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Existence and uniqueness of iterative positive solutions for singular Hammerstein integral equations

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Abstract

In this article, we study the existence and the uniqueness of iterative positive solutions for a class of nonlinear singular integral equations in which the nonlinear terms may be singular in both time and space variables. By using the fixed point theorem of mixed monotone operators in cones, we establish the conditions for the existence and uniqueness of positive solutions to the problem. Moreover, we derive various properties of the positive solutions to the equation and establish their dependence on the model parameter. The theorem obtained in this paper is more general and complements many previous known results including singular and nonlinear cases. Application of the results to the study of differential equations are also given in the article.

Keywords: Mixed monotone operator, fixed point theorem, iterative positive solution, singular integral equations, boundary value problem, cone.
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1. Introduction

Integral equations and boundary value problems for nonlinear fractional differential equations have been addressed by many researchers in recent decades, which is partly due to their numerous applications in many branches of science and engineering including fluid flow, rheology, diffusive transport akin to diffusion, electrical networks, probability, etc. For more details, we refer the reader to [3, 11, 12, 17, 25–27, 40, 41, 45] and the references therein. In particular, boundary value problems with integral boundary conditions for ordinary differential equations often arise in many fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, theorem-elasticity, and plasma physics (see [3, 5, 17, 27, 43]). The existence and uniqueness of positive solutions for such problems have become an important area of investigation in recent years. And these studies were mainly based on following methods: the properties of the Green function and the fixed point index theory (see [8, 29, 32, 33, 35, 38, 43]), the fixed point theorem of mixed monotone operators (see [4, 15, 18, 19, 22, 37, 41, 42]),

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the method of lower and upper solutions (see [40]), properties of operators (completely continuous, k-set-contractive, condensing and the potential tool of the axiomatic measures of non-compactness) (see [20, 21, 28, 30]), the fixed point theorem of cone expansion and compression (see [17]), the contraction mapping principle and successive approximation method (see [9]).

In this paper, we study the existence and uniqueness of the iterative positive solutions for the following class of singular Hammerstein integral equations:

$$x(t) = \lambda h(t, x(t), x(t)) + \lambda \int_0^1 G(t,s)\,[p(s)f(s,x(s),x(s)) + q(s)g(s,x(s),x(s))]\,ds, \quad 0 \leqslant t \leqslant 1,\ \lambda > 0, \tag{1.1}$$

where $p, q : (0, 1) \to [0, +\infty)$ are continuous, and $p(t), q(t)$ are allowed to be singular at $t = 0$ or $t = 1$, $f, g, h : (0, 1) \times (0, +\infty) \times (0, +\infty) \to [0, +\infty)$ are continuous, and also $f(t, u, v)$, $g(t, u, v)$ and $h(t, u, v)$ may be singular at $t = 0, 1$ and $v = 0$.

There are various results in literature related to the uniqueness of solutions for singular and nonlinear boundary value problems (see [1]). In [16], Li et al. studied the existence of solutions for the following integral equation in the Banach space $C(G)$:

$$u(x) = \int_G k(x,y) f(y, u(y))\, dy, \quad x \in G,$$

where G is a bounded closed subset of R^N with the measure mes $G > 0$, $k : G \times G \to R^1$ is nonnegative, continuous and symmetric, i.e., $k(x, y) = k(y, x)$ for all $x, y \in G$, and $k \not\equiv 0$ on $G \times G$. $f : G \times R^1 \to R^1$ is continuous. Furthermore, they applied these results to the following fourth-order boundary value problem of ordinary differential equation (BVP):

$$\begin{cases} u^{(4)}(t) + \beta u''(t) - \alpha u(t) = f(t, x(t)), & t \in [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{1.2}$$

where $f : [0, 1] \times R^1 \to R^1$ is continuous, $\alpha, \beta \in R^1$ satisfy $\beta < 2\pi^2$, $\beta^2 + 4\alpha \geqslant 0$, $\alpha/\pi^4 + \beta/\pi^2 < 1$. By using the strongly monotone operator principle and the critical point theory, respectively, they established some conditions on f which guarantee that BVP (1.2) has a unique solution, at least one nonzero solution, and infinitely many solutions respectively.

In [14], making use of the De Blasi measure of weak noncompactness, the authors establish a variant of the topological fixed point theorem of Krasnoselskii for the sum of two operators without requiring weak compactness or weak continuity assumptions. Using the result they studied the solvability of the following variant of Hammersteins integral equation

$$\psi(t) = g(t, \psi(t)) + \lambda \int_\Omega \zeta(t,s) f(s, \psi(s))ds,$$

in $L_1(\Omega, X)$, the space of Lebesgue integrable functions on a measurable subset Ω of R^N with values in a finite dimensional Banach space X, where g is a function satisfying a contraction condition with respect to the second variable while $f(\cdot, \cdot)$ (resp. $\zeta(\cdot, \cdot)$) is a nonlinear (resp. measurable) function.

In [38], by using the fixed point index method, Zhai et al. established the existence of at least one or at least two symmetric positive solutions for the following boundary value problem:

$$\begin{cases} x^{(4)}(t) = p(t) f(t, x(t)), & t \in [0,1], \\ x(0) = x(1) = x'(0) = x'(1) = 0, \end{cases} \tag{1.3}$$

where $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous and $f(t, x) = f(1 - t, x)$, for each $(t, x) \in [0, 1] \times [0, +\infty)$. Here a symmetric positive solution of the problem (1.3) means a solution x of the problem (1.3) satisfying

$$x(t) = x(1 - t),\ t \in [0, 1] \quad \text{and} \quad x(t) > 0,\ t \in (0, 1).$$

In [27], by means of the monotone iteration method, Sun and Zhao investigated the existence of positive solutions for the following fractional differential equation with integral boundary conditions:

$$\begin{cases} D^{\alpha}_{0^+}x(t) + q(t)f(t,x(t)) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = 0, \quad x(1) = \displaystyle\int_0^1 g(s)x(s)ds, \end{cases}$$

where $2 < \alpha \leqslant 3$, $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville derivative of order α, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, and $g, q : (0,1) \to [0,+\infty)$ are also continuous with $g(t), q(t) \in L^1(0,1)$.

Zhang et al. [43], by using the properties of the Green function and the fixed point index theory, considered the existence of a positive solution of the following nonlinear fractional differential equation with integral boundary conditions:

$$\begin{cases} D^{\alpha}_{0^+}x(t) + h(t)f(t,x(t)) = 0, \quad 0 < t < 1, \\ x(0) = x'(0) = x''(0) = 0, \quad x(1) = \lambda\displaystyle\int_0^{\eta} x(s)ds, \end{cases}$$

where $3 < \alpha \leqslant 4$, $0 < \eta \leqslant 1$, $0 \leqslant \frac{\lambda\eta^{\alpha}}{\alpha} < 1$, $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative, $h : (0,1) \to [0,+\infty)$ is continuous with $h \in L^1(0,1)$, and $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is also continuous.

In [37], Yuan et al. established the existence and uniqueness of positive solution for a singular $2m$-th order continuous Lidstone boundary value problem by the fixed point theorem of mixed monotone operators:

$$\begin{cases} (-1)^m x^{(2m)}(t) = \lambda f(t,x(t)), \quad t \in (0,1), \ \lambda > 0, \\ x^{(2i)}(0) = x^{(2i)}(1) = 0, \quad 0 \leqslant i \leqslant m-1, \end{cases}$$

where $m \geqslant 2$, $f \in C([0,1] \times (0,+\infty), [0,+\infty))$, and $f(t,u)$ may be singular at $u = 0$.

Recently, by the mixed monotone method, Cao et al. [4] presented a uniqueness result for the following singular integral equation:

$$x(t) = \lambda\int_0^1 G(t,s)f(s,x(s))ds, \ \ 0 \leqslant t \leqslant 1, \ \lambda > 0,$$

where $f \in C((0,1) \times (0,+\infty), (0,+\infty))$, and $f(t,u)$ may be singular at $t = 0$, $t = 1$, and/or $u = 0$.

However, up to now, the singular integral equations have seldom been considered by using the fixed point theorem, especially when $f(t,u,v)$, $g(t,u,v)$, and $h(t,u,v)$ in integral equation (1.1) have singularity at $t = 0$ or 1 and $v = 0$. In this article, we apply the fixed point theorem of mixed monotone operators to establish the existence and uniqueness of the iterative solutions for the singular Hammerstein integral equation (1.1).

Our work has many new features. In comparison with the results in [4, 27, 38, 43], we use a new method to deal with problem (1.1), and do not need to suppose the existence of upper-lower solutions. Furthermore, $f(t,u,v)$, $g(t,u,v)$, and $h(t,u,v)$ not only depend on t and u as well as v, but also are allowed to be singular in both time and space variables. Comparing with the results in [27, 37, 38, 43], we do not need to require $f(t,u,v)$, $g(t,u,v)$, and $h(t,u,v)$ to be continuous at $t = 0$ or 1, and at $v = 0$. Moreover, different from the above work mentioned, in this paper we also successively establish some sequences for approximating the unique positive solution. More specifically, the main new features presented in this article are as follows. Firstly, the Hammerstein integral equation has a more general form in which $p(t)$, $q(t)$ are allowed to be singular at $t = 0, 1$, and f, g, h may be singular in both time and space variables, that is, $f(t,u,v)$, $g(t,u,v)$, and $h(t,u,v)$ may be singular at $t = 0$ or 1 and $v = 0$. Secondly, our analysis relies on a fixed point theorem of a sum operator. By using the fixed point theorem of mixed monotone operators in partial ordering Banach spaces, we establish the existence and uniqueness of a positive solution for the Hammerstein integral equation (1.1). Our results not only guarantee the existence of a unique positive solution, but also can be applied to construct an iterative scheme for approximating it. Moreover, we derive various properties of the positive solution to the Hammerstein integral equation and establish their dependence on the equation parameter. We have also successfully demonstrated that our results can be applied to study the existence of solution of a class of singular nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions and the $(k, n-k)$ conjugate boundary value problems for nonlinear ordinary differential equations. We should also emphasize that this paper extends and improves many known results including singular and nonsingular cases.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are to be used to prove our main results. In Section 3, we discuss the existence and uniqueness of positive solution of the Hammerstein integral equation (1.1) and also construct successively some sequences for approximating the unique positive solution. In Section 4, we demonstrate the application of our main results to the study the existence of solution of a class of singular nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions and the $(k, n-k)$ conjugate boundary value problems for nonlinear ordinary differential equations.

2. Preliminaries and lemmas

In this section, we present briefly some definitions, lemmas, and basic results that are to be used in the article for the convenience of readers, and we refer the readers to [6, 7, 12, 22, 23, 25, 26] for more details.

Let $(E, \|\cdot\|)$ be a real Banach space. We denote the zero element of E by θ. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies

(1) $x \in P,\ \lambda \geqslant 0 \Rightarrow \lambda x \in P$;

(2) $x \in P,\ -x \in P \Rightarrow x = \theta$.

The Banach space E is partially ordered by a cone $P \subset E$, that is, $x \leqslant y$ if and only if $y - x \in P$. The cone P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$; the smallest such N is called the normality constant of P.

For $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leqslant x \leqslant x_2\}$ is called the order interval between x_1 and x_2. For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Let $h \in P$, $h \neq \theta$, define $P_h = \{x \in P \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is a component of P. Let $D \subseteq E$, $A : D \times D \to E$ is said to be a mixed monotone operator if $A(x, y)$ is increasing in x, and decreasing in y, that is, $x_i, y_i \in D$ $(i = 1, 2)$, $x_1 \leqslant x_2$, $y_1 \geqslant y_2$ imply $A(x_1, y_1) \leqslant A(x_2, y_2)$. Element $x \in D$ is called a fixed point of A if $A(x, x) = x$. For more details, see for example, [6, 7].

Lemma 2.1 ([22, 42]). *Let $h \in P$, $h \neq \theta$. Suppose $A, B : P_h \times P_h \to P_h$ is two mixed monotone operators and satisfy the following conditions:*

(1) *for all $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that, for all $x, y \in P_h$,*
$$A(tx, t^{-1}y) \geqslant \varphi(t)A(x, y);$$

(2) *for all $t \in (0, 1)$ and $x, y \in P_h$,*
$$B(tx, t^{-1}y) \geqslant tB(x, y);$$

(3) *there exists a constant $\delta > 0$ such that, for all $x, y \in P_h$,*
$$A(x, y) \geqslant \delta B(x, y).$$

Then the operator equation

$$A(x,x)+B(x,x)=x,$$

has a unique solution x^* *in* P_h, *and for any initial values* $x_0, y_0 \in P_h$, *by constructing successively the sequences as follows*

$$x_n = A(x_{n-1}, y_{n-1}) + B(x_{n-1}, y_{n-1}),$$

$$y_n = A(y_{n-1}, x_{n-1}) + B(y_{n-1}, x_{n-1}), \quad n = 1,2,\cdots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *in* E *as* $n \to \infty$.

Remark 2.2. If we take $\varphi(t) = t^\gamma$ $(0<\gamma<1)$ in Lemma 2.1, the conclusions also hold, that is [22, Corollary 3.7].

Lemma 2.3 ([22, 42])**.** *Suppose that all the conditions of Lemma 2.1 are satisfied, then the operator equation*

$$A(x,x)+B(x,x)=\lambda x, \quad \lambda>0,$$

has a unique solution x_λ *which satisfies*

(1) *if there exists* $\beta \in (0,1)$ *such that,* $\varphi(t) \geqslant \frac{t^\beta - t}{\delta} + t^\beta$ *for* $t \in (0,1)$, *then* x_λ *is continuous in* $\lambda \in (0,+\infty)$, *that is,* $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ *implies* $\|x_\lambda - x_{\lambda_0}\| \to 0$;

(2) *if* $\varphi(t) > \frac{t^{\frac{1}{2}} - t}{\delta} + t^{\frac{1}{2}}$ *for* $t \in (0,1)$, *then* x_λ *is strictly decreasing in* λ, *that is,* $0<\lambda_1<\lambda_2$ *implies* $x_{\lambda_1} > x_{\lambda_2}$;

(3) *if there exists* $\beta \in (0,\frac{1}{2})$ *such that,* $\varphi(t) \geqslant \frac{t^\beta - t}{\delta} + t^\beta$ *for* $t \in (0,1)$, *then* $\lim\limits_{\lambda\to\infty} \|x_\lambda\| = 0$, $\lim\limits_{\lambda\to 0^+} \|x_\lambda\| = \infty$.

3. Main results

In this section, we discuss the Hammerstein integral equation (1.1). Throughout this section we assume that

(i) $G(t,s) : [0,1]\times[0,1] \to [0,+\infty)$ is continuous;

(ii) there exist $a, m, n \in C[0,1]$ with $a(t), m(t), n(t) > 0$ for $t \in (0,1)$ such that,

$$a(t)m(s) \leqslant G(t,s) \leqslant a(t)n(s), \quad \forall\, t \in [0,1], \quad s \in [0,1];$$

(iii) $0 < a(t) < 1$, $t \in (0,1)$.

Theorem 3.1. *Assume that the following conditions hold:*

(H_1) $p, q : (0,1) \to [0,+\infty)$ *are continuous, and* $p(t)$, $q(t)$ *are allowed to be singular at* $t=0$ *or* $t=1$;

(H_2) $f, g, h : (0,1)\times(0,+\infty)\times(0,+\infty) \to [0,+\infty)$ *are continuous and* $f(t,u,v)$, $g(t,u,v)$ *and* $h(t,u,v)$ *may be singular at* $t=0$ *or 1 and* $v=0$;

(H_3) *for fixed* $t \in (0,1)$, $v \in (0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$ *and* $h(t,u,v)$ *are increasing in* $u \in (0,+\infty)$. *For fixed* $t \in (0,1)$, $u \in (0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$ *and* $h(t,u,v)$ *are decreasing in* $v \in (0,+\infty)$;

(H_4) *for all* $l \in (0,1)$, *there exists* $\varphi_1(l) \in (l,1]$ *such that for all* $t \in (0,1)$ *and* $u, v \in (0,+\infty)$,

$$f(t, lu, l^{-1}v) \geqslant \varphi_1(l) f(t,u,v). \tag{3.1}$$

For all $l \in (0,1)$, *there exists* $\varphi_2(l) \in (l,1]$ *such that for all* $t \in (0,1)$ *and* $u, v \in (0,+\infty)$,

$$h(t, lu, l^{-1}v) \geqslant \varphi_2(l) h(t,u,v), \tag{3.2}$$

and for all $l \in (0,1)$, *for all* $t \in (0,1)$, *and* $u, v \in (0,+\infty)$,

$$g(t, lu, l^{-1}v) \geqslant l g(t,u,v); \tag{3.3}$$

(H_5)

$$\int_0^1 n(s)p(s)\frac{1}{\varphi_1(a(s))}f(s,1,1)ds < +\infty,$$
$$\int_0^1 n(s)q(s)\left(a(s)\right)^{-1}g(s,1,1)ds < +\infty;$$

(H_6) *there exists a constant* $\delta > 0$ *such that for all* $t \in (0,1)$ *and* $u, v \in (0,+\infty)$, $f(t,u,v) \geqslant \delta g(t,u,v)$.

Then the singular Hammerstein integral equation (1.1) *has a unique positive solution* $x^*_\lambda(t)$, *which satisfies* $\frac{1}{d}a(t) \leqslant x^*_\lambda(t) \leqslant da(t)$, $t \in [0,1]$, *for a constant* $d > 0$. *Moreover, for any initial values* $x_0, y_0 \in P_h$, $h = a(t)$, *the sequences* $\{x_n(t)\}$, $\{y_n(t)\}$ *of successive approximations defined by*

$$\begin{aligned} x_n(t) =& \lambda h\left(s, x_{n-1}(s), y_{n-1}(s)\right) + \lambda\int_0^1 G(t,s)p(s)f\left(s, x_{n-1}(s), y_{n-1}(s)\right)ds \\ &+ \lambda\int_0^1 G(t,s)q(s)g\left(s, x_{n-1}(s), y_{n-1}(s)\right)ds, \\ y_n(t) =& \lambda h\left(s, y_{n-1}(s), x_{n-1}(s)\right) + \lambda\int_0^1 G(t,s)p(s)f\left(s, y_{n-1}(s), x_{n-1}(s)\right)ds \\ &+ \lambda\int_0^1 G(t,s)q(s)g\left(s, y_{n-1}(s), x_{n-1}(s)\right)ds, \quad n = 1,2,\cdots, \end{aligned}$$

both converge uniformly to $x^*_\lambda(t)$ *on* $[0,1]$ *as* $n \to \infty$. *Moreover, the unique positive solution* $x^*_\lambda(t)$ *has the following properties:*

(1) *if there exists* $\beta \in (0,1)$ *such that* $\varphi_i(t) \geqslant \frac{t^\beta - t}{\delta} + t^\beta$ $(i = 1,2)$ *for* $t \in (0,1)$, *then* x_λ *is continuous in* $\lambda \in (0,\infty)$, *that is* $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ *implies* $\|x^*_\lambda - x^*_{\lambda_0}\| \to 0$;

(2) *if* $\varphi_i(t) > \frac{t^{\frac{1}{2}} - t}{\delta} + t^{\frac{1}{2}}$ $(i = 1,2)$ *for* $t \in (0,1)$, *then* x_λ *is strictly increasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *implies* $x^*_{\lambda_1} < x^*_{\lambda_2}$;

(3) *if there exists* $\beta \in (0,\frac{1}{2})$ *such that* $\varphi_i(t) \geqslant \frac{t^\beta - t}{\delta} + t^\beta$ $(i = 1,2)$ *for* $t \in (0,1)$, *then* $\lim\limits_{\lambda\to\infty}\|x^*_\lambda\| = \infty$, $\lim\limits_{\lambda\to 0^+}\|x^*_\lambda\| = 0$.

Proof. Let $E = C[0,1]$ and $\|x\| = \sup\limits_{0\leqslant t\leqslant 1}|x(t)|$. Obviously, $(E, \|\cdot\|)$ is a Banach space. Let

$$P = \{x \in E : x(t) \geqslant 0, t \in [0,1]\}.$$

Clearly P is a normal cone of the Banach space E. Let $h(t) = a(t)$, and we define

$$P_h = \left\{x \in C[0,1] \mid \exists D \geqslant 1 : \frac{1}{D}a(t) \leqslant x(t) \leqslant Da(t), t \in [0,1]\right\}.$$

Define three operators $A_\lambda, B_\lambda, C_\lambda : P_h \times P_h \to P$ by

$$\begin{aligned} A_\lambda(x,y)(t) &= \lambda\int_0^1 G(t,s)p(s)f(s,x(s),y(s))ds, \\ B_\lambda(x,y)(t) &= \lambda\int_0^1 G(t,s)q(s)g(s,x(s),y(s))ds, \\ C_\lambda(x,y)(t) &= \lambda h(t,x(t),y(t)), \end{aligned}$$

respectively. Then it is easy to prove that x is the solution of the singular Hammerstein integral equation (1.1) if it satisfies $x = A_\lambda(x,x) + B_\lambda(x,x) + C_\lambda(x,x)$.

(1) Firstly, we show that $A_\lambda, B_\lambda, C_\lambda : P_h \times P_h \to P$ are well-defined. From (H_4) we have that, for all $l \in (0,1)$, $t \in (0,1)$, and $u, v \in (0,+\infty)$,

$$f(t,u,v) = f(t, ll^{-1}u, l^{-1}lv) \geqslant \varphi_1(l) f(t, l^{-1}u, lv), \tag{3.4}$$

$$g(t,u,v) = g(t, ll^{-1}u, l^{-1}lv) \geqslant l g(t, l^{-1}u, lv), \tag{3.5}$$

$$h(t,u,v) = h(t, ll^{-1}u, l^{-1}lv) \geqslant \varphi_2(l) h(t, l^{-1}u, lv). \tag{3.6}$$

So by (3.4), (3.5), (3.6) for all $l \in (0,1)$, $t \in (0,1)$, and $u, v \in (0,+\infty)$, we have

$$f(t, l^{-1}u, lv) \leqslant \frac{1}{\varphi_1(l)} f(t,u,v), \quad g(t, l^{-1}u, lv) \leqslant l^{-1} g(t,u,v), \quad h(t, l^{-1}u, lv) \leqslant \frac{1}{\varphi_2(l)} h(t,u,v). \tag{3.7}$$

Taking $u = v = 1$ in (3.1), (3.2), (3.3) and (3.7) respectively, we have

$$f(t,l,l^{-1}) \geqslant \varphi_1(l) f(t,1,1), \quad f(t,l^{-1},l) \leqslant \frac{1}{\varphi_1(l)} f(t,1,1), \quad \forall l \in (0,1), \quad t \in (0,1), \tag{3.8}$$

$$g(t,l,l^{-1}) \geqslant l g(t,1,1), \quad g(t,l^{-1},l) \leqslant l^{-1} g(t,1,1), \quad \forall l \in (0,1), \quad t \in (0,1),$$

$$h(t,l,l^{-1}) \geqslant \varphi_2(l) h(t,1,1), \quad h(t,l^{-1},l) \leqslant \frac{1}{\varphi_2(l)} h(t,1,1), \quad \forall l \in (0,1), \quad t \in (0,1). \tag{3.9}$$

For any $x, y \in P_h$, we can choose a constant $D_1 \geqslant 1$ be such that $\frac{1}{D_1} a(t) \leqslant x, y \leqslant D_1 a(t)$. On the one hand, from (H_3), (H_4), (3.7) and (3.8), we have

$$\begin{aligned} f(t,x(t),y(t)) &\leqslant f\left(t, D_1 a(t), D_1^{-1} a(t)\right) \leqslant f\left(t, D_1 (a(t))^{-1}, D_1^{-1} a(t)\right) \\ &\leqslant \frac{1}{\varphi_1(a(t))} f\left(t, D_1, D_1^{-1}\right) \leqslant \frac{1}{\varphi_1(a(t)) \varphi_1\left(D_1^{-1}\right)} f(t,1,1), \quad t \in (0,1), \end{aligned} \tag{3.10}$$

$$\begin{aligned} f(t,x(t),y(t)) &\geqslant f\left(t, D_1^{-1} a(t), D_1 a(t)\right) \geqslant f\left(t, D_1^{-1} a(t), D_1 (a(t))^{-1}\right) \\ &\geqslant \varphi_1(a(t)) f\left(t, D_1^{-1}, D_1\right) \geqslant \varphi_1\left(a(t)\right) \varphi_1\left(D_1^{-1}\right) f(t,1,1), \quad t \in (0,1). \end{aligned} \tag{3.11}$$

On the other hand, from (H_3), (H_4), (3.7) and (3.9), we get

$$\begin{aligned} g(t,x(t),y(t)) &\leqslant g\left(t, D a(t), D_1^{-1} a(t)\right) \leqslant g\left(t, D a(t)^{-1}, D_1^{-1} a(t)\right) \\ &\leqslant (a(t))^{-1} g\left(t, D, D_1^{-1}\right) \leqslant (a(t))^{-1} D_1 g(t,1,1) \leqslant (a(t))^{-1} D_1^2 g(t,1,1), \quad t \in (0,1), \end{aligned} \tag{3.12}$$

$$\begin{aligned} g(t,x(t),y(t)) &\geqslant g\left(t, D_1^{-1} a(t), D_1 a(t)\right) \geqslant g\left(t, D_1^{-1} a(t), D_1 (a(t))^{-1}\right) \\ &\geqslant a(t) g\left(t, D_1^{-1}, D_1\right) \geqslant a(t) D_1^{-1} g(t,1,1) \geqslant a(t) D_1^{-2} g(t,1,1), \quad t \in (0,1). \end{aligned} \tag{3.13}$$

From (H_3), (H_4), (3.7) and (3.9), we get

$$\begin{aligned} h(t,x(t),y(t)) &\leqslant h\left(t, D_1 a(t), D_1^{-1} a(t)\right) \leqslant h\left(t, D_1 (a(t))^{-1}, D_1^{-1} a(t)\right) \\ &\leqslant \frac{1}{\varphi_2(a(t))} h\left(t, D_1, D_1^{-1}\right) \leqslant \frac{1}{\varphi_2(a(t)) \varphi_2\left(D_1^{-1}\right)} h(t,1,1), \quad t \in (0,1), \end{aligned} \tag{3.14}$$

$$\begin{aligned} h(t,x(t),y(t)) &\geqslant h\left(t, D_1^{-1} a(t), D_1 a(t)\right) \geqslant h\left(t, D_1^{-1} a(t), D_1 (a(t))^{-1}\right) \\ &\geqslant \varphi_2(a(t)) h\left(t, D_1^{-1}, D_1\right) \geqslant \varphi_2\left(a(t)\right) \varphi_2\left(D_1^{-1}\right) h(t,1,1), \quad t \in (0,1). \end{aligned} \tag{3.15}$$

By (ii), (H_5), (3.10) and (3.12), we get

$$\begin{aligned}\lambda\int_0^1 G(t,s)p(s)f(s,x(s),y(s))ds &\leqslant \lambda\int_0^1 G(t,s)p(s)\frac{1}{\varphi_1(a(s))\varphi_1(D_1^{-1})}f(s,1,1)ds\\ &\leqslant a(t)\lambda\int_0^1 n(s)p(s)\frac{1}{\varphi_1(a(s))\varphi_1(D_1^{-1})}f(s,1,1)ds\\ &< +\infty,\end{aligned}$$

$$\begin{aligned}\lambda\int_0^1 G(t,s)q(s)g(s,x(s),y(s))ds &\leqslant \lambda\int_0^1 G(t,s)q(s)\left(a(s)\right)^{-1}D_1^2 g(s,1,1)ds\\ &\leqslant a(t)\lambda\int_0^1 n(s)q(s)\left(a(s)\right)^{-1}D_1^2 g(s,1,1)ds\\ &< +\infty.\end{aligned}$$

From (i) we have that $A_\lambda, B_\lambda, C_\lambda : P_h\times P_h \to P$ are well-defined.

(2) Secondly, we prove that $A_\lambda, B_\lambda, C_\lambda : P_h\times P_h \to P_h$. Let $D \geqslant 1$ be such that

$$\begin{aligned}D > \max\Bigg\{&\lambda\int_0^1 n(s)p(s)\frac{1}{\varphi_1(a(s))\varphi_1(D_1^{-1})}f(s,1,1)ds,\\ &\left(\lambda\int_0^1 m(s)p(s)\varphi_1\left(a(t)\right)\varphi_1\left(D_1^{-1}\right)f(s,1,1)ds\right)^{-1},\\ &\lambda\int_0^1 n(s)q(s)(a(s))^{-1}D_1^2 g(s,1)ds,\\ &\left(\lambda\int_0^1 m(s)q(s)a(s)D_1^{-2}g(s,1)ds\right)^{-1},\\ &\frac{\lambda}{a(t)\varphi_2(a(t))\varphi_2\left(D_1^{-1}\right)}h(t,1,1),\\ &\left(\lambda\varphi_2\left(D_1^{-1}\right)h(t,1,1)\right)^{-1}\Bigg\}.\end{aligned} \tag{3.16}$$

Then from (ii), (3.10), (3.11), (3.12), (3.13) and (3.16), for all $t\in[0,1]$, $x,y\in P_h$, we have

$$\begin{aligned}A_\lambda(x,y)(t) &= \lambda\int_0^1 G(t,s)p(s)f(s,x(s),y(s))ds\\ &\leqslant \lambda\int_0^1 G(t,s)p(s)\frac{1}{\varphi_1(a(s))\varphi_1(D_1^{-1})}f(s,1,1)ds\\ &\leqslant \lambda a(t)\int_0^1 n(s)p(s)\frac{1}{\varphi_1(a(s))\varphi_1(D_1^{-1})}f(s,1,1)ds\\ &\leqslant Da(t),\end{aligned}$$

$$\begin{aligned}A_\lambda(x,y)(t) &= \lambda\int_0^1 G(t,s)p(s)f(s,x(s),y(s))ds\\ &\geqslant \lambda\int_0^1 G(t,s)p(s)\varphi_1\left(a(t)\right)\varphi_1\left(D_1^{-1}\right)f(s,1,1)ds\\ &\geqslant \lambda a(t)\int_0^1 m(s)p(s)\varphi_1\left(a(t)\right)\varphi_1\left(D_1^{-1}\right)f(s,1,1)ds\\ &\geqslant \frac{1}{D}a(t),\end{aligned}$$

$$\begin{aligned}
B_\lambda(x,y)(t) &= \lambda \int_0^1 G(t,s)q(s)g(s,x(s),y(s))ds \\
&\leqslant \lambda \int_0^1 G(t,s)q(s)a(s)^{-1}D_1^2 g(s,1,1)ds \\
&\leqslant \lambda a(t) \int_0^1 n(s)q(s)a(s)^{-1}D_1^2 g(s,1,1)ds \\
&\leqslant Da(t), \\
B_\lambda(x,y)(t) &= \lambda \int_0^1 G(t,s)q(s)g(s,x(s),y(s))ds \\
&\geqslant \lambda \int_0^1 G(t,s)q(s)a(s)D_1^{-2} g(s,1,1)ds \\
&\geqslant \lambda a(t) \int_0^1 m(s)q(s)a(s)D_1^{-2} g(s,1,1)ds \\
&\geqslant \frac{1}{D}a(t).
\end{aligned}$$

From (ii), (3.14), (3.15) and (3.16) for all $t \in (0,1)$, $x, y \in P_h$, we have

$$C_\lambda(x,y)(t) = \lambda h(t,x(t),y(t)) \leqslant \frac{\lambda}{\varphi_2(a(t))\varphi_2\left(D_1^{-1}\right)} h(t,1,1) \leqslant Da(t),$$

$$\begin{aligned}
C_\lambda(x,y)(t) &= \lambda h(t,x(t),y(t))ds \\
&\geqslant \lambda \varphi_2\left(a(t)\right) \varphi_2\left(D_1^{-1}\right) h(t,1,1) \\
&\geqslant \lambda a(t) \varphi_2\left(D_1^{-1}\right) h(t,1,1) \\
&\geqslant \frac{1}{D}a(t).
\end{aligned}$$

So $A_\lambda, B_\lambda, C_\lambda : P_h \times P_h \to P_h$.

(3) Next by (H_3), it is easy to prove that $A_\lambda, B_\lambda, C_\lambda : P_h \times P_h \to P_h$ are three mixed monotone operators.

(4) From (H_4), for any $t \in [0,1]$, $l \in (0,1)$ and $x, y \in P_h$, we have

$$\begin{aligned}
A_\lambda(lx, l^{-1}y)(t) &= \lambda \int_0^1 G(t,s) f\left(s, lx(s), l^{-1}y(s)\right) ds \\
&\geqslant \varphi_1(l)\lambda \int_0^1 G(t,s) f\left(s, x(s), y(s)\right) ds \\
&= \varphi_1(l) A_\lambda(x,y)(t), \\
B_\lambda(lx, l^{-1}y)(t) &= \lambda \int_0^1 G(t,s) g\left(s, lx, l^{-1}y(s)\right) ds \\
&\geqslant \lambda \int_0^1 G(t,s) g\left(s, x(s), y(s)\right) ds \\
&= l B_\lambda(x,y)(t),
\end{aligned}$$

$$C_\lambda(lx, l^{-1}y)(t) = \lambda h\left(t, lx, l^{-1}y(t)\right) \geqslant \varphi_2(l)\lambda h\left(t, x(t), y(t)\right) = \varphi_2(l)C_\lambda(x, y)(t),$$

that is, $A_\lambda(lx, l^{-1}y) \geqslant \varphi_1(l)A_\lambda(x, y)$, $B_\lambda(lx, l^{-1}y) \geqslant lB_\lambda(x, y)$, $C_\lambda(lx, l^{-1}y) \geqslant \varphi_2(l)C_\lambda(x, y)$ for all $l \in (0, 1)$, $x, y \in P_h$.

Let $\varphi(l) := \min\{\varphi_1(l), \varphi_2(l)\}$. For all $l \in (0, 1)$, $x, y \in P_h$, we have

$$\begin{aligned} A_\lambda(lx, l^{-1}y) + C_\lambda(lx, l^{-1}y) &\geqslant \varphi_1(l)A_\lambda(x, y) + \varphi_2(l)C_\lambda(x, y) \\ &\geqslant \varphi(l)\left[A_\lambda(x, y) + C_\lambda(x, y)\right]. \end{aligned}$$

(5) By (H_6), for all $t \in [0, 1]$, $x, y \in P_h$, we have

$$\begin{aligned} A_\lambda(x, y)(t) =& \lambda \int_0^1 G(t, s)f(s, x(s), y(s))ds \\ \geqslant& \delta\lambda \int_0^1 G(t, s)g(s, x(s), y(s))ds \\ =& \delta B_\lambda(x, y)(t). \end{aligned}$$

Thus, $A_\lambda(x, y) + C_\lambda(x, y) \geqslant \delta B_\lambda(x, y)$ for all $x, y \in P_h$. Substitute $\frac{1}{\lambda}$ for λ, then by Lemma 2.3 the conclusions of Theorem 3.1 hold. □

Remark 3.2. If we take $\varphi(t) = t^\gamma$ $(0 < \gamma < 1)$ in Theorem 3.1, then a structure of the proof implies a special version of the main results in Theorem 3.1.

4. Application to boundary value problems

In this section, we apply the results in Section 3 to study the existence of solution of a class of singular nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions and the $(k, n-k)$ conjugate boundary value problems for nonlinear ordinary differential equations. In recent years, the study of positive solutions for singular nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions has attracted considerable attention, and many results have been achieved, and here we refer the reader to [23, 36] and the references therein for more details. The subject of $(k, n-k)$ conjugate boundary value problems have a wide range of applications, for example in material mechanics and physical sciences, such as be used to model the deformation of elastic beams. Recently, many people paid attention to existence result of solution of $(k, n-k)$ conjugate boundary value problems, such as [10, 13, 18, 31, 34, 36, 44]. If $n = 4$ and $k = 2$, then the conjugate boundary value problem reduces to the fourth order problem, because it describes the deflection of an elastic beam under a certain force. The existence and uniqueness of positive solutions for the elastic beam equations have been studied extensively, see for example [1, 2, 24] and the references therein.

However, not much work has been done to utilize the fixed point results on mixed monotone operators in partial ordering Banach spaces to study the existence and uniqueness of positive solutions for the singular nonlinear fractional differential equations with Riemann-Stieltjes integral boundary conditions and the $(k, n-k)$ conjugate boundary value problems for nonlinear ordinary differential equations. This motivates us to investigate boundary value problems (4.1) and (4.2) by using our new fixed point theorems presented in Section 3. We will demonstrate that our results not only can guarantee the existence of a unique positive solution, but also can be applied to construct an iterative scheme for approximating the solution.

4.1. Nonlinear fractional boundary value problem

Consider the following nonlinear fractional boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}x(t) + \lambda p(t)f(t, x(t), x(t)) + \lambda q(t)g(t, x(t), x(t)) = 0, \quad t \in (0, 1), \\ x(0) = x'(0) = 0, \quad D_{0+}^{\beta}x(1) = \int_0^1 D_{0+}^{\beta}x(t)dA(t), \end{cases} \tag{4.1}$$

where $2 < \alpha \leqslant 3$, $0 < \beta \leqslant 1$, $\lambda > 0$ are real numbers and $\int_0^1 D_{0+}^{\beta} x(t)dA(t)$ denotes a Riemann-Stieltjes integral.

Definition 4.1 ([25]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 4.2 ([25]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 4.3 ([39]). *Let* $y \in C(0,1) \cap L^1[0,1]$ *and* $\delta_0 := \int_0^1 t^{\alpha-\beta-1} dA(t) < 1$, *then the unique solution of*

$$\begin{cases} D_{0+}^{\alpha} x(t) + y(t) = 0, & t \in (0,1), \\ x(0) = x'(0) = 0, & D_{0+}^{\beta} x(1) = \int_0^1 D_{0+}^{\beta} x(t)dA(t), \end{cases}$$

is $x(t) = \int_0^1 G(t,s)y(s)ds$, *in which*

$$G(t,s) = K(t,s) + \frac{t^{\alpha-1}}{1-\delta_0} \int_0^1 H(t,s)dA(t), \quad t, s \in [0,1],$$

where

$$K(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

and

$$H(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ [t(1-s)]^{\alpha-\beta-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Lemma 4.4 ([39]). $H(t,s)$ *has the following property for any* $t, s \in [0,1]$,

$$\min\{\alpha-\beta-1, 1\} t^{\alpha-\beta-1}(1-t)(1-s)^{\alpha-\beta-1} s \leqslant \Gamma(\alpha)H(t,s) \leqslant \max\{\alpha-\beta-1, 1\}(1-s)^{\alpha-\beta-1}.$$

Lemma 4.5 ([39]). *The function* $G(t,s)$ *satisfies:*

$$t^{\alpha-1}\beta s(1-s)^{\alpha-\beta-1} \leqslant \Gamma(\alpha)K(t,s) \leqslant t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \quad t, s \in [0,1].$$

Lemma 4.6 ([39]). *Let*

$$m(s) := \frac{1}{\Gamma(\alpha)} \left[\beta + \frac{\min\{\alpha-\beta-1, 1\} \int_0^1 t^{\alpha-\beta-1}(1-t)dA(t)}{1-\delta_0}\right] s(1-s)^{\alpha-\beta-1},$$

$$n(s) := \frac{1}{\Gamma(\alpha)} \left[1 + \frac{\min\{\alpha-\beta-1, 1\} \int_0^1 dA(t)}{1-\delta_0}\right] (1-s)^{\alpha-\beta-1}.$$

Then the function $G(t,s)$ *has the following property:*

$$t^{\alpha-1} m(s) \leqslant G(t,s) \leqslant t^{\alpha-1} n(s), \quad t, s \in [0,1].$$

Remark 4.7. From Lemma 4.3, we know that $x(t)$ is the solution of the singular fractional differential equation (4.1) if and only if it satisfies the following integral equation:

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)\left[\lambda p(s) f(s,x(s),x(s)) + \lambda q(s) g(s,x(s),x(s))\right] ds \\ &= \lambda \int_0^1 G(t,s) p(s) f(s,x(s),x(s)) ds + \lambda \int_0^1 G(t,s) q(s) g(s,x(s),x(s)) ds, \end{aligned}$$

where $G(t,s)$ is given as in Lemma 4.3. It follows from Lemma 4.4 and Theorem 3.1 that the following theorem holds.

Theorem 4.8. *Assume that the following conditions hold:*

(H_1) $p, q : (0,1) \to [0,+\infty)$ *are continuous and* $p(t), q(t)$ *are allowed to be singular at* $t = 0$ *or* $t = 1$;

(H_2) $f, g : (0,1) \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ *are continuous and* $f(t,u,v)$, $g(t,u,v)$ *may be singular at* $t = 0$ *or 1 and* $v = 0$;

(H_3) *for fixed* $t \in (0,1)$ *and* $v \in (0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$ *are increasing in* $u \in (0,+\infty)$. *For fixed* $t \in (0,1)$ *and* $u \in (0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$ *are decreasing in* $v \in (0,+\infty)$;

(H_4) *there exists a constant* $\gamma \in (0,1)$ *such that for all* $l \in (0,1)$, $t \in (0,1)$, *and* $u, v \in (0,+\infty)$,

$$f(t, lu, l^{-1}v) \geqslant l^{\gamma} f(t,u,v),$$

and for all $l \in (0,1)$, *for all* $t \in (0,1)$ *and* $u, v \in (0,+\infty)$,

$$g(t, lu, l^{-1}v) \geqslant l g(t,u,v);$$

(H_5)

$$\begin{aligned} &\int_0^1 n(s) p(s) \left(s^{\alpha-1}\right)^{-\gamma} f(s,1,1) ds < +\infty, \\ &\int_0^1 n(s) q(s) \left(s^{\alpha-1}\right)^{-1} g(s,1,1) ds < +\infty; \end{aligned}$$

(H_6) *there exists a constant* $\delta > 0$ *such that for all* $t \in (0,1)$ *and* $u, v \in (0,+\infty)$, $f(t,u,v) \geqslant \delta g(t,u,v)$.

Then the nonlinear fractional boundary value problem (4.1) *has a unique positive solution* $x^*(t)$, *which satisfies* $\frac{1}{d} t^{\alpha-1} \leqslant x^*(t) \leqslant d t^{\alpha-1}$, $t \in [0,1]$, *for a constant* $d > 0$. *Moreover, for any initial values* $x_0(t), y_0(t) \in P_h$, $h(t) = t^{\alpha-1}$, *the sequences* $\{x_n(t)\}$, $\{y_n(t)\}$ *of successive approximations defined by*

$$\begin{aligned} x_n(t) &= \lambda \int_0^1 G(t,s) p(s) f(s, x_{n-1}(s), y_{n-1}(s)) ds + \lambda \int_0^1 G(t,s) q(s) g(s, x_{n-1}(s), y_{n-1}(s)) ds, \\ y_n(t) &= \lambda \int_0^1 G(t,s) p(s) f(s, y_{n-1}(s), x_{n-1}(s)) ds + \lambda \int_0^1 G(t,s) q(s) g(s, y_{n-1}(s), x_{n-1}(s)) ds, \quad n = 1, 2, \cdots, \end{aligned}$$

both converge uniformly to $x^*(t)$ *on* $[0,1]$ *as* $n \to \infty$. *Moreover, the unique positive solution* $x_\lambda^*(t)$ *has the following properties:*

(1) *if there exists* $\beta_0 \in (0,1)$ *such that* $\frac{\delta t^{\gamma} + t}{1+\delta} \geqslant t^{\beta_0}$, *for* $t \in (0,1)$, *then* x_λ *is continuous in* $\lambda \in (0,+\infty)$, *that is* $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ *implies* $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$;

(2) *if* $\frac{\delta t^{\gamma} + t}{1+\delta} > t^{\frac{1}{2}}$, *for* $t \in (0,1)$, *then* x_λ *is strictly increasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1}^* < x_{\lambda_2}^*$;

(3) *if there exists* $\beta_0 \in (0, \frac{1}{2})$ *such that* $\frac{\delta t^{\gamma}+t}{1+\delta} \geqslant t^{\beta_0}$, *for* $t \in (0,1)$, *then* $\lim\limits_{\lambda\to\infty} \|x_\lambda^*\| = +\infty$, $\lim\limits_{\lambda\to 0^+} \|x_\lambda^*\| = 0$.

Example 4.9. As an example, take $p(t) = q(t) = t^{\alpha-1}(1-t)^{-\frac{1}{3}}$, $f(t,u,v) = \sqrt[3]{\frac{u}{tv}}$, $g(t,u,v) = \sqrt[3]{\frac{u}{tv(v+1)}}$, $2 < \alpha \leqslant 3$, $0 < \beta \leqslant 1$. Now we show in the following that all the conditions of Theorem 4.8 are satisfied.

(1) It is obvious that $p(t), q(t)$ are singular at $t = 0$. The functions $f, g : (0,1) \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ are continuous.

(2) It is obvious that for fixed $t \in (0,1)$ and $v \in (0,+\infty)$, $f(t,u,v)$ and $g(t,u,v)$ are increasing in $u \in (0,+\infty)$; for fixed $t \in (0,1)$ and $u \in (0,+\infty)$, $f(t,u,v)$ and $g(t,u,v)$ are decreasing in $v \in (0,+\infty)$.

(3) Take $\gamma = \frac{2}{3} \in (0,1)$, for all $t \in (0,1)$ and $u, v \in (0,+\infty)$, we have

$$f(t,lu,l^{-1}v) = \sqrt[3]{\frac{lu}{tl^{-1}v}} \geqslant l^{\frac{2}{3}}\sqrt[3]{\frac{u}{tv}} = l^{\gamma} f(t,u,v),$$

and, for all $t \in (0,1)$ and $u, v \in (0,+\infty)$,

$$g(t,lu,l^{-1}v) = \sqrt[3]{\frac{lu}{tl^{-1}v(l^{-1}v+1)}} \geqslant l\sqrt[3]{\frac{u}{tv(v+1)}} = lg(t,u,v).$$

(4) It is easy to prove that

$$\begin{aligned}
&\int_0^1 m(s)p(s)\left(s^{\alpha-1}\right)^{-\gamma} f(s,1,1)ds \\
&\quad < \int_0^1 m(s)p(s)\left(s^{\alpha-1}\right)^{-1} f(s,1,1)ds \\
&\quad = \int_0^1 \frac{1}{\Gamma(\alpha)}\left[\beta + \frac{\min\{\alpha-\beta-1,1\}\int_0^1 t^{\alpha-\beta-1}(1-t)dA(t)}{1-\delta_0}\right] s(1-s)^{\alpha-\beta-1}s^{\alpha-1}(1-s)^{-\frac{1}{3}}\frac{1}{s^{\alpha-1}}s^{-\frac{1}{3}}ds \\
&\quad = \frac{1}{\Gamma(\alpha)}\left[\beta + \frac{\min\{\alpha-\beta-1,1\}\int_0^1 t^{\alpha-\beta-1}(1-t)dA(t)}{1-\delta_0}\right]\int_0^1 s^{\frac{2}{3}}(1-s)^{\alpha-\beta-\frac{4}{3}}ds \\
&\quad < +\infty,
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 n(s)q(s)\left(s^{\alpha-1}\right)^{-1} g(s,1,1)ds \\
&\quad = \int_0^1 \frac{1}{\Gamma(\alpha)}\left[1 + \frac{\min\{\alpha-\beta-1,1\}\int_0^1 dA(t)}{1-\delta_0}\right](1-s)^{\alpha-\beta-1}s^{\alpha-1}(1-s)^{-\frac{1}{3}}\frac{1}{s^{\alpha-1}}s^{-\frac{1}{3}}ds \\
&\quad = \frac{1}{\Gamma(\alpha)}\left[1 + \frac{\min\{\alpha-\beta-1,1\}\int_0^1 dA(t)}{1-\delta_0}\right]\int_0^1 s^{-\frac{1}{3}}(1-s)^{\alpha-\beta-\frac{4}{3}}ds \\
&\quad < +\infty.
\end{aligned}$$

(5) Take $\delta = 1 > 0$ such that for all $t \in (0,1)$, $u, v \in (0,+\infty)$,

$$f(t,u,v) = \sqrt[3]{\frac{u}{tv}} \geqslant \sqrt[3]{\frac{u}{tv(v+1)}} = g(t,u,v).$$

Therefore, the assumptions of Theorem 4.8 are satisfied. Thus, the conclusions follow from Theorem 4.8.

Remark 4.10. In the example of Example 4.9, $p(t), q(t)$ are singular at $t=0$ and $t=1$, and $f(t,u,v)$, $g(t,u,v)$ are singular at $t=0$ and $v=0$. But [22, Theorem 4.4] cannot be used to solve this kind of singular problem, thus we generalized and improved it.

Example 4.11. Another example, take $p(t)=q(t)=t^{\alpha-1}(1-t)^{-\frac{1}{3}}$, $f(u,v)=u^{\frac{1}{2}}+(v+1)^{-\frac{1}{2}}$,

$$g(u,v)=\begin{cases} u+\dfrac{1}{v+1}, & 0<u<1,\\ u^{\frac{1}{3}}+\dfrac{1}{v+1}, & u>1.\end{cases}$$

It is easy to prove that the assumptions of Theorem 4.8 are satisfied. Thus, the conclusions follow from Theorem 4.8.

4.2. $(k, n-k)$ *conjugate boundary value problems for nonlinear ordinary differential equations*

We consider the existence of solution of the following $(k, n-k)$ conjugate boundary value problems for nonlinear ordinary differential equations

$$\begin{cases} (-1)^{n-k}x^{(n)}(t)=\lambda p(t)f(t,x(t),x(t))+\lambda q(t)g(t,x(t),x(t)), & 0<t<1,\ \lambda\in(0,+\infty),\\ x^{i}(0)=x^{j}(1)=0, & 0\leqslant i\leqslant k-1,\ 0\leqslant j\leqslant n-k-1,\end{cases} \tag{4.2}$$

where $n\geqslant 2$, $1\leqslant k\leqslant n-1$, are fixed integers.

In the following, Let $C[0,1]$ denote the Banach space of continuous functions with the uniform norm $\|x\|=\sup\limits_{t\in[0,1]}|(t)|$, $x\in C[0,1]$.

Lemma 4.12 ([13])**.** *The following homogeneous boundary value problem*

$$\begin{cases} (-1)^{n-k}x^{(n)}(t)=f(t,x(t)), & 0<t<1,\ n\geqslant 2,\ 1\leqslant k\leqslant n-1,\\ x^{i}(0)=x^{j}(1)=0, & 0\leqslant i\leqslant k-1,\ 0\leqslant j\leqslant n-k-1,\end{cases}$$

has a unique solution

$$x(t)=\int_0^1 G(t,s)f(s,x(s))ds,$$

where

$$G(t,s)=\begin{cases} \dfrac{1}{(k-1)!(n-k)!}\displaystyle\int_0^{t(1-s)} u^{k-1}(u+s-t)^{n-k-1}du, & 0\leqslant t\leqslant s\leqslant 1,\\ \dfrac{1}{(k-1)!(n-k)!}\displaystyle\int_0^{s(1-t)} u^{n-k-1}(u+t-s)^{k-1}du, & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

Lemma 4.13 ([18])**.** *The function* $G(t,s)$ *defined as above has the following properties:*

$$\frac{\beta_0}{n-1}z(t)s^{n-k}(1-s)^k\leqslant G(t,s)\leqslant \alpha_0 z(t)s^{n-k-1}(1-s)^{k-1},\quad 0\leqslant t,s\leqslant 1,$$

where

$$\beta_0=\frac{1}{(k-1)!(n-k)!},\quad z(t)=t^k(1-t)^{n-k},\quad \alpha_0=\frac{1}{\min\{k,n-k\}(k-1)!(n-k)!}.$$

Let $m(s)=\frac{\beta_0}{n-1}s^{n-k}(1-s)^k$, $n(s)=\alpha_0 s^{n-k-1}(1-s)^{k-1}$, $s\in[0,1]$. *We have*

$$m(s)z(t)\leqslant G(t,s)\leqslant n(s)z(t),\quad 0\leqslant t,s\leqslant 1.$$

Remark 4.14. From Lemma 4.12, we know that $x(t)$ is the solution of the singular differential equation (4.2) if and only if it satisfies the following integral equation:

$$x(t) = \lambda \int_0^1 G(t,s)p(s)f(s,x(s),x(s))ds + \lambda \int_0^1 G(t,s)q(s)g(s,x(s),x(s))ds,$$

where $G(t,s)$ is given as in Lemma 4.12. It follows from Lemma 4.12 and Theorem 3.1 that the following theorem holds.

Theorem 4.15. *Assume that the following conditions hold:*

(H_1) *for fixed* $t \in (0,1)$, $v \in (0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$ *are increasing in* $u \in (0,+\infty)$; *for fixed* $t \in (0,1)$, $u \in (0,+\infty)$, $f(t,u,v)$, $g(t,u,v)$ *are decreasing in* $v \in (0,+\infty)$;

(H_2) *there exists a constant* $\gamma \in (0,1)$ *such that, for all* $l \in (0,1)$, $t \in (0,1)$, *and* $u,v \in (0,+\infty)$,

$$f(t,lu,l^{-1}v) \geqslant l^{\gamma} f(t,u,v),$$

and for all $l \in (0,1)$, *for all* $t \in (0,1)$ *and* $u,v \in (0,+\infty)$,

$$g(t,lu,l^{-1}v) \geqslant lg(t,u,v);$$

(H_3)

$$\int_0^1 n(s)p(s)\,(z(s))^{-\gamma}\, f(s,1,1)ds < +\infty,$$
$$\int_0^1 n(s)q(s)\,(z(s))^{-1}\, g(s,1,1)ds < +\infty;$$

(H_4) *there exists a constant* $\delta > 0$ *such that for all* $t \in (0,1)$ *and* $u,v \in (0,+\infty)$, $f(t,u,v) \geqslant \delta g(t,u,v)$.

Then the nonlinear fractional boundary value problem (4.2) *has a unique positive solution* $x_\lambda^*(t)$, *which satisfies* $\frac{1}{d}z(t) \leqslant x_\lambda^*(t) \leqslant dz(t)$, $t \in [0,1]$, *for a constant* $d > 0$. *Moreover, for any initial values* $x_0(t), y_0(t) \in P_h$, $h(t) = z(t)$, *the sequences* $\{x_n(t)\}$ *and* $\{y_n(t)\}$ *of successive approximations defined by*

$$x_n(t) = \lambda \int_0^1 G(t,s)p(s)f(s,x_{n-1}(s),y_{n-1}(s))ds + \lambda \int_0^1 G(t,s)q(s)g(s,x_{n-1}(s),y_{n-1}(s))ds,$$
$$y_n(t) = \lambda \int_0^1 G(t,s)p(s)f(s,y_{n-1}(s),x_{n-1}(s))ds + \lambda \int_0^1 G(t,s)q(s)g(s,y_{n-1}(s),x_{n-1}(s))ds, \quad n = 1,2,\cdots,$$

both converge uniformly to $x_\lambda^*(t)$ *on* $[0,1]$ *as* $n \to \infty$. *Furthermore, the unique positive solution* $x_\lambda^*(t)$ *has the following properties:*

(1) *if there exists* $\beta \in (0,1)$ *such that* $\frac{\delta t^\gamma + t}{1+\delta} \geqslant t^\beta$, *for* $t \in (0,1)$, *then* x_λ *is continuous in* $\lambda \in (0,+\infty)$, *that is* $\lambda \to \lambda_0$ ($\lambda_0 > 0$) *implies* $\|x_\lambda^* - x_{\lambda_0}^*\| \to 0$;

(2) *if* $\frac{\delta t^\gamma + t}{1+\delta} > t^{\frac{1}{2}}$, *for* $t \in (0,1)$, *then* x_λ *is strictly increasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *implies* $x_{\lambda_1}^* < x_{\lambda_2}^*$;

(3) *if there exists* $\beta \in (0,\frac{1}{2})$ *such that* $\frac{\delta t^\gamma + t}{1+\delta} \geqslant t^\beta$, *for* $t \in (0,1)$, *then* $\lim\limits_{\lambda\to\infty} \|x_\lambda^*\| = +\infty$, $\lim\limits_{\lambda\to 0^+} \|x_\lambda^*\| = 0$.

Example 4.16. If $n = 4$ and $k = 2$, then the problem (4.2) reduces to the following integral fourth-order two-point boundary value problem:

$$\begin{cases} x^{(4)}(t) = \lambda p(t)f(t,x(t),x(t)) + \lambda q(t)g(t,x(t),x(t)), \quad t \in [0,1],\ \lambda \in (0,+\infty), \\ x(0) = x(1) = x'(0) = x'(1) = 0, \end{cases}$$

which describes the deformation of an elastic beam with both endpoints fixed, where $p, q : (0,1) \to [0,+\infty)$, $f, g : (0,1) \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ are continuous, and $f(t,x,x) = f(1-t,x,x)$, $g(t,x,x) = g(1-t,x,x)$, for each $(t,x,x) \in (0,1) \times (0,+\infty) \times (0,+\infty)$. Here a symmetric positive solution of the problem (4.2) means a solution x of the problem (4.2) satisfying

$$x(t) = x(1-t),\ t \in (0,1) \quad \text{and} \quad x(t) > 0,\ t \in (0,1).$$

If all the conditions of Theorem 4.8 are satisfied then the result follows from Theorem 4.8.

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References

[1] R. P. Agarwal, *On fourth order boundary value problems arising in beam analysis*, Differential Integral Equations, **2** (1989), 91–110. 1, 4
[2] R. P. Agarwal, Y. M. Chow, *Iterative methods for a fourth order boundary value problem*, J. Comput. Appl. Math., **10** (1984), 203–217. 4
[3] A. Cabada, G.-T. Wang, *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl., **389** (2012), 403–411. 1
[4] Z.-W. Cao, D.-Q. Jiang, C.-J. Yuan, D. O'Regan, *Existence and uniqueness of solutions for singular integral equation*, Positivity, **12** (2008), 725–732. 1, 1
[5] Y.-J. Cui, L.-S. Liu, X.-Q. Zhang, *Uniqueness and existence of positive solutions for singular differential systems with coupled integral boundary value problems*, Abstr. Appl. Anal., **2013** (2013), 9 pages. 1
[6] D.-J. Guo, Y. J. Cho, J. Zhu, *Partial ordering methods in nonlinear problems*, Nova Science Publishers, Inc., Hauppauge, NY, (2004). 2
[7] D.-J. Guo, V. Lakshmikantham, *Nonlinear problems in abstract cones*, Notes and Reports in Mathematics in Science and Engineering, Academic Press, Inc., Boston, MA, (1988). 2
[8] X.-A. Hao, L.-S. Liu, Y.-H. Wu, Q. Sun, *Positive solutions for nonlinear nth-order singular eigenvalue problem with nonlocal conditions*, Nonlinear Anal., **73** (2010), 1653–1662. 1
[9] H. H. G. Hashem, *On successive approximation method for coupled systems of Chandrasekhar quadratic integral equations*, J. Egyptian Math. Soc., **23** (2015), 108–112. 1
[10] W.-H. Jiang, J.-L. Zhang, *Positive solutions for $(k, n-k)$ conjugate boundary value problems in Banach spaces*, Nonlinear Anal., **71** (2009), 723–729. 4
[11] M. Jleli, B. Samet, *Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method*, Nonlinear Anal. Model. Control, **20** (2015), 367–376. 1
[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, (2006). 1, 2
[13] K. Q. Lan, *Multiple positive solutions of conjugate boundary value problems with singularities*, Appl. Math. Comput., **147** (2004), 461–474. 4, 4.12
[14] K. Latrach, M. A. Taoudi, *Existence results for a generalized nonlinear Hammerstein equation on L_1 spaces*, Nonlinear Anal., **66** (2007), 2325–2333. 1
[15] P.-D. Lei, X.-N. Lin, D.-Q. Jiang, *Existence and uniqueness of positive solutions for singular nonlinear elliptic boundary value problems*, Nonlinear Anal., **69** (2008), 2773–2779. 1
[16] F.-Y. Li, Y.-H. Li, Z.-P. Liang, *Existence of solutions to nonlinear Hammerstein integral equations and applications*, J. Math. Anal. Appl., **323** (2006), 209–227. 1
[17] H.-D. Li, L.-S. Liu, Y.-H. Wu, *Positive solutions for singular nonlinear fractional differential equation with integral boundary conditions*, Bound. Value Probl., **2015** (2015), 15 pages. 1
[18] X.-N. Lin, D.-Q. Jiang, X.-Y. Li, *Existence and uniqueness of solutions for singular $(k, n-k)$ conjugate boundary value problems*, Comput. Math. Appl., **52** (2006), 375–382. 1, 4, 4.13
[19] X.-N. Lin, D.-Q. Jiang, X.-Y. Li, *Existence and uniqueness of solutions for singular fourth-order boundary value problems*, J. Comput. Appl. Math., **196** (2006), 155–161. 1
[20] L.-S. Liu, F. Guo, C.-X. Wu, Y.-H. Wu, *Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces*, J. Math. Anal. Appl., **309** (2005), 638–647. 1
[21] L.-S. Liu, C.-X. Wu, F. Guo, *Existence theorems of global solutions of initial value problems for nonlinear integrodifferential equations of mixed type in Banach spaces and applications*, Comput. Math. Appl., **47** (2004), 13–22. 1

[22] L.-S. Liu, X.-Q. Zhang, J. Jiang, Y.-H. Wu, *The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems*, J. Nonlinear Sci. Appl., **9** (2016), 2943–2958. 1, 2, 2.1, 2.2, 2.3, 4.10

[23] A. Lomtatidze, L. Malaguti, *On a nonlocal boundary value problem for second order nonlinear singular differential equations*, Georgian Math. J., **7** (2000), 133–154. 2, 4

[24] M.-H. Pei, S. K. Chang, *Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem*, Math. Comput. Modelling, **51** (2010), 1260–1267. 4

[25] I. Podlubny, *Fractional differential equations*, An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA, (1999). 1, 2, 4.1, 4.2

[26] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives*, Theory and applications, Edited and with a foreword by S. M. Nikol'skiĭ, Translated from the 1987 Russian original, Revised by the authors, Gordon and Breach Science Publishers, Yverdon, (1993). 2

[27] Y.-P. Sun, M. Zhao, *Positive solutions for a class of fractional differential equations with integral boundary conditions*, Appl. Math. Lett., **34** (2014), 17–21. 1, 1

[28] Y.-Q. Wang, L.-S. Liu, Y.-H. Wu, *Positive solutions for a class of fractional boundary value problem with changing sign nonlinearity*, Nonlinear Anal., **74** (2011), 6434–6441. 1

[29] Y.-Q. Wang, L.-S. Liu, Y.-H. Wu, *Positive solutions for a nonlocal fractional differential equation*, Nonlinear Anal., **74** (2011), 3599–3605. 1

[30] J. R. L. Webb, *Uniqueness of the principal eigenvalue in nonlocal boundary value problems*, Discrete Contin. Dyn. Syst. Ser. S, **1** (2008), 177–186. 1

[31] J. R. L. Webb, *Nonlocal conjugate type boundary value problems of higher order*, Nonlinear Anal., **71** (2009), 1933–1940. 4

[32] J. R. L. Webb, *Existence of positive solutions for a thermostat model*, Nonlinear Anal. Real World Appl., **13** (2012), 923–938. 1

[33] J. R. L. Webb, *Positive solutions of nonlinear differential equations with Riemann-Stieltjes boundary conditions*, Electron. J. Qual. Theory Differ. Equ., **2016** (2016), 13 pages. 1

[34] P. J. Y. Wong, *Triple positive solutions of conjugate boundary value problems*, II, Comput. Math. Appl., **40** (2000), 537–557. 4

[35] Z.-L. Yang, *Positive solutions for a system of nonlinear Hammerstein integral equations and applications*, Appl. Math. Comput., **218** (2012), 11138–11150. 1

[36] B. Yang, *Upper estimate for positive solutions of the* $(p, n-p)$ *conjugate boundary value problem*, J. Math. Anal. Appl., **390** (2012), 535–548. 4

[37] C.-J. Yuan, X.-D. Wen, D.-Q. Jiang, *Existence and uniqueness of positive solution for nonlinear singular* $2m$*th-order continuous and discrete Lidstone boundary value problems*, Acta Math. Sci. Ser. B Engl. Ed., **31** (2011), 281–291. 1, 1

[38] C.-B. Zhai, R.-P. Song, Q.-Q. Han, *The existence and the uniqueness of symmetric positive solutions for a fourth-order boundary value problem*, Comput. Math. Appl., **62** (2011), 2639–2647. 1, 1, 1

[39] H.-E. Zhang, *Iterative solutions for fractional nonlocal boundary value problems involving integral conditions*, Bound. Value Probl., **2016** (2016), 13 pages. 4.3, 4.4, 4.5, 4.6

[40] X.-G. Zhang, L.-S. Liu, B. Wiwatanapataphee, Y.-H. Wu, *The eigenvalue for a class of singular* p*-Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary condition*, Appl. Math. Comput., **235** (2014), 412–422. 1

[41] X.-G. Zhang, L.-S. Liu, Y.-H. Wu, *The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium*, Appl. Math. Lett., **37** (2014), 26–33. 1

[42] X.-Q. Zhang, L.-S. Liu, Y.-H. Wu, *Fixed point theorems for the sum of three classes of mixed monotone operators and applications*, Fixed Point Theory Appl., **2016** (2016), 22 pages. 1, 2.1, 2.3

[43] X.-Q. Zhang, L. Wang, Q. Sun, *Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter*, Appl. Math. Comput., **226** (2014), 708–718. 1, 1

[44] M.-C. Zhang, Y.-M. Yin, Z.-L. Wei, *Existence of positive solution for singular semi-positone* $(k, n-k)$ *conjugate* m*-point boundary value problem*, Comput. Math. Appl., **56** (2008), 1146–1154. 4

[45] Y.-L. Zhao, H.-B. Chen, L. Huang, *Existence of positive solutions for nonlinear fractional functional differential equation*, Comput. Math. Appl., **64** (2012), 3456–3467. 1