_1^n(t)) - f_2(t) + M_2^* \circ \partial_2 J(t, M_1 u_1^n(t), M_2 u_2^n(t)), & \forall n \in \mathbf{N}. \end{cases}$$

That is, $\mathbf{p}^n(t) = (p_1^n(t), p_2^n(t)) \in A(t, \mathbf{u}(t)) - f(t) + \Theta(t, \mathbf{u}^n(t))$, $n \in \mathbf{N}$ for a.e. $t \in (0, T)$.

Next, to show $\|\mathbf{p}^n(t)\|_{V^*} \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $t \in (0, T)$, it is sufficient to show $\|p_i^n(t)\|_{V_i^*} \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $t \in (0, T)$ where $i = 1, 2$. Indeed, let $t \in (0, T)$ be fixed. Then, $\|p_i^n(t)\|_{V_i^*} \rightarrow 0$ as $n \rightarrow \infty$ if and only if for any $\varepsilon > 0$ there exists an integer $N \geq 1$ such that $\|p_i^n(t)\|_{V_i^*} < \varepsilon$ for all $n \geq N$. If for some $t_0 \in (0, T)$, $\|p_i^n(t_0)\|_{V_i^*} \not\rightarrow 0$ as $n \rightarrow \infty$, then there exists $\varepsilon_i > 0$ and for each $k \geq 1$ there exists $p_i^{n_k}(t_0)$ such that

$$\|p_i^{n_k}(t_0)\|_{V_i^*} \geq \varepsilon_i.$$

Note that V_i is reflexive, i.e., $V_i = V_i^{**}$. According to the normalized duality mapping \mathcal{J}_i from V_i^* to its dual $V_i^{**}(= V_i)$ defined by

$$\mathcal{J}_i(v_i) = \{x_i \in V_i : \langle v_i, x_i \rangle_{V_i^* \times V_i} = \|v_i\|_{V_i^*}^2 = \|x_i\|_{V_i}^2\}, \quad \forall v_i \in V_i^*,$$

we know that for each $n \in \mathbf{N}$ there exists $j_i(p_i^n(t_0)) \in \mathcal{J}_i(p_i^n(t_0))$ such that

$$\langle p_i^n(t_0), j_i(p_i^n(t_0)) \rangle_{V_i^* \times V_i} = \|p_i^n(t_0)\|_{V_i^*}^2 = \|j_i(p_i^n(t_0))\|_{V_i}^2.$$

Putting $t = t_0$ and $v_i = u_i^n(t_0) - j_i(p_i^n(t_0))$ in (4.6), we get

$$\langle p_i^n(t_0), -j_i(p_i^n(t_0)) \rangle_{V_i^* \times V_i} \geq -\varepsilon_n \alpha_i(t_0, -j_i(p_i^n(t_0))), \quad \forall n \in \mathbf{N},$$

which immediately implies that

$$-\|p_i^n(t_0)\|_{V_i^*}^2 \geq -\varepsilon_n \alpha_i(t_0, -j_i(p_i^n(t_0))), \quad \forall n \in \mathbf{N}. \quad (4.7)$$

Taking into account $\|\frac{j_i(p_i^{n_k}(t_0))}{\|p_i^{n_k}(t_0)\|_{V_i^*}}\|_{V_i} = 1$ for all $k \geq 1$, we obtain that $\|\varepsilon_{n_k} \frac{j_i(p_i^{n_k}(t_0))}{\|p_i^{n_k}(t_0)\|_{V_i^*}}\|_{V_i} \rightarrow 0$ as $k \rightarrow \infty$. Utilizing the continuity and positive homogeneity of $\alpha_i(t_0, \cdot)$, we deduce from (4.7) that

$$\begin{aligned} \varepsilon_i &\leq \|p_i^{n_k}(t_0)\|_{V_i^*} \leq \frac{\varepsilon_{n_k}}{\|p_i^{n_k}(t_0)\|_{V_i^*}} \alpha_i(t_0, -j_i(p_i^{n_k}(t_0))) \\ &= \alpha_i(t_0, -\varepsilon_{n_k} \frac{j_i(p_i^{n_k}(t_0))}{\|p_i^{n_k}(t_0)\|_{V_i^*}}) \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

which reaches a contradiction. This means that $\|p_i^n(t)\|_{V_i^*} \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $t \in (0, T)$ where $i = 1, 2$. Hence, the sequence $\{\mathbf{u}^n\}$ with $\mathbf{u}^n = (u_1^n, u_2^n)$ is an α -approximating sequence for DSIP. So, it follows from the strong α -well-posedness of DSIP that $\mathbf{u}^n(t) \rightarrow \mathbf{u}(t)$ in V for a.e. $t \in (0, T)$. Therefore, the system of time-dependent hemivariational inequalities STDHVI is strongly α -well-posed. This completes the proof. \square

Utilizing the similar arguments to those in the proof of Theorem 4.6, one can easily prove the following equivalence between the strong α -well-posedness in the generalized sense of STDHVI and the strong α -well-posedness in the generalized sense of the system DSIP.

Theorem 4.7. *Let locally Lipschitz functional $J(t, \cdot, \cdot)$ be regular on X for a.e. $t \in (0, T)$. Then, the system STDHVI of time-dependent hemivariational inequalities is strongly α -well-posed in the generalized sense if and only if the derived system DSIP of inclusion problems is strongly α -well-posed in the generalized sense.*

Remark 4.8. Compared with [32, Theorems 4.5-4.6], our Theorems 4.6 and 4.7 use the system STDHVI of time-dependent hemivariational inequalities in place of the system SHVI of hemivariational inequalities, the derived system DSIP corresponding to STDHVI in place of the derived system DSIP corresponding to SHVI and the α -well-posedness (resp., the α -well-posedness in the generalized sense) in place of the well-posedness (resp., the well-posedness in the generalized sense), respectively. All in all, our Theorems 4.6 and 4.7 improve, extend and develop [32, Theorems 4.5-4.6] to a great extent.

5. Concluding remarks

The present paper generalizes the concept of α -well-posedness to a system STDHVI of time-dependent hemivariational inequalities without Volterra integral terms in Banach spaces. We give several definitions of α -well-posedness and, with assumptions (HAB), (HJ) on the operators involved in STDHVI, give some metric characterizations of α -well-posedness for STDHVI considered. On the other hand, by establishing an equivalence result between the system STDHVI and a derived system DSIP of inclusion problems, we prove that the strong α -well-posedness (resp., in the generalized sense) of STDHVI is equivalent to the strong α -well-posedness (resp., in the generalized sense) of DSIP.

Several problems related to the α -well-posedness of systems of time-dependent hemivariational inequalities remain to be considered in the future study. The first one is to exploit some conditions under which the strong (resp., weak) α -well-posedness of systems of time-dependent hemivariational inequalities is equivalent to the existence and uniqueness of their solutions. The second one is to generalize the study of well-posedness (including α -well-posedness) to systems of time-dependent hemivariational inequalities involving both nonsmooth functionals and proper, convex and lower semicontinuous functionals, which are referred to as systems of time-dependent variational-hemivariational inequalities. In addition, it is well-known that there are many other concepts of well-posedness in the literature on optimization problems and variational inequalities, such as well-posedness by perturbations [2, 5] and Levitin-Polyak well-posedness [3, 7, 18, 19]. Without question, extending these concepts of well-posedness to the study of systems of time-dependent hemivariational inequalities is very interesting and quite valuable in the future.

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