**Boundedness of high order commutators of Marcinkiewicz integrals associated with Schrödinger operators**

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**Abstract**

Suppose \( L = -\Delta + V \) is a Schrödinger operator on \( \mathbb{R}^n \), where \( n \geq 3 \) and the nonnegative potential \( V \) belongs to reverse Hölder class \( RH_q \). Let \( b \) belong to a new Campanato space \( \Lambda^\theta_\beta(\rho) \), and let \( \mu^L_j \) be the Marcinkiewicz integrals associated with \( L \). In this paper, we establish the boundedness of the \( m \)-order commutators \([b^m, \mu^L_j]\) from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \), where \( 1/q = 1/p - m\beta/n \) and \( 1 < p < n/(m\beta) \). As an application, we obtain the boundedness of \([b^m, \mu^L_j]\) on the generalized Morrey spaces related to certain nonnegative potentials. ©2017 All rights reserved.

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1. Introduction and results

In this paper we consider the Schrödinger operator

\[ L = -\Delta + V \quad \text{on} \quad \mathbb{R}^n, \quad n \geq 3, \]

where \( V \) is a nonnegative potential. We will assume that \( V \) belongs to a reverse Hölder class \( RH_q \) for some \( q \geq n/2 \), that is to say, \( V \) satisfies the reverse Hölder inequality

\[ \left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq C \int_B V(y) \, dy \]

for any balls \( B \subset \mathbb{R}^n \).

As in [10], for a given potential \( V \in RH_q \) with \( q \geq n/2 \), we define the auxiliary function

\[ \rho(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) \, dy \leq 1 \right\}, \quad x \in \mathbb{R}^n. \]

It is well-known that \( 0 < \rho(x) < \infty \) for any \( x \in \mathbb{R}^n \).

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Let $\theta > 0$ and $0 < \beta < 1$, in view of [6], the new Campanato class $\Lambda_{\beta}^{\theta}(\rho)$ consists of the locally integrable functions $b$ such that
\[
\frac{1}{|B(x, r)|^{1+\beta/n}} \int_{B(x, r)} |b(y) - b_b| dy \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta
\]
for all $x \in \mathbb{R}^n$ and $r > 0$. A seminorm of $b \in \Lambda_{\beta}^{\theta}(\rho)$, denoted by $|b|_{\beta, \rho}^{\theta}$, is given by the infimum of the constants in the inequalities above.

Note that if $\theta = 0$, $\Lambda_{\beta}^{\theta}(\rho)$ is the classical Campanato space; if $\beta = 0$, $\Lambda_{\beta}^{\theta}(\rho)$ is exactly the space $\text{BMO}_{\rho}(\rho)$ introduced in [1].

We define the Marcinkiewicz integral associated with the Schrödinger operator $L$ by
\[
\mu^L_f(x) = \left(\int_0^\infty \left\| \int_{|x-y| \leq t} K^L_j(x, y) f(y) dy \right\|^2 \frac{dt}{t^3}\right)^{1/2},
\]
where $K_j^L(x, y) = \tilde{K}_j^L(x, y) |x-y|$ and $\tilde{K}_j^L(x, y)$ is the kernel of $R_j^L = \frac{\partial}{\partial x_j} A^{-1/2}$. Then $K_{\beta}^L(x, y) = \tilde{K}_{\beta}^L(x, y) |x-y| = \frac{(x-y_j)/|x-y|}{|x-y|^{n-1}}$. Obviously, $\mu^L_f(x)$ is the classical Marcinkiewicz integral. Therefore, it will be an interesting thing to study the property of $\mu^L$.

The area of Marcinkiewicz integral associated with the Schrödinger operator has been under intensive research recently. Gao and Tang in [5] showed that $\mu^L$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and bounded from $L^1(\mathbb{R}^n)$ to weak $L^1(\mathbb{R}^n)$, when $b$ belongs to $\text{BMO}_{\rho}(\rho)$, Chen and Zou in [3] proved that the commutator $[b, \mu^L]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chen and Jin in [2] investigated the boundedness of $[b, \mu^L]$ on some Morrey space related to nonnegative potential $V$. In this paper, we consider the boundedness of $m$-order commutator $[b^m, \mu^L]$ on $L^p(\mathbb{R}^n)$ when $b$ belongs to the new Campanato class $\Lambda_{\beta}^{\theta}(\rho)$, and get the following result.

**Theorem 1.1.** Let $V \in RH_n$. Then for any $b \in \Lambda_{\beta}^{\theta}(\rho), 0 < \beta < 1$, the commutator $[b^m, \mu^L]$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, and
\[
||[b^m, \mu^L]f||_{L^q(\mathbb{R}^n)} \leq C(|b|_{\beta, \rho}^{\theta}m||f||_{L^p(\mathbb{R}^n)}),
\]
where $\frac{1}{q} = \frac{1}{p} - \frac{m\beta}{n}, 1 < p < \frac{n}{m\beta}$.

The classical Morrey space was introduced by Morrey in [8], since then a large number of investigations have been given to them by mathematicians. It is well-known that the classical Morrey space plays an important role in the theory of partial differential equations. In [2], Chen and Jin showed the boundedness of $[b, \mu^L]$ on the Morrey spaces related to certain nonnegative potentials. In [9], we introduced the generalized Morrey space related to nonnegative potential $V$, which covers the general Morrey space; see [2, 7, 8, 11].

**Definition 1.2** ([9]). Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty), 1 \leq p < \infty, \alpha \geq 0$, and $V \in RH_q(q \geq n/2)$. We denote by $M^{\varphi}_{\alpha, \rho}(\mathbb{R}^n)$ the generalized Morrey space related to nonnegative potential $V$, the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm
\[
||f||_{M^{\varphi}_{\alpha, \rho}} = \sup_{x \in \mathbb{R}^n, r > 0} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x, r)^{-1} r^{-n/p} ||f||_{L^p(B(x, r))}.
\]
As an application of Theorem 1.1, we consider the boundedness of \([b^m, \mu^L]\) on \(M^{\alpha, \nu}_{p, \varphi}(\mathbb{R}^n)\), get the following result.

**Theorem 1.3.** Let \(V \in RH_n, b \in \Lambda^D_\beta(\rho), 0 < \beta < 1, \text{ and } (\varphi_1, \varphi_2)\) satisfies the condition

\[
\int_r^\infty \text{ess inf}_{t < s < \infty} \varphi_1(x, s)s^n dt \leq c_0 \varphi_2(x, r), \quad (1.1)
\]

where \(c_0\) does not depend on \(x\) and \(r\). Then the operator \([b^m, \mu^L]\) is bounded from \(M^{\alpha, \nu}_{p, \varphi_1}(\mathbb{R}^n)\) to \(M^{\alpha, \nu}_{q, \varphi_2}(\mathbb{R}^n)\), and

\[
\|([b^m, \mu^L]f)\|_{M^{\alpha, \nu}_{p, \varphi_2}} \leq C([b]_\beta)^m \|f\|_{M^{\alpha, \nu}_{p, \varphi_1}},
\]

where \(\frac{1}{q} = \frac{1}{p} - \frac{m\beta}{n}, 1 < p < \frac{n}{m\beta} .

In this paper, we shall use the symbol \(A \lesssim B\) to indicate that there exists a universal positive constant \(C\), independent of all important parameters, such that \(A \leq CB. A \approx B\) means that \(A \lesssim B\) and \(B \lesssim A\).  

2. Some preliminaries

**Proposition 2.1 ([10]).** Let \(V \in RH_{n/2}\). For the function \(\rho\) there exist \(C\) and \(k_0 \geq 1\) such that

\[
C^{-1} \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{-k_0} \leq \rho(y) \leq C \rho(x) \left(1 + \frac{|x - y|}{\rho(x)}\right)^{k_0}
\]

for all \(x, y \in \mathbb{R}^n\).

Assume that \(Q = B(x_0, \rho(x_0))\), for \(x \in Q\), Proposition 2.1 tells us that \(\rho(x) \approx \rho(y)\), if \(|x - y| < \rho(x)|.\)

**Lemma 2.2 ([11]).** Let \(k \in \mathbb{N}\) and \(x \in 2^{k+1}B(x_0, r) \setminus 2^kB(x_0, r)\). Then we have

\[
\frac{1}{\left(1 + \frac{2^{kr}}{\rho(x)}\right)^N} \lesssim \frac{1}{\left(1 + \frac{2^r}{\rho(x)}\right)^{N/(k_0 + 1)}},
\]

**Proposition 2.3 ([4]).** There exists a sequence of points \(\{x_k\}_{k=1}^\infty\) in \(\mathbb{R}^n\), so that the family of critical balls \(Q_k = B(x_k, \rho(x_k))\), \(k \geq 1\), satisfies

(i) \(\bigcup_k Q_k = \mathbb{R}^n\);

(ii) there exists \(N = N(\rho)\) such that for every \(k \in \mathbb{N}\), \(\text{card}\{j : 4Q_j \cap 4Q_k\} \leq N\).

For \(\alpha > 0\), \(g \in L^1_{\text{loc}}(\mathbb{R}^n)\), and \(x \in \mathbb{R}^n\), we introduce the following maximal functions

\[
M_{p, \alpha}g(x) = \sup_{x \in B \subseteq B_{p, \alpha}} \frac{1}{|B|} \int_B |g(y)| dy,
\]

\[
M^z_{p, \alpha}g(x) = \sup_{x \in B \subseteq B_{p, \alpha}} \frac{1}{|B|} \int_B |g(y) - g_B| dy,
\]

where \(B_{p, \alpha} = \{B(z, r) : z \in \mathbb{R}^n\text{ and } r \leq \alpha \rho(y)\}\).

We have the following Fefferman-Stein type inequality.

**Proposition 2.4 ([1]).** For \(1 < p < \infty\), there exist \(\delta\) and \(\beta\) such that if \(\{Q_k\}_{k=1}^\infty\) is a sequence of balls as in Proposition 2.3 then

\[
\int_{\mathbb{R}^n} |M_{p, \delta}g(x)|^p dx \lesssim \int_{\mathbb{R}^n} |M^z_{p, \beta}g(x)|^p dx + \sum_k |Q_k| \left(\frac{1}{|Q_k|} \int_{2Q_k} |g|\right)^p
\]

for all \(g \in L^1_{\text{loc}}(\mathbb{R}^n)\).
We give an inequality for the function $b \in \Lambda^0_{\beta}(\rho)$.

**Lemma 2.5 ([6]).** Let $1 \leq s < \infty, b \in \Lambda^0_{\beta}(\rho)$, and $B = B(x, r)$. Then
\[
\left(\frac{1}{|2kB|} \int_{2kB} |b(y) - b_B|^s dy\right)^{1/s} \leq C|b|^0_{\beta}(2^k r)^{\theta'} \left(1 + \frac{2^k r}{\rho(x)}\right)^{\theta''}
\]
for all $k \in \mathbb{N}$, where $\theta' = (k_0 + 1)\theta$ and $k_0$ is the constant appearing in Proposition 2.1.

The following proposition gives some estimates on the kernel of $\mu^1_i$.

**Proposition 2.6 ([10]).** Suppose $V \in RH_q$.

(i) If $q \geq n$, then for every $N$, there exists a constant $C$ such that
\[
|K_{\beta}^1(x, z)| \leq \frac{C \left(1 + |x - z|/\rho(x)\right)^{-N}}{|x - z|^{n-1}}.
\]

(ii) If $q \geq n$, then for every $N$ and $0 < \delta < 1 - n/q$, there exists a constant $C$ such that
\[
|K_{\beta}^1(x, z) - K_{\beta}^1(y, z)| \leq \frac{C|x - y|^\delta \left(1 + |x - z|/\rho(x)\right)^{-N}}{|x - z|^{n-1+\delta}},
\]
where $|x - y| < \frac{2}{3}|x - z|$.

### 3. Proof of Theorem 1.1

We first prove the following lemmas.

**Lemma 3.1.** Let $V \in RH_n, b \in \Lambda^0_{\beta}(\rho)$, and $Q = B(x_0, \rho(x_0))$. Then for any $1 < s < \infty$,
\[
\frac{1}{|Q|^s} \int_Q \left(||b^m \mu^1_i|| f\right)^s \leq \left(||b^0_{\beta}|| \inf_{x \in Q} M_{m, \beta, s}(f)(x) + \sum_{\gamma=0}^{m-1} ||b^\gamma_{\beta}|| \inf_{x \in Q} M_{(m-\gamma)\beta, s}(b^\gamma, \mu^1_i) f(x)\right)^s
\]
holds for all $f \in L^s(\mathbb{R}^n)$, where
\[
M_{m, \beta, s}(f)(x) = \sup_{x \in B} \left(\frac{1}{|B|^{1+m\beta s/n}} \int_B |f(y)|^s dy\right)^{1/s}.
\]

**Proof.** By Binomial Theorem we have
\[
(b(y) - b(z))^m = \sum_{l=1}^{m} C_{l, m}(b(y) - \lambda)^l (\lambda - b(z))^{m-l} + (\lambda - b(z))^m
\]
\[
= \sum_{l=1}^{m} C_{l, m}(b(y) - \lambda)^l (\lambda - b(y) + b(y) - b(z))^{m-l} + (\lambda - b(z))^m
\]
\[
= \sum_{l=1}^{m} \sum_{h=0}^{m-l} C_{l, m, h}(b(y) - \lambda)^{l+h} (b(y) - b(z))^{m-l-h} + (\lambda - b(z))^m
\]
\[
= \sum_{\gamma=0}^{m-1} C_{\gamma, m}(b(y) - \lambda)^{m-\gamma} (b(y) - b(z))^\gamma + (\lambda - b(z))^m,
\]
then

\[ [b^m, \mu^l]f(y) = \left( \int_0^\infty \left| \int_{|y-z| \leq t} K_t^y(y, z)(b(y) - b(z))^m f(z) \, dz \right|^2 \frac{dt}{t^3} \right)^{1/2} \]

\[ \leq \sum_{\gamma=0}^{m-1} C_{\gamma, m} |b(y) - \lambda|^{m-\gamma} |b^\gamma, \mu^l| f(y) + \mu^l((b - \lambda)^m f)(y). \]

Let \( \lambda = b_{2Q} \). Then by Hölder’s inequality and Lemma 2.5 we get

\[ \frac{1}{|Q|} \int_Q \left| \sum_{\gamma=0}^{m-1} C_{\gamma, m} (b(y) - \lambda)^{m-\gamma} |b^\gamma, \mu^l| f(y) \right| \, dy \]

\[ \leq \sum_{\gamma=0}^{m-1} \left( \frac{1}{|Q|} \int_Q |b(y) - b_{2Q}|^{m-\gamma} |b^\gamma, \mu^l| f(y) \, dy \right)^{1/s'} \left( \frac{1}{|Q|} \int_Q ||b^\gamma, \mu^l|| f(y) || \, dy \right)^{1/s} \]

\[ \leq \sum_{\gamma=0}^{m-1} \left( |b|^\gamma \right)^{m-\gamma} \inf_{x \in Q} M_{m-\gamma, \beta, s}(|b^\gamma, \mu^l| f)(x), \]

where \( 1 < s < \infty \), and \( 1/s + 1/s' = 1 \).

For the second term, we split \( f = f_1 + f_2 \) with \( f_1 = f_{\chi_{2Q}} \). Let \( 1 < \tilde{s} < s < \infty \), and \( \nu = s \tilde{s}/(s - \tilde{s}) \), by the boundedness of \( \mu^l \) on \( L^\infty(\mathbb{R}^n) \), Hölder’s inequality, and Lemma 2.5 we obtain

\[ \frac{1}{|Q|} \int_Q |\mu^l((b - b_{2Q})^m f_1)(y)| \, dy \leq \left( \frac{1}{|Q|} \int_Q |\mu^l((b - b_{2Q})^m f_1)(y)|| \, dy \right)^{1/s} \]

\[ \leq \left( \frac{1}{|Q|} \int_{2Q} |(b(y) - b_{2Q})^m f(y)|| \, dy \right)^{1/s} \]

\[ \leq \left( \frac{1}{|Q|} \int_{2Q} |f(y)|| \, dy \right)^{1/s} \left( \frac{1}{|Q|} \int_{2Q} |b(y) - b_{2Q}|^\nu \, dy \right)^{1/\nu} \]

\[ \leq (|b|^\beta)^m \inf_{x \in Q} M_{m, \beta, s} f(x). \]

For the remaining term, note that \( \rho(y) \approx \rho(x_0) \) for any \( y \in Q \), by Proposition 2.6, Minkowski’s inequality, and Lemma 2.5 we get

\[ |\mu^l((b - b_{2Q})^m f_2)(y)| \leq \left( \int_0^\infty \left| \int_{|y-z| \leq t} |f_2(z)||b(z) - b_{2Q}|^m \, dz \right|^2 \frac{dt}{t^3} \right)^{1/2} \]

\[ \leq \int_{\mathbb{R}^n} \left| \frac{f_2(z)||b(z) - b_{2Q}|^m}{|y-z|^{n-1}(1 + |y-z|/\rho(y))^N} \left( \int_{|y-z| \leq t} \frac{dt}{t^3} \right)^{1/2} \right| \, dz \]

\[ \leq \rho(x_0)^N \left( \int_{2Q} \left| \frac{f(z)||b(z) - b_{2Q}|^m}{|y-z|^{n+N}} \right| \, dz \right) \]

\[ \leq \rho(x_0)^N \int_0^\infty \left( \frac{2k\rho(x_0)}{2^{k+1}Q} \right)^{-N} \left( \int_{2k+1Q \setminus 2kQ} |f(z)||b(z) - b_{2Q}|^m \, dz \right) \]
\[ \leq \sum_{k=1}^{\infty} 2^{-kn} \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)||b(z) - b_{2B}|^m \, dz \right) \]

This finishes the proof of Lemma 3.1. \( \square \)

**Lemma 3.2.** Let \( V \in \text{RH}_1 \) and \( b \in \Lambda_{\rho}^0 (\rho) \), then for any \( s > 1 \) and \( \gamma \geq 1 \), there exists a constant \( C \) such that

\[ |\mu_1^s ((b - b_B)^m f_2)(u) - \mu_1^s ((b - b_B)^m f_2)(z)| \leq C (|b|_{B_\rho}^m) \inf_{x \in B} M_{m, \beta, s} f(x) \]

holds for all \( f \in L_{\text{loc}}^1 (\mathbb{R}^n) \), \( u, z \in B = B(x_0, r) \) with \( r < \gamma \rho(x_0) \) and \( f_2 = f \chi_{(2B)^c} \).

**Proof.** We write

\[ |\mu_1^s ((b - b_B)^m f_2)(u) - \mu_1^s ((b - b_B)^m f_2)(z)| \]

\[ \leq \left( \int_0^\infty \left( \int_{|u - y| < t < |z - y|} |K_1^s (u, y) f_2(y)(b(y) - b_{2B})|^m \, dy \right) \frac{dt}{t^3} \right)^{1/2} \]

\[ + \left( \int_0^\infty \left( \int_{|z - y| < t < |u - y|} |K_1^s (u, y) f_2(y)(b(y) - b_{2B})|^m \, dy \right) \frac{dt}{t^3} \right)^{1/2} \]

\[ + \left( \int_0^\infty \left( \int_{|u - y| < t < |z - y| < t} |K_1^s (u, y) - K_1^s (z, y)||f_2(y)(b(y) - b_{2B})|^m \, dy \right) \frac{dt}{t^3} \right)^{1/2} \]

\[ = J_1 + J_2 + J_3. \]

Due to the estimates for \( J_1 \) and \( J_2 \) which are similar, then we only consider \( J_1 \). Let \( Q = B(x_0, \gamma \rho(x_0)) \). Since \( u, z \in Q \), then \( \rho(u) \approx \rho(x_0) \) and \( |u - y| \approx |z - y| \). By Minkowski’s inequality and Proposition 2.6 we have

\[ J_1 \leq \int_{(2B)^c} |K_1^s (u, y) f(y)(b(y) - b_{2B})|^m \left( \int_{|u - y| < t < |z - y|} \frac{dt}{t^3} \right)^{1/2} \, dy \]

\[ \leq \frac{1}{2^{j_0} B} \int_{(2B)^c} |K_1^s (u, y) f(y)(b(y) - b_{2B})|^m \, dy \]

\[ \leq \frac{1}{2^{j_0} B} \int_{Q \setminus 2B} \frac{|f(y)||b(y) - b_{2B}|^m}{|u - y|^{n+1/2}} \, dy + \frac{1}{2^{j_0} B} \int_{Q \setminus 2B} \frac{|f(y)||b(y) - b_{2B}|^m}{|u - y|^{n+1/2} + N} \, dy \]

\[ = J_{11} + J_{12}. \]

Let \( j_0 \) be the least integer such that \( 2^{j_0} \geq \gamma \rho(x_0)/r \). Splitting into annuli, we have

\[ J_{11} \leq \sum_{j=2}^{j_0} \frac{2^{-j/2}}{2^{j} B} \int_{2^{j} B} |f(y)||b(y) - b_{2B}|^m \, dy. \]
By Hölder’s inequality, Lemma 2.5, and noting that $2^j r < \gamma \rho(x_0)$ for $j < j_0$, then we have

$$
\frac{1}{|2B|} \int_{2B} |f(y)||b(y)| - b_{2B}|^m dy \lesssim ((b)^0_\beta)^m (2^j r)^m \rho(x_0) \left( \frac{1}{\rho(x_0)} \right)^{m \theta'} \left( \frac{1}{|2B|} \int_{2B} |f(y)|^s dy \right)^{1/s} \lesssim ((b)^0_\beta)^m \inf_{x \in B} M_{m \beta, s} f(x).
$$

Thus

$$
J_{11} \lesssim ((b)^0_\beta)^m \sum_{j=2}^{j_0} 2^{-j/2} \inf_{x \in B} M_{m \beta, s} f(x) \lesssim ((b)^0_\beta)^m \inf_{x \in B} M_{m \beta, s} f(x).
$$

Note that for $j \geq j_0$,

$$
\frac{1}{|2B|} \int_{2B} |f(y)||b(y)| - b_{2B}|^m dy \lesssim ((b)^0_\beta)^m (2^j r)^m \rho(x_0) \left( \frac{1}{\rho(x_0)} \right)^{m \theta'} \left( \frac{1}{|2B|} \int_{2B} |f(y)|^s dy \right)^{1/s} \lesssim ((b)^0_\beta)^m \left( \frac{2^j r}{\rho(x_0)} \right)^{m \theta'} \inf_{x \in B} M_{m \beta, s} f(x).
$$

Then, by choosing $N \geq m \theta'$ we get

$$
J_{12} \leq \rho(x_0)^N \int_{Q_\varepsilon} \left| f(y)||b(y)| - b_{2B}|^m \right|_{|u - y|^{n+1/2+N}} dy \leq \rho(x_0)^N \sum_{j=j_0}^{\infty} 2^{-j(1/2+N)} \frac{1}{|2B|} \int_{2B} |f(y)||b(y)| - b_{2B}|^m dy \lesssim ((b)^0_\beta)^m \sum_{j=j_0}^{\infty} 2^{-j/2} \left( \frac{\rho(x_0)}{2^j r} \right)^{N-m \theta'} \inf_{x \in B} M_{m \beta, s} f(x) \lesssim ((b)^0_\beta)^m \inf_{x \in B} M_{m \beta, s} f(x).
$$

For $J_3$, note that $\rho(u) = \rho(x_0), |u - y| \approx |z - y|$, and $|u - z| < \frac{1}{2} |u - y|$, then by Minkowski’s inequality and Proposition 2.6, similar to the estimates for $J_{11}$ and $J_{12}$, we have

$$
J_3 \leq \int_{(2B)^c} \left( \frac{\rho(y)}{|u - y|^{n+\delta + N}} \right) \left( \int_{|u - y| \leq t, |z - y| \leq t} \frac{dt}{t^\gamma} \right)^{1/2} dy \lesssim \frac{1}{|u - y|^{n+\delta}} \int_{(2B)^c} \left| f(y)||b(y)| - b_{2B}|^m \right| \left| K^\gamma \left( \frac{t}{r} \right) \right| dy \lesssim r^\delta \rho(x_0)^N \int_{Q_{\varepsilon}^{2B}} \left| f(y)||b(y)| - b_{2B}|^m \right|_{|u - y|^{n+\delta+N}} dy \lesssim ((b)^0_\beta)^m \inf_{x \in B} M_{m \beta, s} f(x).
$$

Then the proof of Lemma 3.2 is completed.

\[\square\]

**Lemma 3.3.** Let $s > 1, B = B(x_0, r)$ with $r \leq \gamma \rho(x_0)$, and $x \in B$. Then

$$
M_{\rho, \eta}^s ((b)^m, \mu_\gamma^s)f(x) \lesssim ((b)^0_\beta)^m M_{m \beta, s} f(x) + \sum_{\gamma = 0}^{m-1} ((b)^0_\beta)^m \gamma M_{(m-\gamma)\beta, s} ((b)^s_\mu^\gamma)f(x).
$$
Proof. Since
\[ [b^m, \mu^1_j]f(y) = \sum_{\gamma=0}^{m-1} C_{\gamma,m}(b(y) - b_{2B})^{m-\gamma}[b^\gamma, \mu^1_j](f)(y) + \mu^1_j((b - b_{2B})^m f)(y), \]
then
\[
\frac{1}{|B|} \int_B \left| [b^m, \mu^1_j]f(y) - ([b^m, \mu^1_j]f)_B \right| dy \\
\lesssim \sum_{\gamma=0}^{m-1} \frac{1}{|B|} \int_B |(b(y) - b_{2B})^{m-\gamma}[b^\gamma, \mu^1_j](f)(y) - ((b(y) - b_{2B})^{m-\gamma}[b^\gamma, \mu^1_j](f))_B| dy \\
+ \frac{1}{|B|} \int_B |\mu^1_j((b - b_{2B})^m f)(y) - (\mu^1_j((b - b_{2B})^m f))_B| dy \\
\lesssim \sum_{\gamma=0}^{m-1} \frac{1}{|B|} \int_B \left| (b(y) - b_{2B})^{m-\gamma}[b^\gamma, \mu^1_j](f)(y) \right| dy \\
+ \frac{1}{|B|} \int_B |\mu^1_j((b - b_{2B})^m f)(y) - (\mu^1_j((b - b_{2B})^m f))_B| dy \\
= \mathcal{K}_1 + \mathcal{K}_2.
\]
By Hölder’s inequality and Lemma 2.5 we get
\[
\mathcal{K}_1 \lesssim \sum_{\gamma=0}^{m-1} \frac{1}{|B|} \int_B |(b(y) - b_{2B})^{m-\gamma}[b^\gamma, \mu^1_j](f)(y)| dy \\
\lesssim \sum_{\gamma=0}^{m-1} \left( \frac{1}{|B|} \int_B \left| (b(y) - b_{2B})^{(m-\gamma)s'} \right| dy \right)^{1/s'} \left( \frac{1}{|B|} \int_B \left| [b^\gamma, \mu^1_j](f)(y) \right|^s dy \right)^{1/s} \\
\lesssim \sum_{\gamma=0}^{m-1} ([b]_{\beta}^0 (m-\gamma) M_{(m-\gamma)p,s}([b^\gamma, \mu^1_j](f))(x)).
\]
For $\mathcal{K}_2$, we split $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$, we have
\[
\mathcal{K}_2 \leq \frac{1}{|B|} \int_B \left| \mu^1_j((b - b_{2B})^m f_1)(y) \right| dy + \frac{1}{|B|} \int_B \left| \mu^1_j((b - b_{2B})^m f_2)(y) - (\mu^1_j((b - b_{2B})^m f))_B \right| dy = \mathcal{K}_{21} + \mathcal{K}_{22}.
\]
As the proof in Lemma 3.1, we obtain
\[
\mathcal{K}_{21} \lesssim ([b]_{\beta}^0)^m M_{m\beta, s}(f)(x).
\]
For $\mathcal{K}_{22}$, by Lemma 3.2, we get
\[
\mathcal{K}_{22} \lesssim \frac{1}{|B|^2} \int_B \int_B \left| \mu^1_j((b - b_B)^m f_2)(u) - \mu^1_j((b - b_B)^m f_2)(y) \right| du dy \lesssim ([b]_{\beta}^0)^m M_{m\beta, s}(f)(x).
\]
Now let us prove Theorem 1.1.
Choose numbers $t_\gamma$ such that $\frac{1}{t_\gamma} = \frac{1}{\gamma} - \frac{m-\gamma}{n}$, $\gamma = 0, 1, \ldots, m - 1$. Then $\frac{1}{q} = \frac{1}{t_\gamma} = \frac{m-\gamma}{\gamma n}$. We need to prove the following inequality
\[
\| [b^m, \mu^1_j]f \|_{L^q(R^n)}^q \lesssim ([b]_{\beta}^0)^{mq} \| f \|_{L^p(R^n)}^q + \sum_{\gamma=0}^{m-1} ([b]_{\beta}^0)^{(m-\gamma)q} \| [b^\gamma, \mu^1_j](f) \|_{L^{t_\gamma}(R^n)}^q, \tag{3.1}
\]
If (3.1) holds, then Theorem 1.1 will be proved by the mathematical induction. In fact, when \( m = 1 \), we have \( \gamma = 0 \) and \( p = t_{\gamma} \). Note that \([b_0^0, \mu_1^1] = \mu_1^1\), by the boundedness of \( \mu_1^1 \) on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \), then \([b_0^0, \mu_1^1] \) is bounded from \( L^p(\mathbb{R}^n) \) into \( L^q(\mathbb{R}^n) \). Suppose that the \( L^p - L^q \) boundedness of \([b_0^0, \mu_1^1] \) holds for
\[
\frac{1}{\ell_{\gamma}} = \frac{1}{p} - \frac{\gamma \beta}{n},
\]
that is
\[
\|[b^0, \mu_1^1](f)\|_{L^q(\mathbb{R}^n)} \lesssim \|(b_0^0)^{\gamma} f\|_{L^p(\mathbb{R}^n)},
\]
where \( \gamma = 2, 3, \ldots, m - 1 \), then by (3.1) we get
\[
\|[b^m, \mu_1^1](f)\|_{L^q(\mathbb{R}^n)} \lesssim \|(b_0^0)^m\|_{L^p(\mathbb{R}^n)}.
\]
In the following, we will focus on the proof of (3.1).

Let \( 1 < s < p < \infty, f \in L^p(\mathbb{R}^n) \). By Proposition 2.4 we have
\[
\|[b^m, \mu_1^1](f)\|_{L^q(\mathbb{R}^n)} \lesssim \|b_0^m\|_{L^p(\mathbb{R}^n)} \|M_{\mu_1^1}(f)\|_{L^q(\mathbb{R}^n)} + \sum_{\gamma=0}^{m-1} \|M_{(m-\gamma)\beta,s}([b_0^0]^{m-\gamma}) \|_{L^q(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.
\]

By Lemma 3.3,
\[
M_{\mu_1^1}^q([b^m, \mu_1^1](f)) \lesssim \|b_0^m\|_{L^p(\mathbb{R}^n)} \|M_{\mu_1^1}(f)\|_{L^q(\mathbb{R}^n)} + \sum_{\gamma=0}^{m-1} \|M_{(m-\gamma)\beta,s}([b_0^0]^{m-\gamma}) \|_{L^q(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.
\]

Then the proof of (3.1) is finished.
4. Proof of Theorem 1.3

To prove Theorem 1.3, we first investigate the following local estimate.

Lemma 4.1. Let $V \in RH_{n}, b \in A_{\beta}^{0}(\rho)$. If $1 < p < \frac{n}{m\beta}, \frac{1}{q} = \frac{1}{p} - \frac{m\beta}{n}$, then the inequality

$$
\| [b^{m}, \mu_{f}^{1}] (f) \|_{L^{q}(B(x_{0},r))} \lesssim \left( \| b \|_{p} \right)^{m} \| f \|_{L^{p}(B(x_{0},r))} \int_{2r}^{\infty} \frac{\| f \|_{L^{p}(B(x_{0},t))}}{t^{\frac{n}{q}}} \, dt
$$

holds for any $f \in L_{loc}^{p}(\mathbb{R}^{n})$.

Proof. We write $f$ as $f = f_{1} + f_{2}$, where $f_{1}(y) = f(y)\chi_{B(x_{0},2r)}(y)$. Then

$$
\| [b^{m}, \mu_{f}^{1}] (f) \|_{L^{q}(B(x_{0},r))} \lesssim \left( \| b \|_{p} \right)^{m} \| f_{1} \|_{L^{q}(B(x_{0},r))} + \| [b^{m}, \mu_{f}^{1}] (f_{2}) \|_{L^{q}(B(x_{0},r))}.
$$

By Theorem 1.1 we know $[b^{m}, \mu_{f}^{1}]$ is bounded from $L^{p}(\mathbb{R}^{n})$ to $L^{q}(\mathbb{R}^{n})$, then we get

$$
\| [b^{m}, \mu_{f}^{1}] (f_{1}) \|_{L^{q}(B(x_{0},r))} \lesssim \left( \| b \|_{p} \right)^{m} \| f_{1} \|_{L^{p}(B(x_{0},r))} \int_{2r}^{\infty} \frac{\| f \|_{L^{p}(B(x_{0},t))}}{t^{\frac{n}{q}}} \, dt
$$

$$
\lesssim \left( \| b \|_{p} \right)^{m} \left( \int_{2r}^{\infty} \frac{\| f \|_{L^{p}(B(x_{0},t))}}{t^{\frac{n}{q}}} \, dt \right) \left( \int_{2r}^{\infty} \frac{\| f \|_{L^{p}(B(x_{0},t))}}{t^{\frac{n}{q}}} \, dt \right)^{1/2}
$$

We now turn to deal with the term $\| [b^{m}, \mu_{f}^{1}] (f_{2}) \|_{L^{q}(B(x_{0},r))}$. By Binomial Theorem, we have

$$
[b^{m}, \mu_{f}^{1}] f_{2}(x) \leq \sum_{\gamma = 0}^{m} C_{\gamma, m} |b(x) - b_{2B}|^{\gamma} \mu((b-b_{2B})^{m-\gamma} f_{2})(x).
$$

By Proposition 2.6 and Lemma 2.2 we have

$$
\sup_{x \in B(x_{0},r)} \mu_{f}^{1}( (b-b_{2B})^{m-\gamma} f_{2})(x) \lesssim \left( \int_{2B}^{\infty} \frac{1}{(1 + \frac{|x-y|}{\rho(x)})^{N}} |b-b_{2B}|^{m-\gamma} |f(y)| \left( \int_{|x_{0}-y|}^{\infty} \frac{dt}{t^{\frac{n}{q}}} \right)^{1/2} \right) dy
$$

$$
\lesssim \sum_{k=1}^{\infty} \left( \frac{k+1}{\rho(x_{0})} \right)^{N/(k+1)} (2^{k+1}r)^{-n} \int_{2^{k+1}B} |b-b_{2B}|^{m-\gamma} |f(y)| dy.
$$

From Lemma 2.5 we get

$$
(2^{k+1}r)^{-n} \int_{2^{k+1}B} |b(y) - b_{2B}|^{m-\gamma} |f(y)| dy
$$

$$
\lesssim \left( \left( (2^{k+1}r)^{-n} \int_{2^{k+1}B} |b(y) - b_{2B}|^{m-\gamma} |f(y)| \, dy \right)^{1/p'} \right)^{1/p'} (2^{k+1}r)^{-\frac{\gamma}{p'}} \| f \|_{L^{p}(B(x_{0},2^{k+1}r))}
$$

$$
\lesssim \left( \left( b \right)_{p}^{\beta} \right)^{m-\gamma} \left( 1 + \frac{2^{k}r}{\rho(x_{0})} \right)^{(m-\gamma)\theta'} (2^{k}r)^{(m-\gamma)\beta-\frac{\gamma}{p'}} \| f \|_{L^{p}(B(x_{0},2^{k+1}r))}.
$$

Then

$$
\sup_{x \in B(x_{0},r)} \mu_{f}^{1}( (b-b_{2B})^{m-\gamma} f_{2})(x)
$$

$$
\lesssim \left( \left( b \right)_{p}^{\beta} \right)^{m-\gamma} \sum_{k=1}^{\infty} \left( 1 + \frac{2^{k}r}{\rho(x_{0})} \right)^{(m-\gamma)\theta'-N/(k+1)} (2^{k}r)^{(m-\gamma)\beta-\frac{\gamma}{p'}} \| f \|_{L^{p}(B(x_{0},2^{k+1}r))};
$$
Notice that
\[ \|(b - b_{2B})^\gamma\|_{L^q(2B)} \lesssim (|b|^0)_{2^n} \rho^\gamma \left(1 + \frac{2r}{\rho(x_0)}\right)^\theta \gamma. \]

Then, taking \( N \geq (k_0 + 1)(m)\theta^t \) and noticing \( m\theta - \frac{N}{q} = -\frac{N}{q} \) we get
\[
\| [b, \mu^1_\phi] (f_2) \|_{L^q(B(x_0, r))} \lesssim (|b|^0)_{2^n} \rho^\gamma \sum_{k=1}^\infty 2^{-\gamma \beta} \left(1 + \frac{2k}{\rho(x_0)}\right)^{m\theta - N/(k_0 + 1)} (2k)^{m\theta - \frac{N}{q}} \| f \|_{L^p(B(x_0, 2k+1)r))}.
\]

Combining (4.1) and (4.2), the proof of Lemma 4.1 is completed. 

\[ \square \]

**Proof of Theorem 1.3.** Note that \( \| f \|_{L^p(B(x_0, t))} \) is a nondecreasing function of \( t \), and \( f \in M^{V, \psi}_{\alpha, \rho} \), then we have
\[
\left(1 + \frac{t}{\rho(x_0)}\right)^\alpha \| f \|_{L^p(B(x_0, t))} \leq \sup_{t < s < \infty} \left(1 + \frac{s}{\rho(x_0)}\right)^\alpha \| f \|_{L^p(B(x_0, s))}.
\]

Since \( \alpha \geq 0 \), and \((\varphi_1, \varphi_2)\) satisfies the condition (1.1), then
\[
\int_2^\infty \frac{\| f \|_{L^p(B(x_0, t))}}{t^{\frac{n}{q}}} dt = \int_2^\infty \frac{\| f \|_{L^p(B(x_0, t))}}{t^{\frac{n}{q}}} \left(1 + \frac{t}{\rho(x_0)}\right)^\alpha \| f \|_{L^p(B(x_0, t))} \frac{\| \varphi_1 \|_{L^p(B(x_0, s))}}{s^{\frac{n}{q}}} dt \\
\lesssim \| f \|_{M^{V, \psi}_{\alpha, \rho}} \int_2^\infty \frac{\| f \|_{L^p(B(x_0, t))}}{t^{\frac{n}{q}}} \left(1 + \frac{t}{\rho(x_0)}\right)^\alpha \| f \|_{L^p(B(x_0, t))} \frac{\| \varphi_1 \|_{L^p(B(x_0, s))}}{s^{\frac{n}{q}}} dt \\
\lesssim \| f \|_{M^{V, \psi}_{\alpha, \rho}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \int_r^\infty \frac{\| f \|_{L^p(B(x_0, t))}}{t^{\frac{n}{q}}} \frac{\| \varphi_1 \|_{L^p(B(x_0, s))}}{s^{\frac{n}{q}}} dt \\
\lesssim \| f \|_{M^{V, \psi}_{\alpha, \rho}} \left(1 + \frac{r}{\rho(x_0)}\right)^{-\alpha} \frac{\| \varphi_2 \|_{M^{V, \psi}_{\alpha, \rho}}}{r}.
\]

Then by Lemma 4.1 we get
\[
\| [b, \mu^1_\phi] (f) \|_{L^q(B(x_0, r))} \lesssim \| f \|_{L^p(B(x_0, 2k+1)r))} \frac{\| \varphi_2 \|_{M^{V, \psi}_{\alpha, \rho}}}{r}.
\]

\[ \square \]
References