Some identities of $\lambda$-Daehee polynomials

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Abstract

In this paper, we give some identities of $\lambda$-Daehee polynomials and investigate a new and interesting identities of $\lambda$-Daehee polynomial arising from the symmetry properties of the $p$-adic invariant integral on $\mathbb{Z}_p$. ©2017 All rights reserved.

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1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{n=0}^{p^N-1} f(x), \quad (\text{see } [15, 16]). \quad (1.1)$$

From (1.1), we note that

$$I_0(f_1) - I_0(f) = f'(0), \quad (1.2)$$

where $f'(0) = \frac{df(x)}{dx} \bigg|_{x=0}$ and $f_1(x) = f(x + 1)$.

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As is well-known, the Dahee polynomials are defined by
\[
\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!},
\]  
(1.3)
(see [1–3, 5–9, 14, 20, 21]). From (1.2), we note that
\[
\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_0(y) = \frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!},
\]
(1.4)
where \(|t|_p < p^{-\frac{1}{r}}
\).

By (1.4), we easily get
\[
\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_0(y) = \int_{\mathbb{Z}_p} e^{(x+y)\log(1+t)} d\mu_0(y)
= \sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)^m d\mu_0(y) \frac{1}{m!} (\log(1+t))^m
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} S_1(n, m) B_m(x) \right) \frac{t^n}{n!},
\]
(1.5)
where \(S_1(n, m)\) is the Stirling number of the first kind and \(B_m(x)\) are the Bernoulli polynomials (see [17]).

From (1.4) and (1.5), we note that
\[
D_n(x) = \sum_{m=0}^{n} S_1(n, m) B_m(x),
\]
(see [10, 12, 23]).

Recently many researchers have studied symmetric identities of special polynomials (see [11, 13, 14, 16, 18, 19, 22]). In this paper, we give some identities of \(\lambda\)-Dahee polynomials and investigate a new and interesting identities of \(\lambda\)-Dahee polynomial arising from the symmetry properties of the p-adic invariant integral on \(\mathbb{Z}_p\).

2. The \(\lambda\)-Dahee polynomials

In this section, we will investigate interesting identities of the \(\lambda\)-Dahee polynomials which are modified by the Dahee polynomials in (1.3).

The \(\lambda\)-Dahee polynomials are defined by the generating function to be
\[
\frac{\lambda \log(1+t)}{(1+t)^{\lambda} - 1}(1+t)^x = \sum_{n=0}^{\infty} D_n(x | \lambda) \frac{t^n}{n!},
\]
(2.1)
when \(x = 0, D_n(0 | \lambda) = D_n(\lambda)\) are called \(\lambda\)-Dahee numbers.

For \(|t|_p < p^{-\frac{1}{r}}\), by (1.2), we get
\[
\int_{\mathbb{Z}_p} (1+t)^{\lambda y+x} d\mu_0(y) = \frac{\lambda \log(1+t)}{(1+t)^{\lambda} - 1}(1+t)^x = \sum_{n=0}^{\infty} D_n(x | \lambda) \frac{t^n}{n!}.
\]

From (2.1), we have
\[
\sum_{n=0}^{\infty} D_n(x | \lambda) \frac{t^n}{n!} = \frac{\lambda \log(1+t)}{(1+t)^{\lambda} - 1}(1+t)^x
= \sum_{m=0}^{\infty} B_m \left( \frac{x}{\lambda} \right) \lambda^m \frac{(\log(1+t))^m}{m!}
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} B_m \left( \frac{x}{\lambda} \right) \lambda^m S_1(n, m) \right) \frac{t^n}{n!}.
\]
(2.2)
Thus, by (2.2), we have the following theorem.

**Theorem 2.1.**

\[ D_n(x | \lambda) = \sum_{m=0}^{n} B_m \left( \frac{x}{\lambda} \right) \lambda^m S_1(n, m), \quad (n \geq 0). \]

Recall that for \( z \in \mathbb{R} \), the Harmonic polynomials \( H_m(z) \) defined as follows

\[ \sum_{n=0}^{\infty} H_n(z) t^n = \frac{-\ln(1-t)}{t(1-t)}(1-t)^z. \]

Observe that

\[ \sum_{n=0}^{\infty} \frac{n! H_n(z)}{n!} t^n = \frac{-\ln(1-t)}{t(1-t)}(1-t)^z \]

\[ = 1 \cdot \ln(1 + (-t)) \frac{(1 + (-t))^{z-1}}{(1 + (-t))^{1-1}} \]

\[ = \sum_{n=0}^{\infty} D_n(z-1 | 1)(-1)^n \frac{t^n}{n!}. \]  

Thus, by (2.3), we have the following theorem.

**Theorem 2.2.** For \( n \in \mathbb{N} \cup \{0\} \), we have

\[ H_n(z) = D_n(z-1 | 1) \frac{(-1)^n}{n!}. \]

3. Some identities of symmetry for \( \lambda \)-Daehee polynomials

In this paper, we give some new identities of symmetry for the \( \lambda \)-Daehee polynomials which are derived from our symmetric properties related to p-adic invariant integral on \( Z_p \). In addition, we investigate some new identities of symmetry for the \( \lambda \)-Daehee polynomial invariant under Dihedral group \( D_4 \) of degree 4 arising from the p-adic invariant integral on \( Z_p \).

In this section, we assume that \( t \in \mathbb{Q}_p \) with \( |t|_p < p^{-\frac{1}{m}} \). For \( \lambda \in \mathbb{Z}_p \), let us take \( f(x) = (1 + t)^{\lambda x} \). Then, by (1.2), we get

\[ \int_{Z_p} (1 + t)^{\lambda x} d \mu_0(x) = \frac{\lambda \log(1 + t)}{(1 + t)^{\lambda - 1} - 1} = \sum_{n=0}^{\infty} D_n(\lambda) \frac{t^n}{n!}, \]

and

\[ \int_{Z_p} (1 + t)^{\lambda y + x} d \mu_0(y) = \frac{\lambda \log(1 + t)}{(1 + t)^{\lambda - 1} + 1} (1 + t)^x = \sum_{n=0}^{\infty} D_n(x | \lambda) \frac{t^n}{n!}. \]

As is well-known, the Bernoulli polynomials are defined by the p-adic invariant integral on \( Z_p \) as follows:

\[ \int_{Z_p} e^{(x+y)t} d \mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \]  

(3.1)

From (3.1), we note that

\[ \int_{Z_p} e^{(y+\frac{x}{\lambda})t} d \mu_0(y) = \sum_{m=0}^{\infty} D_m(x) \frac{1}{m!} (e^{x/\lambda t} - 1)^m \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} D_m(x) S_2(n, m) \lambda^{-n} \right) \frac{t^n}{n!}. \]  

(3.2)
where $S_2(n, m)$ is the Stirling number of the second kind.

By (3.1) and (3.2), we get
\[
\lambda^n B_n \left( \frac{x}{\lambda} \right) = \sum_{m=0}^{n} D_m(x) S_2(n, m), \quad (n \geq 0).
\]

We assume that $w_1, w_2, w_3, w_4 \in \mathbb{N}$. From (1.1), we note that
\[
\int_{Z_p} (1 + t)^{\lambda w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_1 w_2 w_4 i + w_1 w_3 w_4 j + w_2 w_3 w_4 k} \ d\mu_0(y)
\]
\[
= \lim_{N \to \infty} \frac{1}{p^N} \sum_{y=0}^{w_4 p^N - 1} (1 + t)^{\lambda w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_1 w_2 w_4 i + w_1 w_3 w_4 j + w_2 w_3 w_4 k}
\]
\[
= \frac{1}{w_4} \lim_{N \to \infty} \frac{1}{p^N} \sum_{y=0}^{w_4 - 1 p^N - 1} \sum_{l=0}^{w_4 - 1} \sum_{j=0}^{w_3 - 1} \sum_{k=0}^{w_1 - 1} (1 + t)^{\lambda w_1 w_2 w_3 (l + w_4 y) + w_1 w_2 w_3 w_4 x + w_1 w_2 w_4 i + w_1 w_3 w_4 j + w_2 w_3 w_4 k}.
\]

Thus, by (3.3), we get
\[
\frac{1}{w_1 w_2 w_3 w_4} \sum_{i=0}^{w_3 - 1} \sum_{j=0}^{w_2 - 1} \sum_{k=0}^{w_1 - 1} \int_{Z_p} (1 + t)^{\lambda w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_1 w_2 w_4 i + w_1 w_3 w_4 j + w_2 w_3 w_4 k} \ d\mu_0(y)
\]
\[
= \frac{1}{w_1 w_2 w_3 w_4} \lim_{N \to \infty} \frac{1}{p^N} \sum_{l=0}^{w_4 - 1} \sum_{j=0}^{w_3 - 1} \sum_{k=0}^{w_1 - 1} \sum_{y=0}^{w_4 p^N - 1} (1 + t)^{\lambda w_1 w_2 w_3 (l + w_4 y) + w_1 w_2 w_3 w_4 x + w_1 w_2 w_4 i + w_1 w_3 w_4 j + w_2 w_3 w_4 k}.
\]

As this expression is invariant under any permutation $\sigma \in D_4$, we have the following theorem.

**Theorem 3.1.** For $w_1, w_2, w_3, w_4 \in \mathbb{N}$, the following expressions
\[
\frac{1}{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}} \sum_{i=0}^{w_{\sigma(3)} - 1} \sum_{j=0}^{w_{\sigma(2)} - 1} \sum_{k=0}^{w_{\sigma(1)} - 1} \int_{Z_p} (1 + t)^{\lambda w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} y + w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} x + w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} i + w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} j + w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} k} \ d\mu_0(y),
\]
are the same for any $\sigma \in D_4$.

Now, we note that
\[
\int_{Z_p} (1 + t)^{\lambda w_1 w_2 w_3 y + w_1 w_2 w_3 w_4 x + w_1 w_2 w_4 i + w_1 w_3 w_4 j + w_2 w_3 w_4 k} \ d\mu_0(y)
\]
\[
= \sum_{n=0}^{\infty} D_n (w_1 w_2 w_3 x + w_1 w_2 w_4 i + w_1 w_3 w_4 j + w_2 w_3 w_4 k \mid \lambda w_1 w_2 w_3) \frac{t^n}{n!}.
\]

Therefore, by Theorem 3.1 and (3.4), we obtain the following theorem.
Theorem 3.2. For \( n \geq 0, w_1, w_2, w_3, w_4 \in \mathbb{N} \), the following expressions

\[
\frac{1}{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}} \sum_{i=0}^{w_{\sigma(3)}-1} \sum_{j=0}^{w_{\sigma(2)}-1} \sum_{k=0}^{w_{\sigma(1)}-1} D_n \left( w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)} | \lambda \sigma(1)w_{\sigma(2)}w_{\sigma(3)} \right)
\]

are the same for any \( \sigma \in D_4 \).

Now, we observe that

\[
\frac{1}{w_1w_2w_3} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_1-1} D_n \left( w_1w_2w_3w_4x + w_1w_2w_4i + w_1w_3w_4j + w_2w_3w_4k | \lambda w_1w_2w_3 \right)
\]

\[= \frac{1}{w_2w_3w_4} \sum_{i=0}^{w_4-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_2-1} D_n \left( w_2w_3w_4w_1x + w_2w_3w_4i + w_2w_4w_1j + w_3w_4w_1k | \lambda w_2w_3w_4 \right)
\]

\[= \frac{1}{w_3w_4w_1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_4-1} \sum_{k=0}^{w_3-1} D_n \left( w_3w_4w_1w_2x + w_3w_4w_2i + w_3w_1w_2j + w_4w_1w_2k | \lambda w_3w_4w_1 \right)
\]

\[= \frac{1}{w_4w_1w_2} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_1-1} \sum_{k=0}^{w_4-1} D_n \left( w_4w_1w_2w_3x + w_4w_1w_3i + w_4w_2w_3j + w_1w_2w_3k | \lambda w_4w_1w_2 \right)
\]

\[= \frac{1}{w_1w_4w_3} \sum_{i=0}^{w_4-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_1-1} D_n \left( w_1w_4w_3w_2x + w_1w_4w_2i + w_1w_3w_2j + w_4w_3w_2k | \lambda w_1w_4w_3 \right).
\]

Therefore, we obtain the following theorem.

Theorem 3.3. For \( n \geq 0 \), we have

\[
\frac{1}{w_1w_2w_3} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_2-1} \sum_{k=0}^{w_1-1} D_n \left( w_1w_2w_3w_4x + w_1w_2w_4i + w_1w_3w_4j + w_2w_3w_4k | \lambda w_1w_2w_3 \right)
\]

\[= \frac{1}{w_2w_3w_4} \sum_{i=0}^{w_4-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_2-1} D_n \left( w_2w_3w_4w_1x + w_2w_3w_4i + w_2w_4w_1j + w_3w_4w_1k | \lambda w_2w_3w_4 \right)
\]

\[= \frac{1}{w_3w_4w_1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_4-1} \sum_{k=0}^{w_3-1} D_n \left( w_3w_4w_1w_2x + w_3w_4w_2i + w_3w_1w_2j + w_4w_1w_2k | \lambda w_3w_4w_1 \right)
\]

\[= \frac{1}{w_4w_1w_2} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_1-1} \sum_{k=0}^{w_4-1} D_n \left( w_4w_1w_2w_3x + w_4w_1w_3i + w_4w_2w_3j + w_1w_2w_3k | \lambda w_4w_1w_2 \right).
\]
4. Conclusion

We consider the \(\lambda\)-Dahee polynomials such as various degenerate special polynomials over years: Koborov polynomials, \(\lambda\)-Bell polynomials, the degenerate Euler polynomials, the degenerate Bernoulli polynomials, the degenerate Genocchi polynomials and the Changhee polynomials have many applications in the mathematics and mathematical physics (see [4]). In Theorems 2.1 and 2.2, we gave some identities of related to the \(\lambda\)-Dahee polynomials. In Theorems 3.1, 3.2 and 3.3 we obtained new and novel symmetry properties related to \(\lambda\)-Dahee polynomials by using the symmetry properties of the \(p\)-adic invariant integral on \(\mathbb{Z}_p\).

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References


