Fixed point results for generalized \((\alpha\eta)\)-\(\Theta\) contractions with applications

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Abstract

The aim of this paper is to define generalized \((\alpha\eta)\)-\(\Theta\) contraction and to extend the results of Jleli and Samet [M. Jleli, B. Samet, J. Inequal. Appl., 2014 (2014), 8 pages] by applying a simple condition on the function \(\Theta\). We also deduce certain fixed and periodic point results for orbitally continuous generalized \(\Theta\)-contractions and certain fixed point results for integral inequalities are derived. Finally, we provide an example to show the significance of the investigation of this paper. ©2017 All rights reserved.

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1. Introduction and preliminaries

Banach’s contraction principle [8] is one of the pivotal results of analysis. It establishes that, given a mapping \(F\) on a complete metric space \((X,d)\) into itself and a constant \(k \in (0,1)\) such that

\[ d(Fx, Fy) \leq kd(x, y), \]

holds for all \(x, y \in X\). Then \(F\) has a unique fixed point in \(X\).

Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [1–13, 17] and references therein). In 2012, Samet et al. [21] introduced the concepts of \(\alpha\psi\)-contractive and \(\alpha\)-admissible mappings and established various fixed point theorems for such mappings defined on complete metric spaces.

Definition 1.1 ([21]). Let \(F\) be a self-mapping on \(X\) and \(\alpha : X \times X \to [0, +\infty)\) be a function. We say that \(F\) is an \(\alpha\)-admissible mapping if

\[ x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Fx, Fy) \geq 1. \]
Afterwards Salimi et al. [20] and Hussain et al. [15, 16] modified the notions of α-admissible mappings and established certain fixed point theorems.

**Definition 1.2** ([20]). Let $F$ be a self-mapping on $X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that $F$ is an $\alpha$-admissible mapping with respect to $\eta$ if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \implies \alpha(Fx, Fy) \geq \eta(Fx, Fy).$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 1.1. Also, if we take $\alpha(x, y) = 1$, then we say that $F$ is a $\eta$-subadmissible mapping.

**Definition 1.3** ([16]). Let $(X, d)$ be a metric space. Let $\alpha, \eta : X \times X \to [0, \infty)$ and $F : X \to X$ be functions. We say $F$ is an $\alpha$-$\eta$-continuous mapping on $(X, d)$, if for given $x \in X$ and sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$, $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \forall n \in \mathbb{N} \implies Fx_n \to Fx$.

A mapping $F : X \to X$ is called orbitally continuous at $p \in X$ if $\lim_{n \to \infty} F^n x = p$ implies that $\lim_{n \to \infty} FF^n x = Fp$. The mapping $F$ is orbitally continuous on $X$ if $F$ is orbitally continuous for all $p \in X$.

**Remark 1.4** ([16]). Let $F : X \to X$ be a self-mapping on an orbitally $F$-complete metric space $X$. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 3, & \text{if } x, y \in O(w), \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \eta(x, y) = 1,$$

where $O(w)$ is an orbit of a point $w \in X$. If $F : X \to X$ is an orbitally continuous map on $(X, d)$, then $F$ is $\alpha$-$\eta$-continuous on $(X, d)$.

Very recently, Jleli and Samet [19] introduced a new type of contraction called $\Theta$-contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

**Definition 1.5.** Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

1. $\Theta$ is nondecreasing;
2. for each sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$, $\lim_{n \to \infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n \to \infty} (\alpha_n) = 0$;
3. there exist $0 < k < 1$ and $1 \in (0, \infty]$ such that $\lim_{\alpha \to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = 1$.

A mapping $F : X \to X$ is said to be $\Theta$-contraction if there exist the function $\Theta$ satisfying $(\Theta_1)$-$(\Theta_3)$ and a constant $k \in (0, 1)$ such that for all $x, y \in X$,

$$d(Fx, Fy) \neq 0 \implies \Theta(d(Fx, Fy)) \leq \left[\Theta(d(x, y))\right]^k.$$

**Theorem 1.6** ([19]). Let $(X, d)$ be a complete metric space and $F : X \to X$ be a $\Theta$-contraction, then $F$ has a unique fixed point.

They showed that any Banach contraction is a particular case of $\Theta$-contraction while there are $\Theta$-contractions which are not Banach contractions. To be consistent with Jleli et al. [19], we denote by the $\Psi$ set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions $(\Theta_1)$-$(\Theta_3)$.

Hussain et al. [17] modified and extended the above result and proved the following fixed point theorem for generalized $\Theta$-contractive condition in the setting of complete metric spaces.

**Theorem 1.7** ([17]). Let $(X, d)$ be a complete metric space and $F : X \to X$ be a self-mapping. If there exist a function $\Theta \in \Psi$ and positive real numbers $\alpha, \beta, \gamma, \delta$ with $0 \leq \alpha + \beta + \gamma + 2\delta < 1$ such that

$$\Theta(d(Fx, Fy)) \leq \left[\Theta(d(x, y))\right]^\alpha \cdot \left[\Theta(d(x, Fx))\right]^\beta \cdot \left[\Theta(d(y, Fy))\right]^\gamma \cdot \left[\Theta(d(x, Fy) + d(y, Fx))\right]^\delta$$

for all $x, y \in X$, then $F$ has a unique fixed point.
Very recently, Ahmad et al. [2, 7] used the following weaker condition instead of the condition \((\Theta_3)\) in Definition 1.5.

\((\Theta_3')\) \(\Theta\) is continuous on \((0, \infty)\).

Consistent with Ahmad et al. [2], we denote by \(\Omega\) the set of all functions satisfying the conditions \((\Theta_1),(\Theta_2)\) and \((\Theta_3')\).

**Example 1.8** ([2]). Let \(\Theta_1(t) = e^{\sqrt{t}}, \Theta_2(t) = e^{\sqrt{t}+t}, \Theta_3(t) = e^{t}, \Theta_4(t) = \cosh t, \Theta_5(t) = 1 + \ln(1 + t)\) and \(\Theta_6(t) = e^{te^t}\) for all \(t > 0\). Then \(\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6 \in \Omega\).

**Example 1.9** ([2]). Note that the conditions \(\Theta_3\) and \(\Theta_3'\) are independent of each other. Indeed, for \(p \geq 1\), \(\Theta(t) = e^{t^p}\) satisfies the conditions \((\Theta_1)\) and \((\Theta_2)\) but it does not satisfy \((\Theta_3)\), while it satisfies the condition \((\Theta_3')\). Therefore \(\Omega \subseteq \Psi\). Again for \(p > 1, m \in (0, \frac{1}{p})\) \(\Theta(t) = 1 + t^m(1 + |t|)\) where \([t]\) denotes the integral part of \(t\), satisfies the conditions \((\Theta_1)\) and \((\Theta_2)\) but it does not satisfy \((\Theta_3')\), while it satisfies the condition \((\Theta_3)\) for any \(k \in (\frac{1}{p}, 1)\). Therefore \(\Psi \not\subseteq \Omega\). Also, if we take \(\Theta(t) = e^{\sqrt{t}}\), then \(\Theta \in \Psi\) and \(\Theta \in \Omega\). Therefore \(\Psi \cap \Omega \neq \emptyset\).

In this paper, we apply the same weaker condition \((\Theta_3')\) to obtain some new fixed point theorems in the context of complete metric spaces.

**2. Main results**

In this section, we define \(\alpha\)-\(\eta\)-\(\Theta\)-contraction for a new family of functions \(\Omega\) and establish certain fixed point theorems in the context of complete metric spaces.

**Definition 2.1.** Let \((X, d)\) be a metric space and \(F\) be a self-mapping on \(X\). Also suppose that \(\alpha, \eta : X \times X \to [0, +\infty)\) be two functions. We say that \(F\) is \(\alpha\)-\(\eta\)-\(\Theta\)-contraction if for \(x, y \in X\) with \(\eta(x, Fx) \leq \alpha(x, y)\) and \(d(Fx, Fy) > 0\), we have

\[
\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^k,
\]

where \(\Theta \in \Omega\) and \(k \in (0, 1)\).

**Theorem 2.2.** Let \((X, d)\) be a complete metric space. Let \(F : X \to X\) be a self-mapping satisfying the following assertions:

(i) \(F\) is \(\alpha\)-admissible mapping with respect to \(\eta\);

(ii) \(F\) is \(\alpha\)-\(\eta\)-\(\Theta\)-contraction;

(iii) there exists \(x_0 \in X\) such that \(\alpha(x_0, Fx_0) \geq \eta(x_0, Fx_0)\);

(iv) \(F\) is \(\alpha\)-\(\eta\)-continuous.

Then \(F\) has a fixed point. Moreover, \(F\) has a unique fixed point when \(\alpha(x, y) \geq \eta(x, x)\) for all \(x, y \in \text{Fix}(F)\).

**Proof.** Let \(x_0 \in X\) such that \(\alpha(x_0, Fx_0) \geq \eta(x_0, Fx_0)\). For such \(x_0\), we define the sequence \(\{x_n\}\) by \(x_n = F^n x_0 = x_{n-1}\). Now, since \(F\) is \(\alpha\)-admissible mapping with respect to \(\eta\), then \(\alpha(x_0, x_1) = \alpha(x_0, Fx_0) \geq \eta(x_0, Fx_0) = \eta(x_0, x_1)\). By continuing this process we have

\[
\eta(x_{n-1}, Fx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n),
\]

for all \(n \in \mathbb{N}\). If there exists \(n_0 \in \mathbb{N}\) such that \(x_{n_0} = x_{n_0+1}\), then \(x_{n_0}\) is a fixed point of \(F\) and we have nothing to prove. Hence, we assume, \(x_n \neq x_{n+1}\) or \(d(Fx_{n-1}, Fx_n) > 0\) for all \(n \in \mathbb{N}\). Since, \(F\) is \(\alpha\)-\(\eta\)-\(\Theta\)-contraction, so we have

\[
1 < \Theta(d(x_n, x_{n+1})) = \Theta(d(Fx_{n-1}, Fx_n)) \leq [\Theta(d(x_{n-1}, x_n))]^k
\]
for all \( n \in \mathbb{N} \). Since \( \Theta \in \Omega \), so by taking limit as \( n \to \infty \) in above inequality, we have

\[
\lim_{n \to \infty} \Theta(d(x_n, x_{n+1})) = 1.
\]

By (\( \Theta_2 \)), we have

\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]  
(2.1)

Now, we claim that \( \{x_n\}_{n=1}^{\infty} \) is a Cauchy sequence. We suppose on the contrary that \( \{x_n\}_{n=1}^{\infty} \) is not a Cauchy sequence, then we assume that there exist \( \varepsilon > 0 \) and sequences \( \{p(n)\}_{n=1}^{\infty} \) and \( \{q(n)\}_{n=1}^{\infty} \) of natural numbers such that for \( p(n) > q(n) > n \), we have

\[
d(x_{p(n)}, x_{q(n)}) \geq \varepsilon.
\]

Then

\[
d(x_{p(n)-1}, x_{q(n)}) < \varepsilon
\]  
(2.2)

for all \( n \in \mathbb{N} \). So, by triangle inequality and (2.2), we have

\[
\varepsilon \leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \leq d(x_{p(n)-1}, x_{p(n)}) + \varepsilon.
\]

By taking the limit and using inequality (2.2), we get

\[
\lim_{n \to \infty} d(x_{p(n)}, x_{q(n)}) = \varepsilon.
\]  
(2.3)

From (2.1), we can choose a natural number \( n_0 \in \mathbb{N} \) such that

\[
d(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4} \quad \text{and} \quad d(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4}
\]  
(2.4)

for all \( n \geq n_0 \). Next, we claim that \( Fx_{p(n)} \neq Fx_{q(n)} \) for all \( n \geq n_0 \), that is

\[
d(x_{p(n)+1}, x_{q(n)+1}) = d(Fx_{p(n)}, Fx_{q(n)}) > 0.
\]  
(2.5)

Arguing by contradiction, there exists \( N_0 \geq n_0 \) such that \( d(x_{p(n)+1}, x_{q(n)+1}) = 0 \). It follows from (2.1), (2.4), and (2.5) that

\[
\varepsilon \leq d(x_{p(n)}, x_{q(n)}) \leq d(x_{p(n)}, x_{p(n)+1}) + d(x_{p(n)+1}, x_{q(n)+1}) + d(x_{p(n)+1}, x_{q(n)})
\]

\[
\leq \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\]

a contradiction. Thus the relation (2.4) holds. Then by the assumption, we get

\[
\Theta(d(Fx_{p(n)}, Fx_{q(n)})) \leq \theta(\Theta(d(x_{p(n)}, x_{q(n)}))) k.
\]  
(2.6)

By taking limit as \( n \to +\infty \) and using (\( \Theta'_2 \)), (2.3) and (2.6), we get

\[
\Theta(\varepsilon) \leq [\Theta(\varepsilon)] k,
\]

which is a contradiction. Thus \( \{x_n\} \) is a Cauchy sequence. Completeness of \( X \) ensures that there exists \( z \in X \) such that \( x_n \to z \) as \( n \to \infty \). Now, since \( F \) is \( \alpha \)-\( \eta \)-continuous and \( \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n) \), so

\[
d(z, Fz) = \lim_{n \to \infty} d(x_n, Fx_n) = \lim_{n \to \infty} d(x_n, x_{n+1}) = d(z, z) = 0.
\]
Hence, $z$ is a fixed point of $F$. Now we show the uniqueness of fixed point. We suppose on the contrary that there exists another fixed point $u$ of $F$ distinct from $z$, that is

$$Fz = z \neq u = Fu \quad \text{that is} \quad Fz \neq Fu.$$ 

Then from assumption of theorem, we obtain

$$\Theta(d(z, u)) = \Theta(d(Fz, Fu)) \leq [\Theta(d(z, u))]^k,$$

which is contradiction because $k \in (0, 1)$. Thus $z$ is the unique fixed point of $F$. \hfill $\square$

**Theorem 2.3.** Let $(X, d)$ be a complete metric space. Let $F : X \to X$ be a self-mapping satisfying the following assertions:

1. $F$ is an $\alpha$-admissible mapping with respect to $\eta$;
2. $F$ is $\alpha$-$\eta$-$\Theta$-contraction;
3. there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq \eta(x_0, Fx_0)$;
4. if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to \infty$, then either

$$\eta(Fx_n, F^2x_n) \leq \alpha(Fx_n, x), \quad \text{or} \quad \eta(F^2x_n, F^3x_n) \leq \alpha(F^2x_n, x),$$

holds for all $n \in \mathbb{N}$.

Then $F$ has a fixed point. Moreover, $F$ has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in \text{Fix}(T)$.

**Proof.** Let $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geq \eta(x_0, Fx_0)$. As in proof of Theorem 2.2 we can conclude that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x^*$ as $n \to \infty$, where, $x_{n+1} = Tx_n$. So, from (iv), either

$$\eta(Fx_n, F^2x_n) \leq \alpha(Fx_n, x^*) \quad \text{or} \quad \eta(F^2x_n, F^3x_n) \leq \alpha(F^2x_n, x^*),$$

holds for all $n \in \mathbb{N}$. This implies

$$\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, x) \quad \text{or} \quad \eta(x_{n+2}, x_{n+3}) \leq \alpha(x_{n+2}, x),$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Fx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x^*),$$

and so from (2.7) we deduce that

$$\Theta(d(Fx_{n_k}, F^*x)) \leq [\Theta(d(x_{n_k}, x^*))]^\lambda < \Theta(d(x_{n_k}, x^*)).$$

From $(\Theta_1)$ we have

$$d(x_{n_k+1}, Fx^*) < d(x_{n_k}, x^*).$$

By taking limit as $k \to \infty$ in the above inequality we get $d(x^*, Fx^*) = 0$, i.e., $x^* = Fx^*$. Uniqueness follows similarly as in Theorem 2.2. \hfill $\square$

Taking $\alpha(x, y) = \eta(x, y) = 1$ for all $x, y \in X$, then we deduce the following result as corollary.
Corollary 2.4. Let \((X, d)\) be a complete metric space and \(F : X \to X\) be a self-mapping. If for all \(x, y \in X\) with \(d(Fx, Fy) > 0\), we have
\[
\Theta(d(Fx, Fy)) \leq \|\Theta(d(x, y))\|^k,
\]
where \(F \in \Omega\). Then \(F\) has a fixed point.

Recall that a self-mapping \(T\) is said to have the property \(P\), if \(\text{Fix}(T^n) = F(T)\) for every \(n \in \mathbb{N}\).

Theorem 2.5. Let \((X, d)\) be a complete metric space and \(F : X \to X\) be an \(\alpha\)-continuous self-mapping. Assume that there exists some \(k \in (0, 1)\) such that
\[
\Theta(d(Fx, F^2x)) \leq \|\Theta(d(x, Fx))\|^k,
\]
holds for all \(x \in X\) with \(d(Fx, F^2x) > 0\) where \(\Theta \in \Omega\). If \(F\) is an \(\alpha\)-admissible mapping and there exists \(x_0 \in X\) such that \(\alpha(x_0, Fx_0) \geq 1\), then \(F\) has the property \(P\).

Proof. Let \(x_0 \in X\) such that \(\alpha(x_0, Fx_0) \geq 1\). For such \(x_0\), we define the sequence \(\{x_n\}\) by \(x_n = F^n x_0 = Fx_{n-1}\). Now, since \(F\) is \(\alpha\)-admissible mapping, so \(\alpha(x_1, x_2) = \alpha(Fx_0, Fx_1) \geq 1\). By continuing this process, we have
\[
\alpha(x_{n-1}, x_n) \geq 1
\]
for all \(n \in \mathbb{N}\). If there exists \(n_0 \in \mathbb{N}\) such that \(x_{n_0} = x_{n_0+1} = Fx_{n_0}\), then \(x_{n_0}\) is fixed point of \(F\) and we have nothing to prove. Hence, we assume, \(x_n \neq x_{n+1}\) or \(d(Fx_{n-1}, F^2x_{n-1}) > 0\) for all \(n \in \mathbb{N} \cup \{0\}\). From (2.8) we have
\[
1 < \Theta(d(Fx_{n-1}, F^2x_{n-1})) \leq \Theta(d(x_{n-1}, Fx_{n-1})))^k,
\]
which implies
\[
1 < \Theta(d(x_n, x_{n+1})) \leq \Theta(d(x_{n-1}, x_n))^k.
\]
and so
\[
1 < \Theta(d(x_n, x_{n+1})) \leq \Theta(d(x_{n-1}, x_n))^k.
\]
Therefore,
\[
1 < \Theta(d(x_n, x_{n+1})) \leq \Theta(d(x_{n-1}, x_n))^k \leq \Theta(d(x_{n-2}, x_{n-1}))^k \leq \cdots \leq \Theta(d(x_0, x_1))^k.
\]
By taking limit as \(n \to \infty\) in above inequality, we have \(\lim_{n \to \infty} \Theta(d(x_n, x_{n+1})) = 1\), and since \(\Theta \in \Omega\) we obtain
\[
\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]
(2.9)

Now, we claim that \(\{x_n\}_{n=1}^{\infty}\) is a Cauchy sequence. We suppose on the contrary that \(\{x_n\}_{n=1}^{\infty}\) is not Cauchy then we assume there exist \(\varepsilon > 0\) and sequences \(\{p(n)\}_{n=1}^{\infty}\) and \(\{q(n)\}_{n=1}^{\infty}\) of natural numbers such that for \(p(n) > q(n) > n\), we have
\[
\varepsilon \leq d(x_{p(n)}, Fx_{q(n)-1}) = d(x_{p(n)}, x_{q(n)}) \geq \varepsilon.
\]
(2.10)
Then
\[
d(x_{p(n)-1}, Fx_{q(n)-1}) < \varepsilon
\]
for all \(n \in \mathbb{N}\). So, by triangle inequality and (2.10), we have
\[
\varepsilon \leq d(x_{p(n)}, Fx_{q(n)-1}) \leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, Fx_{q(n)-1}) \leq d(x_{p(n)}, x_{p(n)-1}) + \varepsilon.
\]
By taking the limit and using inequality (2.9), we get
\[
\lim_{n \to \infty} d(x_{p(n)}, Fx_{q(n)-1}) = \varepsilon.
\]
On the other hand, from (2.9) there exists a natural number \( n_0 \in \mathbb{N} \) such that
\[
d(x_{p(n)}, x_{p(n)+1}) < \frac{\varepsilon}{4} \quad \text{and} \quad d(x_{q(n)}, x_{q(n)+1}) < \frac{\varepsilon}{4}
\] (2.11)
for all \( n \geq n_0 \). Next, we claim that
\[
d(Fx_{p(n)}, F^2x_{q(n)-1}) = d(x_{p(n)+1}, Fx_{q(n)}) > 0
\] (2.12)
for all \( n \geq n_0 \). We suppose on the contrary that there exists \( m \geq n_0 \) such that
\[
d(Fx_{p(m)}, F^2x_{q(m)-1}) = d(x_{p(m)+1}, Fx_{q(m)}) = 0.
\] (2.13)
Then from (2.11), (2.12) and (2.13), we have
\[
\varepsilon \leq d(x_{p(m)}, Fx_{q(m)-1}) \leq d(x_p(m), x_{p(m)+1}) + d(x_{q(m)+1}, Fx_{q(m)-1}) \\
\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, Fx_{q(m)-1}) \\
= d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, Fx_{q(m)}) + d(x_{q(m)+1}, Fx_{q(m)}) \\
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\]
which is a contradiction. Thus
\[
d(Fx_{p(n)}, F^2x_{q(n)-1}) = d(x_{p(n)+1}, Fx_{q(n)}) > 0,
\]
\[
\Theta(d(Fx_{p(n)}, F^2x_{q(n)-1})) \leq [\Theta(d(x_{p(n)}, Fx_{q(n)-1}))]^k,(2.14)
\]
is established which further implies that
\[
\Theta(d(x_{p(n)+1}, x_{q(n)+1})) \leq [\Theta(d(x_{p(n)}, x_{q(n)}))]^k.
\]
From (\( \Theta \_3 \)), (2.10) and (2.14), we get
\[
\Theta(\varepsilon) \leq [\Theta(\varepsilon)]^k,
\]
which is a contradiction because \( k \in (0, 1). \) Thus we proved that \( \{x_n\} \) is a Cauchy sequence. Completeness of \( X \) ensures that there exists \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to \infty \). Now, since \( F \) is \( \alpha \)-continuous and \( \alpha(x_{n-1}, x_n) \geq 1 \) then, \( x_{n+1} = F x_n \to F x^* \) as \( n \to \infty \). That is, \( x^* = F x^* \). Thus \( F \) has a fixed point and \( F(F^n) = F(F) \) for \( n = 1 \). Let \( n > 1 \). Assume contrarily that \( w \in F(F^n) \) and \( w \notin F(F) \). Then, \( d(w, Fw) > 0 \). Now we have
\[
1 < \Theta(d(w, Fw)) = \Theta(d(F(F^{n-1}w), F^2(F^{n-1}w))) \\
\leq [\Theta(d(F^{n-1}w, F^n w))]^k \\
\leq [\Theta(d(F^{n-2}w, F^{n-1}w))]^k \leq \ldots \\
\leq [\Theta(d(w, Fw))]^k.
\]
By taking limit as \( n \to \infty \) in the above inequality we have \( \Theta(d(w, Fw)) = 1 \). Hence, by (\( \Theta \_2 \)) we get, \( d(w, Fw) = 0 \) which is a contradiction. Therefore, \( F(F^n) = F(F) \) for all \( n \in \mathbb{N} \).

Let \( (X, d, \leq) \) be a partially ordered metric space. Recall that \( F : X \to X \) is nondecreasing if for all \( x, y \in X \), \( x \leq y \) implies \( F(x) \leq F(y) \) and ordered \( \Theta \)-contraction if for \( x, y \in X \) with \( x \leq y \) and \( d(Fx, Fy) > 0 \), we have
\[
\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^k,
\]
where \( \Theta \in \Omega \). Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [1, 14, 16, 18] and references therein). From Theorems 2.2-2.5, we derive the following new results in partially ordered metric spaces.
Theorem 2.6. Let \((X, d, \preceq)\) be a complete partially ordered metric space. Assume that the following assertions hold true:

(i) \(F\) is nondecreasing and ordered \(\Theta\)-contraction;

(ii) there exists \(x_0 \in X\) such that \(x_0 \preceq Fx_0\);

(iii) either for a given \(x \in X\) and sequence \(\{x_n\}\)

\[
x_n \to x \text{ as } n \to \infty \quad \text{and} \quad x_n \preceq x_{n+1}, \quad \forall n \in \mathbb{N}
\]

we have \(F_{x_n} \to Fx\),

or if \(\{x_n\}\) is a sequence such that \(x_n \preceq x_{n+1}\) with \(x_n \to x\) as \(n \to \infty\), then either

\[
F_{x_n} \preceq x \quad \text{or} \quad F^2_{x_n} \preceq x,
\]

holds for all \(n \in \mathbb{N}\).

Then \(F\) has a fixed point.

Theorem 2.7. Let \((X, d, \preceq)\) be a complete partially ordered metric space. Assume that the following assertions hold true:

(i) \(F\) is nondecreasing and satisfies (2.8) for all \(x \in X\) with \(d(Fx, F^2x) > 0\) where \(\Theta \in \Omega\) and \(\tau > 0\);

(ii) there exists \(x_0 \in X\) such that \(x_0 \preceq Fx_0\);

(iii) for a given \(x \in X\) and sequence \(\{x_n\}\)

\[
x_n \to x \text{ as } n \to \infty \quad \text{and} \quad x_n \preceq x_{n+1} \quad \text{for all } n \in \mathbb{N}
\]

we have \(F_{x_n} \to Fx\).

Then \(F\) has a property \(P\).

As an application of our results proved above, we deduce certain Suzuki-Samet type fixed point theorems.

Theorem 2.8. Let \((X, d)\) be a complete metric space and \(F\) be a continuous self-mapping on \(X\). If for \(x, y \in X\) with

\[
\frac{1}{2}d(x, Fx) \leq d(x, y) \quad \text{and} \quad d(Fx, Fy) > 0
\]

we have

\[
\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^k,
\]

where \(\Theta \in \Omega\). Then \(F\) has a unique fixed point.

Proof. Define, \(\alpha, \eta : X \times X \to [0, \infty)\) by

\[
\alpha(x, y) = d(x, y) \quad \text{and} \quad \eta(x, y) = \frac{1}{2}d(x, y)
\]

for all \(x, y \in X\). Now, since \(\frac{1}{2}d(x, y) \leq d(x, y)\) for all \(x, y \in X\), so \(\eta(x, y) \leq \alpha(x, y)\) for all \(x, y \in X\). That is, conditions (i) and (iii) of Theorem 2.2 hold true. Since \(F\) is continuous, so \(F\) is \(\alpha-\eta\)-continuous. Let

\[
\eta(x, Fx) \leq \alpha(x, y) \quad \text{with} \quad d(Fx, Fy) > 0.
\]

Equivalently, if \(\frac{1}{2}d(x, Fx) \leq d(x, y)\) with \(d(Fx, Fy) > 0\), then we have

\[
\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^k.
\]

That is, \(F\) is \(\alpha-\eta-\Theta\)-contraction mapping. Hence, all conditions of Theorem 2.2 hold and \(F\) has a unique fixed point. \(\square\)
Theorem 2.9. Let \((X, d)\) be a complete metric space and \(F\) be a self-mapping on \(X\). Assume that there exists some \(k \in (0, 1)\) such that

\[
\frac{1}{2(1+\tau)} d(x, Fx) \leq d(x, y) \implies \Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^k
\]

(2.15)

for \(x, y \in X\) with \(d(Fx, Fy) > 0\) where \(\Theta \in \Omega\). Then \(F\) has a unique fixed point.

Proof. Define \(\alpha, \eta : X \times X \to [0, \infty)\) by

\[
\alpha(x, y) = d(x, y) \quad \text{and} \quad \eta(x, y) = \frac{1}{2(1+\tau)} d(x, y)
\]

for all \(x, y \in X\) where \(\tau > 0\). Now, since \(\frac{1}{2(1+\tau)} d(x, y) \leq d(x, y)\) for all \(x, y \in X\), so \(\eta(x, y) \leq \alpha(x, y)\) for all \(x, y \in X\). That is, conditions (i) and (iii) of Theorem 2.3 hold true. Let \(\{x_n\}\) be a sequence with \(x_n \to x\) as \(n \to \infty\). Assume that \(d(Fx_n, F^2x_n) = 0\) for some \(n\). Then \(Fx_n = F^2x_n\). That is, \(Fx_n\) is a fixed point of \(F\) and we have nothing to prove. Hence we assume, \(Fx_n \neq F^2x_n\) for all \(n \in \mathbb{N}\). Since \(\frac{1}{2(1+\tau)} d(Fx_n, F^2x_n) \leq d(Fx_n, F^2x_n)\) for all \(n \in \mathbb{N}\), then from (2.15) we get

\[
\Theta(d(F^2x_n, F^3x_n)) \leq [\Theta(d(Fx_n, F^2x_n))]^k < \Theta(d(Fx_n, F^2x_n)),
\]

and so from (\(\Theta_1\)) we get,

\[
d(F^2x_n, F^3x_n) < d(Fx_n, F^2x_n).
\]

(2.16)

Assume there exists \(n_0 \in \mathbb{N}\) such that

\[
\eta(Fx_{n_0}, F^2x_{n_0}) > \alpha(Fx_{n_0}, x) \quad \text{and} \quad \eta(F^2x_{n_0}, F^3x_{n_0}) > \alpha(F^2x_{n_0}, x),
\]

then

\[
\frac{1}{2(1+\tau)} d(Fx_{n_0}, F^2x_{n_0}) > d(Fx_{n_0}, x) \quad \text{and} \quad \frac{1}{2(1+\tau)} d(F^2x_{n_0}, F^3x_{n_0}) > d(F^2x_{n_0}, x),
\]

so by (2.16) we have,

\[
d(Fx_{n_0}, F^2x_{n_0}) \leq d(Fx_{n_0}, x) + d(F^2x_{n_0}, x)
\]

\[
< \frac{1}{2(1+\tau)} d(Fx_{n_0}, F^2x_{n_0}) + \frac{1}{2(1+\tau)} d(F^2x_{n_0}, F^3x_{n_0})
\]

\[
< \frac{1}{2(1+\tau)} d(Fx_{n_0}, F^2x_{n_0}) + \frac{1}{2(1+\tau)} d(Fx_{n_0}, F^2x_{n_0})
\]

\[
= \frac{2}{2(1+\tau)} d(Fx_{n_0}, F^2x_{n_0}) \leq d(Fx_{n_0}, F^2x_{n_0}),
\]

which is a contradiction. Hence, either

\[
\eta(Fx_n, F^2x_n) \leq \alpha(Fx_n, x) \quad \text{or} \quad \eta(F^2x_n, F^3x_n) \leq \alpha(F^2x_n, x),
\]

holds for all \(n \in \mathbb{N}\). That is condition (iv) of Theorem 2.3 holds. Let \(\eta(x, Fx) \leq \alpha(x, y)\). So, \(\frac{1}{2(1+\tau)} d(x, Fx) \leq d(x, y)\). Then from (2.15) we get \(\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^k\). Hence, all conditions of Theorem 2.3 hold and \(F\) has a unique fixed point. \(\Box\)

3. Applications

Theorem 3.1. Let \((X, d)\) be a complete metric space and \(F : X \to X\) be a self-mapping satisfying the following assertions:

(i) for \(x, y \in O(w)\) with \(d(Fx, Fy) > 0\) we have

\[
\Theta(d(Fx, Fy)) \leq [\Theta(d(x, y))]^k,
\]

where \(\Theta \in \Omega\) and \(k \in (0, 1)\);
(ii) $F$ is an orbitally continuous function.

Then $F$ has a fixed point. Moreover, $F$ has a unique fixed point when $\text{Fix}(F) \subseteq O(w)$.

Proof. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$
\alpha(x, y) = \begin{cases} 
3, & \text{if } x, y \in O(w), \\
0, & \text{otherwise},
\end{cases}
\quad \text{and} \quad \eta(x, y) = 1,
$$

where $O(w)$ is an orbit of a point $w \in X$. From Remark 1.4 we know that $F$ is an $\alpha$-$\eta$-continuous mapping. Let $\alpha(x, y) \geq \eta(x, y)$, then $x, y \in O(w)$. So $F_x, F_y \in O(w)$. That is, $\alpha(F_x, F_y) \geq \eta(F_x, F_y)$. Therefore, $F$ is an $\alpha$-admissible mapping with respect to $\eta$. Since $w, Fw \in O(w)$, then $\alpha(w, Fw) \geq \eta(w, Fw)$. Let $\alpha(x, y) \geq \eta(x, Fx)$ and $d(Fx, Fy) > 0$. Then, $x, y \in O(w)$ and $d(Fx, Fy) > 0$. Therefore from (i) we have

$$
\Theta(d(Fx, Fy)) \leq \|\Theta(d(x, y))\|^k,
$$

which implies $F$ is $\alpha$-$\eta$-$\Theta$-contraction mapping. Hence, all conditions of Theorem 2.2 hold true and $F$ has a fixed point. If $\text{Fix}(F) \subseteq O(w)$, then, $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in \text{Fix}(F)$ and so from Theorem 2.2 $F$ has a unique fixed point.

**Theorem 3.2.** Let $(X, d)$ be a complete metric space and $F : X \to X$ be a self-mapping satisfying the following assertions:

(i) for $x \in X$ with $d(Fx, F^2x) > 0$ we have,

$$
\Theta(d(Fx, F^2x)) \leq \|\Theta(d(x, Fx))\|^k,
$$

where $\Theta \in \Omega$ and $k \in (0, 1)$;

(ii) $F$ is an orbitally continuous function.

Then $F$ has the property $P$.

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by

$$
\alpha(x, y) = \begin{cases} 
1, & \text{if } x \in O(w), \\
0, & \text{otherwise},
\end{cases}
$$

where $w \in X$. Let $\alpha(x, y) \geq 1$, then $x, y \in O(w)$. So $F_x, F_y \in O(w)$. That is, $\alpha(F_x, F_y) \geq 1$. Therefore, $F$ is $\alpha$-admissible mapping. Since $w, Fw \in O(w)$, so $\alpha(w, Fw) \geq 1$. By Remark 1.4 we conclude that $F$ is $\alpha$-continuous mapping. If $x \in X$ with $d(Fx, F^2x) > 0$, then, from (i) we have

$$
\Theta(d(Fx, F^2x)) \leq \|\Theta(d(x, Fx))\|^k.
$$

Thus all conditions of Theorem 2.5 hold true and $F$ has the property $P$. 

We can easily deduce following results involving integral inequalities.

**Theorem 3.3.** Let $(X, d)$ be a complete metric space and $F$ be a continuous self-mapping on $X$. If for $x, y \in X$ with

$$
\int_0^1 d(Fx, Fy) \rho(t) \, dt \leq \int_0^1 d(x, y) \rho(t) \, dt \quad \text{and} \quad \int_0^1 d(Fx, Fy) \rho(t) \, dt > 0,
$$

we have

$$
\Theta(\int_0^1 d(Fx, Fy) \rho(t) \, dt) \leq \|\Theta(\int_0^1 d(x, y) \rho(t) \, dt)\|^k,
$$

where $\Theta \in \Omega$, $k \in (0, 1)$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^1 \rho(t) \, dt > 0$ for $\varepsilon > 0$. Then $F$ has a unique fixed point.
Theorem 3.4. Let \((X, d)\) be a complete metric space and \(F\) be a self-mapping on \(X\). Assume that there exists some \(k \in (0, 1)\) such that

\[
\frac{1}{2t(1+t)} \int_0^t d(Fx, Fy) \rho(t)dt \leq \int_0^t d(x, y) \rho(t)dt \Rightarrow \Theta(\int_0^t d(Fx, Fy) \rho(t)dt) \leq [\Theta(\int_0^t d(x, y) \rho(t)dt)]^k
\]

for \(x, y \in X\) with \(\int_0^t d(Fx, Fy) \rho(t)dt > 0\) where \(\Theta \in \Omega\) and \(\rho : [0, \infty) \to [0, \infty)\) is a Lebesgue-integrable mapping satisfying \(\int_0^\varepsilon \rho(t)dt > 0\) for \(\varepsilon > 0\). Then \(F\) has a unique fixed point.

Theorem 3.5. Let \((X, d)\) be a complete metric space and \(F : X \to X\) be a self-mapping satisfying the following assertions:

(i) for \(x, y \in O(w)\) with \(\int_0^t d(Fx, Fy) \rho(t)dt > 0\) we have

\[
\Theta(\int_0^t d(Fx, Fy) \rho(t)dt) \leq [\Theta(\int_0^t d(x, y) \rho(t)dt)]^k,
\]

where \(\Theta \in \Omega\), \(k \in (0, 1)\) and \(\rho : [0, \infty) \to [0, \infty)\) is a Lebesgue-integrable mapping satisfying \(\int_0^\varepsilon \rho(t)dt > 0\) for \(\varepsilon > 0\);

(ii) \(F\) is an orbitally continuous function.

Then \(F\) has a fixed point. Moreover, \(F\) has a unique fixed point when \(\text{Fix}(F) \subseteq O(w)\).

Theorem 3.6. Let \((X, d)\) be a complete metric space and \(F : X \to X\) be a self-mapping satisfying the following assertions:

(i) for \(x \in X\) with \(\int_0^t d(Fx, F^2x) \rho(t)dt > 0\) we have

\[
\Theta(\int_0^t d(Fx, F^2x) \rho(t)dt) \leq [\Theta(\int_0^t d(x, Fx) \rho(t)dt)]^k,
\]

where \(\Theta \in \Omega\), \(k \in (0, 1)\) and \(\rho : [0, \infty) \to [0, \infty)\) is a Lebesgue-integrable mapping satisfying \(\int_0^\varepsilon \rho(t)dt > 0\) for \(\varepsilon > 0\);

(ii) \(F\) is an orbitally continuous function.

Then \(F\) has the property P.

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