Hardy type estimates for commutators of fractional integrals associated with Schrödinger operators

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Abstract

We consider the Schrödinger operator $L = -\Delta + V$ on $\mathbb{R}^n$, where $n \geq 3$ and the nonnegative potential $V$ belongs to reverse Hölder class $RH_{q_1}$, for some $q_1 > \frac{2}{n}$. Let $I_\alpha$ be the fractional integral associated with $L$, and let $b$ belong to a new Campanato space $\Lambda^{\frac{\alpha}{n-\alpha}}_\theta(V)$. In this paper, we establish the boundedness of the commutators $[b, I_\alpha]$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ whenever $\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta}{n}$, $1 < p < \frac{n}{\alpha + \beta}$. When $\frac{n}{n + \beta} < p \leq 1$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta}{n}$, we show that $[b, I_\alpha]$ is bounded from $H^p_\alpha(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Moreover, we also prove that $[b, I_\alpha]$ maps $H^p_\alpha(\mathbb{R}^n)$ continuously into weak $L^p(\mathbb{R}^n)$.

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1. Introduction and results

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$. The function $V$ is nonnegative, $V \neq 0$, and belongs to a reverse Hölder class $RH_{q_1}$, for some $q_1 > \frac{2}{n}$, that is, there exists a constant $C$ such that

$$\left( \frac{1}{|B|} \int_B V(y)^{q_1} dy \right)^{1/q_1} \leq \frac{C}{|B|} \int_B V(y) dy$$

for every ball $B \subset \mathbb{R}^n$.

Suppose $V \in RH_{q_1}$ with $q_1 > n/2$. The fractional integral associated with $L$ is defined by

$$I_\alpha f(x) = L^{-\alpha/2} f(x) = \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{\alpha/2+1}}$$

for $0 < \alpha < n$. If $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $I_\alpha$ is the Riesz potential $I_\alpha$, that is,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

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As in [10], for a given potential \( V \in RH_{q_1} \) with \( q_1 > n/2 \), we define the auxiliary function

\[
\rho(x) = \sup \left\{ r > 0 : \frac{1}{\nu^{n-2}} \int_{B(x, r)} V(y) dy \leq 1 \right\}, \quad x \in \mathbb{R}^n.
\]

It is well-known that \( 0 < \rho(x) < \infty \) for any \( x \in \mathbb{R}^n \).

Let \( \theta > 0 \) and \( 0 < \beta < 1 \), according to [7], the new Campanato class \( \Lambda^0_\beta(\rho) \) consists of the locally integrable functions \( b \) such that

\[
\frac{1}{|B(x, r)|^{1+\beta/n}} \int_{B(x, r)} |b(y) - b_B| dy \leq C \left( 1 + \frac{r}{\rho(x)} \right) \theta
\]

holds for all \( x \in \mathbb{R}^n \) and \( r > 0 \). A seminorm of \( b \in \Lambda^0_\beta(\rho) \), denoted by \( |b|^0_\beta \), is given by the infimum of the constants in the inequalities above.

Note that if \( \theta = 0 \), \( \Lambda^0_\beta(\rho) \) is the classical Campanato space. If \( \beta = 0 \), \( \Lambda^0_\beta(\rho) \) is exactly the space \( \text{BMO}_\theta(\rho) \) introduced in [1].

We recall the Hardy space associated with Schrödinger operator \( \mathcal{L} \) which had been studied by Dziubański and Zienkiewicz in [4] and [5]. Because \( V \in L^q_{1\text{loc}}(\mathbb{R}^n) \), the Schrödinger operator \( \mathcal{L} \) generates a \((C_0)\) contraction semigroup \( \{T^\mathcal{L}_s : s > 0 \} \) \( \{e^{-s\mathcal{L}} : s > 0 \} \). The maximal function associated with \( \{T^\mathcal{L}_s : s > 0 \} \) is defined by \( M^\mathcal{L} f(x) = \sup_{s > 0} |T^\mathcal{L}_s f(x)| \). We always denote \( \eta = 2 - n/q_1 \) and \( \delta' = \min(1, \eta) \).

For \( \frac{n}{n+\beta} < p < 1 \), the Hardy space \( H^p_\mathcal{L} (\mathbb{R}^n) \) associated with Schrödinger operator \( \mathcal{L} \) is defined as follows.

**Definition 1.1.** We say that \( f \) is an element of \( H^p_\mathcal{L} (\mathbb{R}^n) \) if the maximal function \( M^\mathcal{L} f \) belongs to \( L^p(\mathbb{R}^n) \). The quasi-norm of \( f \) is defined by \( \|f\|_{H^p_\mathcal{L} (\mathbb{R}^n)} = \|M^\mathcal{L} f\|_{L^p(\mathbb{R}^n)} \).

We introduce the concept of \( H^p, \mathcal{L} ,q\)-atom.

**Definition 1.2.** Let \( \frac{n}{n+\delta'} < p < 1 \leq q \leq \infty \). A function \( a \in L^2(\mathbb{R}^n) \) is called an \( H^p, \mathcal{L} ,q\)-atom if \( r < \rho(x_0) \) and the following conditions hold:

(i) \( \text{supp} \ a \subset B(x_0, r) \),

(ii) \( \|a\|_{L^q(\mathbb{R}^n)} \leq |B(x_0, r)|^{1/q-1/p} \),

(iii) if \( r < \rho(x_0)/4 \), then \( \int_{B(x_0, r)} a(x) dx = 0 \).

We have the following atomic characterization of Hardy space.

**Proposition 1.3 ([5]).** Let \( \frac{n}{n+\delta'} < p \leq 1 \leq q < \infty \). Then \( f \in H^p_\mathcal{L} (\mathbb{R}^n) \) if and only if \( f \) can be written as \( f = \sum \lambda_j a_j \), where \( a_j \) are \( H^p, \mathcal{L} ,q\)-atoms, \( \sum_j |\lambda_j|^p < \infty \), and the sum converges in the \( H^p_\mathcal{L} (\mathbb{R}^n) \) quasi-norm. Moreover

\[
\|f\|_{H^p_\mathcal{L} (\mathbb{R}^n)} \approx \inf \left\{ \left( \sum_j |\lambda_j|^p \right)^{1/p} \right\},
\]

where the infimum is taken over all atomic decompositions of \( f \) into \( H^p, \mathcal{L} ,q\)-atoms.

The above atomic decomposition of \( H^p_\mathcal{L} (\mathbb{R}^n) \) implies that the Hardy space \( H^p_\mathcal{L} (\mathbb{R}^n) \) is larger than the classical Hardy space \( H^p(\mathbb{R}^n) \). Especially, the Hardy space \( H^p_\mathcal{L} (\mathbb{R}^n) \) is exactly the local Hardy space \( h^p(\mathbb{R}^n) \) introduced by Goldberg in [6] when the potential \( V \) is a positive constant.

Let us consider the commutator associated with the Riesz potential \( I_\alpha \) and locally integrable function \( b \), \( [b, I_\alpha] f(x) = b(x) I_\alpha f(x) - I_\alpha (b f)(x) \). When \( b \in \text{BMO} \), Chanillo proved in [3] that \( [b, I_\alpha] \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) with \( 1/q = 1/p - \alpha/n, 1 < p < n/\alpha \). When \( b \) belongs to the Campanato space \( \Lambda_\beta, 0 < \beta < 1 \), Paluszyński in [9] showed that \( [b, I_\alpha] \) is bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \), where \( 1/q = \).
1/p − (α + β)/n, 1 < p < n/(α + β). Furthermore, Lu et al. in [8] considered the boundedness of [b, Iα] on the classical Hardy spaces when b ∈ Λβ, 0 < β ≤ 1. They proved that if \( \frac{n}{n + p} < p \leq 1 \) and \( 1/q = 1/p - (\alpha + \beta)/n, [b, I\alpha] \) maps \( H^p(\mathbb{R}^n) \) continuously into \( L^q(\mathbb{R}^n) \). At the endpoint \( p = \frac{n}{n + \beta} \), they also showed that \([b, I\alpha] \) maps \( H^p(\mathbb{R}^n) \) continuously into weak \( L^{n/(n-\alpha)}(\mathbb{R}^n) \).

When \( b \in \text{BMO}_0(\rho), \) Bui in [2] obtained the boundedness of \([b, I\alpha] \) from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) with \( 1/q = 1/p - \alpha/n, 1 < p < n/\alpha \).

Inspired by the above results, in this paper, we are interested in the boundedness of \([b, I\alpha] \) when \( b \) belongs to the new Campanato class \( \Lambda^0_{\alpha}(\rho) \). The results of this paper are as follows.

**Theorem 1.6.** Let \( 0 < \alpha < n \), and let \( V \in RH_{q_1} \) with \( q_1 > n/2 \). Then for any \( b \in \Lambda^0_{\alpha}(\rho), 0 < \beta < 1 \), the commutator \([b, I\alpha] \) is bounded from \( L^p(\mathbb{R}^n) \) into \( L^q(\mathbb{R}^n) \), where \( \ell q = \frac{1}{p} - \frac{\alpha + \beta}{n}, 1 < p < \frac{n}{n + \beta} \).

**Theorem 2.1.** Let \( 0 < \alpha < n \), and let \( V \in RH_{q_1} \) with \( q_1 > n/2 \). Suppose \( b \in \Lambda^0_{\alpha}(\rho), 0 < \beta < \delta' \). If \( \frac{n}{n + \beta} < p \leq 1 \) and \( \ell q = \frac{1}{p} - \frac{\alpha + \beta}{n} \), then the commutator \([b, I\alpha] \) is bounded from \( H^p_{L^\infty}(\mathbb{R}^n) \) into \( L^q_{L^\infty}(\mathbb{R}^n) \).

**Theorem 2.6.** Let \( 0 < \alpha < n \), and let \( V \in RH_{q_1} \) with \( q_1 > n/2 \). Suppose \( b \in \Lambda^0_{\alpha}(\rho), 0 < \beta < \delta' \). Then the commutator \([b, I\alpha] \) is bounded from \( H^{\tilde{n}}_{L^\infty}(\mathbb{R}^n) \) into weak \( L^{n/\alpha}(\mathbb{R}^n) \).

Finally, we make some conventions on the notation. Throughout the whole paper, we always use \( C \) to denote a positive constant, that is independent of the main parameters involved but whose value may differ from line to line. We shall use the symbol \( A \lesssim B \) to indicate that there exists a constant \( C \) such that \( A \leq CB \). \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \).

### 2. Some preliminaries

We would like to recall some important properties concerning the auxiliary function which will play an important role to obtain the main results.

**Proposition 2.1 ([10]).** Let \( V \in RH_{n/2} \). For the auxiliary function \( \rho \) there exist \( C \) and \( k_0 \geq 1 \) such that

\[
C^{-1} \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{-k_0} \leq \rho(y) \leq C \rho(x) \left( 1 + \frac{|x - y|}{\rho(x)} \right)^{k_0}
\]

for all \( x, y \in \mathbb{R}^n \).

A ball \( B(x, \rho(x)) \) is called critical. Assume that \( Q = B(x_0, \rho(x_0)) \), for \( x \in Q \), the inequality above tells us that \( \rho(x) \approx \rho(y) \) if \( |x - y| < C \rho(x) \).

It is easy to get the following results from Proposition 2.1.

**Lemma 2.2.** Let \( k \in \mathbb{N} \) and \( x \in 2^{k+1}B(x_0, r) \setminus 2^kB(x_0, r) \). Then we have

\[
\frac{1}{\left( 1 + \frac{2kr}{\rho(x)} \right)^N} \lesssim \frac{1}{\left( 1 + \frac{2kr}{\rho(x_0)} \right)^{N/(k_0 + 1)}}.
\]

**Proposition 2.3 ([4]).** There exists a sequence of points \( \{x_k\}_{k=1}^\infty \) in \( \mathbb{R}^n \), so that the family of critical balls \( Q_k = B(x_k, \rho(x_k)) \), \( k \geq 1 \), satisfies

(i) \( \bigcup_k Q_k = \mathbb{R}^n \);

(ii) there exists \( N = N(\rho) \) such that for every \( k \in \mathbb{N}, \text{card}(j : 4Q_j \cap 4Q_k) \leq N \).

Given \( \alpha > 0 \), we define the following maximal functions for \( g \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
M_{\rho, \alpha}g(x) = \sup_{x \in B \subseteq \mathcal{B}_{\rho, \alpha}} \frac{1}{|B|} \int_B |g(y)| \, dy,
\]

\[
M_{\rho, \alpha}^{|g|}g(x) = \sup_{x \in B \subseteq \mathcal{B}_{\rho, \alpha}} \frac{1}{|B|} \int_B |g(y) - gb| \, dy,
\]

where \( \mathcal{B}_{\rho, \alpha} = \{B(z, r) : z \in \mathbb{R}^n \text{ and } r \leq \alpha \rho(y)\} \).
Proposition 2.4 ([1]). For $1 < p < \infty$, there exist $\eta$ and $\gamma$ such that if $(Q_k)_k$ is a sequence of balls as in Proposition 2.3, then
\[
\int_{\mathbb{R}^n} |M_{p, \eta} g(x)|^p \, dx \lesssim \int_{\mathbb{R}^n} |M_{p, \gamma}^* g(x)|^p \, dx + \sum_k |Q_k| \left( \frac{1}{|Q_k|} \int_{2Q_k} |g| \right)^p
\]
for all $g \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Let us recall the property of $b \in \Lambda^0_{R} (\rho)$.

Lemma 2.5 ([7]). Let $1 \leq s < \infty, b \in \Lambda^0_{R} (\rho)$, and $B = B(x, r)$. Then
\[
\left( \frac{1}{|2^k B|} \int_{2^k B} |b(y) - b_B|^s \, dy \right)^{1/s} \leq [b]^0_{\beta} (2^k r)^\beta \left( 1 + \frac{2^k r}{\rho(x)} \right)^{\theta'}
\]
for all $k \in \mathbb{N}$, where $\theta' = (k_0 + 1)\theta$, and $k_0$ is the constant appearing in Proposition 2.1.

Proposition 2.6 ([5]). Let $p_t(x,y)$ be the kernels associated with the semigroups $\{e^{-t\cdot} \}_{t > 0}$. If $V \in \text{RH}_{q_1}$ with $q_1 > n/2$, then for every $0 < \delta' < \delta > 0$ and every $N > 0$ there exists a constant $C > 0$ such that for $|y - z| < \frac{1}{2}|x - y|$, we have
\[
|p_t(x, y) - p_t(z, z)| = |p_t(x, y) - p_t(y, x) - p_t(z, z)| \lesssim \frac{1}{t^{n/2}} \left( \frac{|y - z|}{\sqrt{t}} \right)^\delta \exp \left( \frac{-|x - y|^2}{5t} \right) \left( 1 + \frac{2^k r}{\rho(x)} + \frac{2^k r}{\rho(y)} \right).
\]

Let $K_\alpha$ be the kernel of $\mathbb{I}_\alpha$. The following results give the estimates on the kernel $K_\alpha(x, y)$.

Lemma 2.7. Suppose $V \in \text{RH}_{q_1}$ with $q_1 > \frac{n}{2}$.

(i) For every $N > 0$, there exists a constant $C$ such that
\[
|K_\alpha(x, y)| \lesssim \frac{1}{\left( 1 + \frac{|x - y|}{\rho(x)} \right)^N} \frac{1}{|x - y|^{n - \alpha}}.
\]

(ii) For every $0 < \delta < \delta'$ there exists a constant $C$ such that for every $N > 0$, we have
\[
|K_\alpha(x, y) - K_\alpha(x, z)| + |K_\alpha(y, x) - K_\alpha(z, x)| \lesssim \frac{1}{\left( 1 + \frac{|x - y|}{\rho(x)} \right)^N} \frac{|y - z|^\delta}{|x - y|^{n + \delta - \alpha}},
\]
where $|y - z| \leq |x - y|/4$.

Proof. We observe that (i) is the result of Proposition 3.3 of [2]. By Proposition 2.4 and the methods used in Proposition 3.3 of [2], we can obtain (ii). \qed

Since $|\mathbb{I}_\alpha(f)(x)| \lesssim I_\alpha(|f|)(x)$, then we get the following.

Corollary 2.8. Suppose $V \in \text{RH}_{q_1}$ with $q_1 > n/2$. Let $0 < \alpha < n$ and let $1 \leq p < q < \infty$ satisfy $1/q = 1/p - \alpha/n$. Then for all $f$ in $L^p(\mathbb{R}^n)$ we have
\[
\|\mathbb{I}_\alpha f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}
\]
when $p > 1$, and also
\[
\|\mathbb{I}_\alpha f\|_{W^{1,q}(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}
\]
when $p = 1$. 
3. Proof of Theorem 1.4

In order to prove Theorem 1.4, we need the following lemmas.

Lemma 3.1. Let \(1 < s < p < n/(\alpha + \beta), \ b \in \Lambda_Q^0(p), \) and \(Q = B(x_0, \rho(x_0)).\) Then

\[
\frac{1}{|Q|} \int_Q |[b, \mathbb{I}_\alpha]f(y)| \, dy \lesssim [b]_\beta^0 \inf_{x \in Q} M_{\alpha + \beta,s}(f)(x),
\]

where

\[
M_{\alpha + \beta,s}(f)(x) = \sup_{x \in B} \left( \frac{1}{|B|^{1-(\alpha + \beta)s/n}} \int_B |f(y)|^s \, dy \right)^{1/s}.
\]

Proof. Since

\[
[b, \mathbb{I}_\alpha]f(y) = (b(y) - b_Q)\mathbb{I}_\alpha f(y) - \mathbb{I}_\alpha((b - b_Q)f)(y),
\]

we have

\[
\frac{1}{|Q|} \int_Q |[b, \mathbb{I}_\alpha]f(y)| \, dy \leq \frac{1}{|Q|} \int_Q |(b(y) - b_Q)\mathbb{I}_\alpha f(y)| \, dy + \frac{1}{|Q|} \int_Q |\mathbb{I}_\alpha((b - b_Q)f)(y)| \, dy = I_1 + I_2.
\]

By Hölder’s inequality and Lemma 2.5, for any \(t > 1\) we get

\[
I_1 \leq \left( \frac{1}{|Q|} \int_Q |b(y) - b_Q|^{t'} \, dy \right)^{1/t'} \left( \frac{1}{|Q|} \int_Q |\mathbb{I}_\alpha f(y)|^t \, dy \right)^{1/t} \lesssim [b]_\beta^0 \rho(x_0)^\beta \left( \left( \frac{1}{|Q|} \int_Q |\mathbb{I}_\alpha f_1(y)|^t \, dy \right)^{1/t} + \left( \frac{1}{|Q|} \int_Q |\mathbb{I}_\alpha f_2(y)|^t \, dy \right)^{1/t} \right) = I_{11} + I_{12},
\]

where \(f = f_1 + f_2\) with \(f_1 = f_{\chi_{2Q}}.\) Choose \(t > 1\) such that \(1/s - 1/t = \alpha/n,\) then by the \((L^s, L^t)\)-boundedness of \(\mathbb{I}_\alpha\) (see Corollary 2.8), we have

\[
I_{11} \lesssim [b]_\beta^0 \rho(x_0)^\beta \left( \frac{1}{|Q|^{1/t}} \int_{2Q} |f(y)|^s \, dy \right)^{1/s} \lesssim [b]_\beta^0 \left( \frac{1}{|2Q|^{1-(\alpha + \beta)s/n}} \int_{2Q} |f(y)|^s \, dy \right)^{1/s} \lesssim [b]_\beta^0 \inf_{x \in Q} M_{\alpha + \beta,s}(f)(x).
\]

Note that

\[
|\mathbb{I}_\alpha f_2(y)| \leq \int_{(2Q)^c} |K_\alpha(y, z)f(z)| \, dz \lesssim \int_{(2Q)^c} \frac{|f(z)|}{\left(1 + \frac{|y - z|}{\rho(y)}\right)^N} \frac{|y - z|^{n - \alpha}}{|y - z|^n} \, dz.
\]

In this situation, we have \(\rho(y) \approx \rho(x_0), |y - z| \approx |x_0 - z|\) for any \(y \in Q\) and \(z \in (2Q)^c.\) So, decomposing \((2Q)^c\) into annuli \(2^k Q \setminus 2^{k-1} Q, k \geq 2,\) we get

\[
|\mathbb{I}_\alpha f_2(y)| \lesssim \sum_{k \geq 2} 2^{-kN} \frac{2^{-kN}}{|2^k Q|^{1 - \alpha/n}} \int_{2^k Q} |f(z)| \, dz.
\]

Then, by Hölder’s inequality we get

\[
I_{12} \lesssim [b]_\beta^0 \rho(x_0)^\beta \sum_{k \geq 2} 2^{-kN} \frac{2^{-kN}}{|2^k Q|^{1 - \alpha/n}} \int_{2^k Q} |f(z)| \, dz
\]
then by Lemma 2.7 we get

\[ \leq [b]^\theta \sum_{k \geq 2} 2^{-kN} \frac{2^{-kN}}{2^kQ |l - (x + \beta)}/n \int_{2^kQ} |f(z)|dz \]

\[ \leq [b]^\theta \sum_{k \geq 2} 2^{-kN} \left( \frac{1}{2^kQ |l - (x + \beta)}/s/n \int_{2^kQ} |f(z)|^s dz \right)^{1/s} \]

\[ \leq [b]^\theta \inf_{x \in Q} M_{\alpha + \beta, s}(f)(x). \]

The estimate for \( I_2 \) can be proceeded in the same way of \( I_1 \). The decomposition \( f = f_1 + f_2 \) gives

\[ I_2 \leq \frac{1}{|Q|} \int_Q |f_1|(y)|dy| + \frac{1}{|Q|} \int_Q |f_2|(y)|dy| = I_{21} + I_{22}. \]

Choose \( r \) such that \( 1 < r < s < p \) and \( 1/r - 1/r_0 = \alpha/n \). By Hölder’s inequality, Lemma 2.5 and \((L^r, L^s)\)-boundedness of \( \mathbb{I}_\alpha \), for some \( u > r \) we have

\[ I_{21} \leq \left( \frac{1}{|Q|} \int_Q |f_1|(y)|dy| \right)^{1/r_0} \]

\[ \leq \frac{1}{|Q|^{1/\alpha/n}} \left( \frac{1}{|Q|} \int_{2Q} |f_1|(y)|dy| \right)^{1/r} \]

\[ \leq \frac{1}{|Q|^{1/\alpha/n}} \left( \frac{1}{|Q|} \int_{2Q} |f(y)|^s dy \right)^{1/s} \left( \frac{1}{|Q|} \int_{2Q} |b(y) - b_Q|^u dy \right)^{1/u} \]

\[ \leq [b]^\theta \inf_{x \in Q} M_{\alpha + \beta, s}(f)(x). \]

The estimate \( I_{22} \leq [b]^\theta \inf_{x \in Q} M_{\alpha + \beta, s}(f)(x) \) can be obtained by the similar approach to ones of \( I_{12} \) and \( I_{21} \). Then we omit the details here. \( \square \)

**Lemma 3.2.** Let \( B = B(x_0, r) \) with \( r \leq \gamma p(x_0) \) and let \( x \in B \), then for any \( y, z \in B \) we have

\[ \int_{\overline{B}_r} |K_\alpha(y, u) - K_\alpha(z, u)||b(u) - b_B||f(u)||du \leq [b]^\theta M_{\alpha + \beta, s}(f)(x). \]

**Proof.** Setting \( Q = B(x_0, \gamma p(x_0)) \), due to the facts that \( \rho(y) \approx \rho(z) \approx \rho(x_0) \) and \( |y - u| \approx |z - u| \approx |x_0 - u| \), then by Lemma 2.7 we get

\[ \int_{\overline{B}_r} |K_\alpha(y, u) - K_\alpha(z, u)||b(u) - b_B||f(u)||du \leq \mathcal{K}_1 + \mathcal{K}_2, \]

where

\[ \mathcal{K}_1 = r^\delta \int_{Q \setminus 2B} \frac{|f(u)||b(u) - b_B|}{|x_0 - u|^{n+\delta - \alpha}} \]

and

\[ \mathcal{K}_2 = r^\delta \rho(x_0)^N \int_{Q \setminus 2B} \frac{|f(u)||b(u) - b_B|}{|x_0 - u|^{n+N+\delta - \alpha}} \]

Let \( j_0 \) be the least integer such that \( 2^{j_0} \geq \gamma p(x_0)/r \). Splitting into annuli, we have

\[ \mathcal{K}_1 \leq \sum_{j=2}^{j_0} 2^{-j(N+\alpha)} \frac{1}{|2^jB|} \int_{2^jB} |f(u)||b(u) - b_B||du. \]
By Hölder’s inequality, Lemma 2.5 and $2^1r < \gamma p(x_0)$ for $j < j_0$, we have

$$
K_1 \lesssim \sum_{j=2}^{j_0} 2^{-j\delta} \left( \frac{1}{|2^j B|} \int_{2^j B} |f(u)^s| \ du \right)^{1/s} \left( \frac{1}{|2^j B|} \int_{2^j B} |b(u) - b_B|^{s'} \ du \right)^{1/s'}
$$

$$
\lesssim |b|^\theta \sum_{j=2}^{j_0} 2^{-j\delta} (2^j r)^{\alpha + \beta} \left( 1 + \frac{2^j r}{\rho(x_0)} \right)^{\theta'} \left( \frac{1}{|2^j B|} \int_{2^j B} |f(u)^s| \ du \right)^{1/s}
$$

$$
\lesssim |b|^\theta M_{\alpha + \beta, s}(f)(x).
$$

Note that

$$
\frac{1}{|2^j B|} \int_{2^j B} |f(u)| b(u) - b_B|du \lesssim |b|^\theta (2^j r)^{\beta} \left( 1 + \frac{2^j r}{\rho(x_0)} \right)^{\theta'} \left( \frac{1}{|2^j B|} \int_{2^j B} |f(u)|^s \ du \right)^{1/s}.
$$

Since $\frac{2^j r}{\rho(x_0)} \geq \gamma$ for $j \geq j_0$, then, by choosing $N > \theta'$ we get

$$
K_2 \lesssim \rho(x_0)^N \sum_{j \geq j_0} 2^{-j\delta} \left( \frac{1}{|2^j B|} \int_{2^j B} |f(u)| b(u) - b_B|du \right)
$$

$$
\lesssim |b|^\theta \left( \frac{2^j r}{\rho(x_0)} \right)^{-\left(N - \theta'\right)} \sum_{j=0}^{\infty} 2^{-j\delta} (2^j r)^{\alpha + \beta} \left( \frac{1}{|2^j B|} \int_{2^j B} |f(u)|^s \ du \right)^{1/s}
$$

$$
\lesssim |b|^\theta M_{\alpha + \beta, s}(f)(x).
$$

\[ \square \]

**Lemma 3.3.** Let $1 < s < p < n/(\alpha + \beta)$, $B = B(x_0, r)$ with $r \leq \gamma p(x_0)$, and $x \in B$. Then

$$
M_{\alpha, r}^s ([b, I_{\alpha}]f)(x) \lesssim |b|^\theta \left( M_{\alpha + \beta, s}(f)(x) + M_{\beta, s}(I_{\alpha} f)(x) \right).
$$

**Proof.** We write

$$
\frac{1}{|B|} \int_B |[b, I_{\alpha}]f(y) - ([b, I_{\alpha}]f)_B|dy \lesssim \frac{2}{|B|} \int_B |(b(y) - b_B)I_{\alpha} f(y)|dy + \frac{2}{|B|} \int_B |I_{\alpha}((b - b_B)f_1)(y)|dy + \frac{1}{|B|} \int_B |I_{\alpha}((b - b_B)f_2)(y) - (I_{\alpha}((b - b_B)f_2))_B|dy
$$

$$
= J_1 + J_2 + J_3,
$$

where $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$.

Since $r \leq \gamma p(x_0)$ and $\rho(x) \approx \rho(x_0)$, by Hölder’s inequality and Lemma 2.5, we get

$$
J_1 \lesssim \left( \frac{1}{|B|} \int_B |b(y) - b_B|^{s'} \ dy \right)^{1/s'} \left( \frac{1}{|B|} \int_B |I_{\alpha} f(y)|^s \ dy \right)^{1/s}
$$

$$
\lesssim |b|^\theta r^\beta \left( \frac{1}{|B|} \int_B |I_{\alpha} f(y)|^s \ dy \right)^{1/s} \lesssim |b|^\theta M_{\beta, s}(I_{\alpha} f)(x).
$$

For some $1 < r < s$, and $1/r - 1/r_0 = \alpha/n$, by Hölder’s inequality and Lemma 2.5, we have

$$
J_2 \lesssim \left( \frac{1}{|B|} \int_B |I_{\alpha}((b - b_B)f_1)(y)|^{r_0} \ dy \right)^{1/r_0}
$$

$$
\lesssim \frac{1}{|B|^{-\alpha/n}} \left( \frac{1}{|2B|} \int_{2B} |(b(y) - b_B)f(y)|^r \ dy \right)^{1/r}
$$
Let \( \beta \) holds, where we have used the finite overlapping property given by Proposition 2.3.

Choosing \( \tau > 1 \), we only need to show that for any \( \alpha \) is a constant independent of \( \beta \).

By Lemma 3.2,

\[
J_3 \leq \int_{B} \int_{B} \int_{(2B)^c} |\alpha(y,u) - \alpha(z,u)||b(u) - b_B||f(u)| dudzdy \\
\leq \int_{(2B)^c} |\alpha(y,u) - \alpha(z,u)||b(u) - b_B||f(u)| du \\
\leq [b]_{\beta}^{q} M_{\alpha + \beta,s}(f)(x).
\]

We now come to prove Theorem 1.4. By proposition 2.4, Lemma 3.1, and Lemma 3.3 we have

\[
\| [b, I_\alpha] f \|_{L_q^q(R^n)} \leq \int_{R^n} |M_{\rho,q}([b, I_\alpha] f)(x)| q \, dx \\
\leq \int_{R^n} |M_{\rho,q}^2([b, I_\alpha] f)(x)| q \, dx + \sum_k |Q_k| \left( \frac{1}{|Q_k|} \int_{2Q_k} |[b, I_\alpha] f(x)| dx \right)^q \\
\leq \int_{R^n} |M_{\rho,q}^2([b, I_\alpha] f)(x)| q \, dx + \sum_k |Q_k| \left( \inf_{y \in 2Q_k} M_{\alpha + \beta,s}(f)(y) \right)^q \\
\leq [b]_{\beta}^{q} \int_{R^n} |M_{\alpha + \beta,s}(f)(x) + M_{\alpha,s}(I_\alpha f)(x)| q \, dx + [b]_{\beta}^{q} \sum_k \int_{2Q_k} |M_{\alpha + \beta,s}(f)(x)| q \, dx \\
\leq [b]_{\beta}^{q} \left( \int_{R^n} |M_{\alpha + \beta,s}(f)(x)| q \, dx + \int_{R^n} |M_{\alpha,s}(I_\alpha f)(x)| q \, dx \right) \\
\leq [b]_{\beta}^{q} \| f \|_{L_p^q(R^n)}^q,
\]

where we have used the finite overlapping property given by Proposition 2.3.

4. Proofs of Theorems 1.5 and 1.6

Let us first prove Theorem 1.5.

Choosing \( \tau > 1 \), we only need to show that for any \( H^0_{\rho,q} \)-atom \( \alpha \),

\[
\| [b, I_\alpha] \alpha \|_{L_q(R^n)} \leq C
\]

holds, where \( C \) is a constant independent of \( \alpha \). Suppose \( \text{supp } \alpha \subset B = B(x_0, r) \) with \( r < \rho(x_0) \). Then

\[
\| [b, I_\alpha] \alpha \|_{L_q(R^n)} \leq \left( \int_{2B} |[b, I_\alpha] \alpha(x)| q \, dx \right)^{1/q} + \left( \int_{(2B)^c} |[b, I_\alpha] \alpha(x)| q \, dx \right)^{1/q} = A_1 + A_2.
\]

Let \( 1/q_1 = 1/\tau - (\alpha + \beta)/n \). By Theorem 1.4 and the size condition of atom \( \alpha \), we have

\[
A_1 \leq \left( \int_{2B} |[b, I_\alpha] \alpha(x)| q_1 \, dx \right)^{1/q_1} (2r)^{\frac{n}{q_1} - \frac{n}{q}} \leq C \left( \int_{2B} |\alpha(x)|^\tau \, dx \right)^{1/\tau} (2r)^{\frac{n}{q} - \frac{n}{q_1}} \leq C(2r)^{\frac{n}{q} - \frac{n}{p}} (2r)^{\frac{n}{q} - \frac{n}{p}} = C.
\]
For $A_2$, we consider two cases, that are $r < \rho(x_0)/4$ and $\rho(x_0)/4 \leq r < \rho(x_0)$.

**Case I:** When $r < \rho(x_0)/4$, by the vanishing condition of $a$, we have

$$||b, I_\alpha||a(x)|| \leq ||b(x) - b_B||_B \int_B |K_\alpha(x,y) - K_\alpha(x,x_0)||a(y)||dy + \int_B |K_\alpha(x,y)(b(y) - b_B)a(y)||dy = A_{21} + A_{22}.$$  

Note that

$$\int_B |a(y)||dy \lesssim r^{n-\frac{n}{p}},$$

and

$$\frac{1}{|2^k B|} \int_{2^k B} |b(x) - b_B|^q dx \lesssim ([b]_p^q)^q \sum_{k > 1} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{-\alpha} \frac{r^\delta}{N/(k_0 + 1) (2^k r)^{n+\delta-\alpha}}.$$  

When $x \in 2^{k+1} B(x_0, r) \setminus 2^k B(x_0, r)$, and $y \in B$, by Lemmas 2.7 and 2.2, we can take $\delta$ such that $0 < \beta < \delta < \delta'$ and

$$|K_\alpha(x,y) - K_\alpha(x,x_0)| \lesssim \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^N/(k_0 + 1) (2^k r)^{n+\delta-\alpha}}.$$  

Noticing $1/q = 1/p - (\alpha + \beta)/n$ and $p > \frac{n}{n+\delta}$, then we get

$$\int_{(2B)_e} (A_{21})^q dx \lesssim r^{(n-\frac{n}{p})} q \left(\sum_{k \geq 1} \frac{1}{(1 + \frac{2^k r}{\rho(x_0)})^N/(k_0 + 1) (2^k r)^{n+\delta-\alpha}} \int_{2^k B} |b(x) - b_B|^q dx \right)$$

$$\lesssim ([b]_p^q)^q \sum_{k \geq 1} \left(1 + \frac{2^k r}{\rho(x_0)}\right)^{-\alpha} \frac{2^k q \left(\frac{n}{p} - n - \delta\right)}{r^\delta}.$$  

For $x \in (2B)_e, y \in B$, we have $|x - y| \approx |x - x_0|$. By Lemmas 2.7 and 2.2,

$$\left(\int_{2^{k+1} B \setminus 2^k B} |K_\alpha(x,y)|^q dx\right)^{1/q} \lesssim \frac{r^\delta}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N/(k_0 + 1) (2^k r)^{n+\delta-\alpha}} \int_{2^{k+1} B \setminus 2^k B} \frac{dx}{|x - x_0|^q (n+\delta-\alpha)}$$

$$\lesssim \frac{2^{-k\delta} r^\delta}{\left(1 + \frac{2^k r}{\rho(x_0)}\right)^N/(k_0 + 1) (2^k r)^{n+\delta-\alpha}}.$$  

By Hölder’s inequality and Lemma 2.5 we get

$$\int_B |b(y) - b_B||a(y)||dy \leq \left(\int_B |a(y)|^\tau dy\right)^{1/\tau} \left(\int_B |b(y) - b_B|^\tau' dy\right)^{1/\tau'}$$

$$\lesssim ([b]_p^q)^{\frac{n}{p} - \frac{n}{p} \beta + \frac{n}{p} \gamma} \left(1 + \frac{r}{\rho(x_0)}\right)^{\theta}$$

$$\lesssim ([b]_p^q)^{\frac{n}{p} - \frac{n}{p} \beta} \left(1 + \frac{r}{\rho(x_0)}\right)^{\theta}.$$
Then, by Minkowski’s inequality we get
\[
\left( \int_{(2B)^c} (A_{22})^q \, dx \right)^{1/q} \lesssim \int_B |b(y) - b_B| \cdot |a(y)| \, dy \left( \sum_{k \geq 1} \int_{2^{k+1}B \setminus 2^kB} |K_\alpha(x, y)|^q \, dx \right)^{1/q}
\]
\[
\lesssim |b| \beta \cdot r^{n-\frac{\alpha}{q}+\beta} \left( 1 + \frac{r}{r_0(x_0)} \right)^{\theta'} \sum_{k \geq 1} \frac{2^{-k\delta}}{(1 + \frac{2^k r}{r_0(x_0)})^{\frac{N}{q_0\tau}} (2^k r)^{\frac{n}{\eta} - \alpha}}
\]
\[
\lesssim |b| \beta \sum_{k \geq 1} 2^{k(n-\frac{\alpha}{q}+\beta+\delta)}
\]
\[
\lesssim |b| \beta.
\]

**Case II:** When $\rho(x_0)/4 \leq r < \rho(x_0)$, this is $\frac{r}{\rho(x_0)} \geq 1/4$. The atom $a$ does not satisfy the vanishing condition. By Minkowski’s inequality,
\[
A_2 \leq \left\{ \int_{(2B)^c} |b(x) - b_B|^q \left| \int_B K_\alpha(x, y) a(y) \, dy \right|^q \, dx \right\}^{1/q}
\]
\[
+ \left\{ \int_{(2B)^c} \left| \int_B K_\alpha(x, y) (b(y) - b_B) a(y) \, dy \right|^q \, dx \right\}^{1/q}
\]
\[
= A_{21} + A_{22}.
\]

When $y \in B, x \in 2^{k+1}B \setminus 2^kB$, we have
\[
|K_\alpha(x, y)| \lesssim \frac{1}{\left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{\frac{N}{q_0\tau}} (2^k r)^{n-\alpha}'}
\]
\[
\int_B |a(y)| \, dy \lesssim r^{n-\frac{\alpha}{q}}
\]

and
\[
\int_{2^kB} |b(x) - b_B|^q \, dx \lesssim (|b| \beta)^q (2^k r)^{n+\beta} \left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{\theta'q}.
\]

Note that $\frac{r}{\rho(x_0)} \geq 1/4$, then
\[
(A_{21})^q \lesssim (|b| \beta)^q \sum_{k \geq 1} \frac{1}{\left( 1 + \frac{2^k r}{\rho(x_0)} \right)^{\frac{N}{q_0\tau}-q\theta'} (2^k r)^{n-\alpha}q r^{(n-\frac{\alpha}{q})q}}
\]
\[
\lesssim (|b| \beta)^q \sum_{k \geq 1} \frac{1}{(2^k r)^{\frac{n}{\eta} - q\theta'} (2^k r)^{(n-\alpha)q} r^{(n-\frac{\alpha}{q})q}} \lesssim (|b| \beta)^q.
\]

Since $N$ can be chosen large enough, the last series converges.

The estimate of $A_{22}$ is exactly the same as $\|A_{22}\|_{L^q((2B)^c)}$, we omit the detail of the proof. Then the proof of Theorem 1.5 is finished.

Finally, we proceed to prove Theorem 1.6.

Let $f \in H^{n/p}_{\mathcal{C}}(\mathbb{R}^n)$, we write $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where each $a_j$ is an $H^{n/p}_{\mathcal{C}}$-atom, $1 < l < \frac{n}{\alpha+\beta}$, and
\[
\left( \sum_{j=-\infty}^{\infty} |\lambda_j|^{n/p+\beta} \right)^{n/p+\beta} \lesssim 2\|f\|_{H^{n/p}_{\mathcal{C}}(\mathbb{R}^n)}.
Suppose that supp \( a_j \subset B_j = B(x_j, r_j) \) with \( r_j < \rho(x_j) \). Write

\[
[b, \mathbb{I}_\alpha]f(x) = \sum_{j=-\infty}^{\infty} \lambda_j [b, \mathbb{I}_\alpha] a_j(x) \chi_{SB_j}(x) + \sum_{j, r_j \geq \rho(x_j)/4} \lambda_j (b(x) - b_{B_j}) \mathbb{I}_\alpha a_j(x) \chi_{SB_j}(x)
\]

\[
+ \sum_{j, r_j < \rho(x_j)/4} \lambda_j (b(x) - b_{B_j}) \mathbb{I}_\alpha a_j(x) \chi_{SB_j}(x) - \sum_{j=-\infty}^{\infty} \lambda_j \mathbb{I}_\alpha (b - b_{B_j}) a_j(x) \chi_{SB_j}(x)
\]

\[
= \sum_{i=1}^{4} \sum_{j=-\infty}^{\infty} \lambda_j A_{ij}(x).
\]

In the following, we always let \( q = \frac{n}{n-\alpha} \). Note that

\[
\left( \int_{B_j} |a_j(x)|^q \, dx \right)^{1/q} \lesssim |B_j|^{\frac{1}{q} - \frac{n+\beta}{n}}.
\]

Choose \( t > \frac{n-\alpha}{n-\alpha-\beta} \) such that \( \frac{1}{qt} = \frac{1}{t} - \frac{\alpha+\beta}{n} \). By Hölder’s inequality and Theorem 1.4 we get

\[
\|A_{1,t}\|_{L^q(\mathbb{R}^n)} \lesssim \left( \int_{SB_j} |[b, \mathbb{I}_\alpha] a_j(x)|^{qt} \, dx \right)^{\frac{1}{qt}} \lesssim \left( \int_{B_j} |a_j(x)|^t \, dx \right)^{1/t} \lesssim [b]_\beta^0 |B_j|^{\frac{1}{q} - \frac{n+\beta}{n}} \lesssim [b]_\beta^0.
\]

Noticing \( 0 < \frac{n}{n+\beta} < 1 \), we get

\[
\left\| \sum_{j=-\infty}^{\infty} \lambda_j A_{1j} \right\|_{L^q(\mathbb{R}^n)} \lesssim \sum_{j=-\infty}^{\infty} |\lambda_j| \|A_{1j}\|_{L^q(\mathbb{R}^n)} \lesssim [b]_\beta^0 \sum_{j=-\infty}^{\infty} |\lambda_j| \lesssim [b]_\beta^0 \left( \sum_{j=-\infty}^{\infty} |\lambda_j|^{\frac{n}{n+\beta}} \right)^{n+\beta \over n} \lesssim [b]_\beta^0 \|f\|_{H^{n+\beta \over n+\beta}(\mathbb{R}^n)}.
\]

Then

\[
\left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} \lambda_j A_{1j} > \frac{\lambda}{4} \right\} \lesssim \frac{\lambda}{\lambda^q} \sum_{j=-\infty}^{\infty} |\lambda_j| \lesssim \frac{\lambda}{\lambda^q} \|f\|_{H^{n+\beta \over n+\beta}(\mathbb{R}^n)}.
\]

Since \( x \in B_j, y \in 2^{k+1}B_j \setminus 2^kB_j \), we have \( |x - y| \approx |x - x_j| \approx 2^kr_j \), and by Lemma 2.2 we get

\[
\frac{1}{1 + \frac{|x-y|}{\rho(x)}} \leq \frac{1}{1 + \frac{2^k r_j}{\rho(x_j)}} \frac{|x|}{|x|} \lesssim \frac{1}{1 + \frac{2^k r_j}{\rho(x_j)}} \frac{|x|}{|x|}.
\]

Note that \( \int_{B_j} |a_j(y)| \, dy \leq r_j^{-\beta} \), and \( r_j / \rho(x_j) \geq 1/4 \). Then

\[
\|A_{2,j}(x)\|_{L^q(\mathbb{R}^n)}^q \lesssim \sum_{k \geq 3} \int_{2^{k+1}B_j \setminus 2^kB_j} |b(x) - b_{B_j}|^q \left( \int_{B_j} \frac{1}{1 + \frac{|x-y|}{\rho(x)}} \frac{1}{|x-y|^{n-\alpha}} |a_j(y)| \, dy \right)^q dx
\]

\[
\lesssim \sum_{k \geq 3} \frac{1}{1 + \frac{2^k r_j}{\rho(x_j)}} \frac{|x|}{|x|} \int_{2^{k+1}B_j} |b(x) - b_{B_j}|^q \left( \int_{B_j} \frac{1}{1 + \frac{|x-y|}{\rho(x)}} \frac{1}{|x-y|^{n-\alpha}} |a_j(y)| \, dy \right)^q dx.
\]
Thus, by the vanishing condition of $a_j$ and $0 < \beta < \delta < \delta'$ we have

$$
\| A_{3,j}(x) \|_{L^q(\mathbb{R}^n)}^q = \sum_{k \geq 3} \int_{2^{k-1}B_j \setminus 2^kB_j} |b(x) - b_{B_j}|^q \left( \int_{B_j} |K_\alpha(x,y) - K_\alpha(x,x_j)||a_j(y)|dy \right)^q dx
$$

$$
\lesssim \sum_{k \geq 3} \frac{1}{\left( 1 + \frac{2k \rho(\mathcal{R}_j)}{\rho(x_j)} \right)^{Nq \mathcal{R}_j}} \frac{r_j^{\delta q}}{(2^k \mathcal{R}_j)^{(n+\delta-\alpha)q}} \int_{2^{k-1}B_j} |b(x) - b_{B_j}|^q dx \left( \int_{B_j} |a_j(y)|dy \right)^q
$$

$$
\lesssim (b)_{\beta}^q \sum_{k \geq 3} \left( \frac{1}{2k^k \rho(\mathcal{R}_j)} \right)^{Nq \mathcal{R}_j} \frac{r_j^{\delta q}}{(2^k \mathcal{R}_j)^{(n+\delta-\alpha)q}} \frac{1}{2^{(\delta-\beta)q}} \lesssim (b)_{\beta}^q
$$

Then

$$
\left\| \sum_{j=-\infty}^{\infty} \lambda_j A_{3,j} \right\|_{L^q(\mathbb{R}^n)} \lesssim [b]_{\beta}^q \|f\|_{H^\frac{n}{n+\delta-\alpha}(\mathbb{R}^n)}^{\frac{n}{n+\delta-\alpha}}.
$$

Therefore

$$
\left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} \lambda_j A_{3,j} > \frac{\lambda}{4} \right\} \lesssim \frac{[b]_{\beta}^q}{\lambda^q} \|f\|_{H^\frac{n}{n+\delta-\alpha}(\mathbb{R}^n)}^{\frac{n}{n+\delta-\alpha}}.
$$

Note that

$$
\| (b - b_{B_j}) a_j \|_{L^1} \leq \left( \int_{B_j} |b(x) - b_{B_j}|^q dx \right)^{1/1'} \left( \int_{B_j} |a_j(x)|^q dx \right)^{1/1'}
$$
Thus, \(\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} |\lambda_j A_j(x)| > \frac{\lambda}{4} \} \leq \left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} |\lambda_j (b - b_{B_j}) a_j| (x) > \frac{\lambda}{4} \right\}\)

By the boundedness of \(I_\alpha\) from \(L^1(\mathbb{R}^n)\) to \(WL^q(\mathbb{R}^n)\) (see Corollary 2.8) we get

\[
\left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} |\lambda_j A_j| > \frac{\lambda}{4} \right\} \leq \left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} |\lambda_j (b - b_{B_j}) a_j| (x) > \frac{\lambda}{4} \right\}
\]

Thus,

\[
\left\{ x \in \mathbb{R}^n : \sum_{i=1}^{4} \sum_{j=-\infty}^{\infty} \lambda_j A_{ij} > \frac{\lambda}{4} \right\} \leq \left\{ x \in \mathbb{R}^n : \sum_{j=-\infty}^{\infty} |\lambda_j A_j| > \frac{\lambda}{4} \right\}
\]

which completes the proof of Theorem 1.6.

References


[4] J. Dziubański, J. Zienkiewicz, Hardy space \(H^1\) associated to Schrödinger operator with potential satisfying reverse Hölder inequality, Rev. Mat. Iberoamericana, 15 (1999), 279–296, 1.2.3


